

# SpheroidalPS

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## Notations

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### Traditional name

Angular spheroidal function of the first kind

### Traditional notation

$PS_{\nu,\mu}(\gamma, z)$

### Mathematica StandardForm notation

SpheroidalPS[ $\nu, \mu, \gamma, z$ ]

## Primary definition

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11.08.02.0001.01

$PS_{\nu,\mu}(\gamma, z)$

$PS_{\nu,\mu}(\gamma, z)$  is the angular spheroidal function of the first kind with variable  $z$  and parameters  $\nu, \mu, \gamma$ . It is defined as the normalizable solution  $w(z) = PS_{\nu,\mu}(\gamma, z)$  of the wave differential equation  $(1 - z^2)w''(z) - 2zw'(z) + (\lambda + \gamma^2(1 - z^2) - \mu^2/(1 - z^2))w(z) = 0$  with parameter  $\lambda$  equal to spheroidal eigenvalue  $\lambda = \lambda_{\nu,\mu}(\gamma)$ . The parameter  $\nu$  enumerates the spheroidal eigenvalues in such a manner that in the spherical limit ( $\gamma \rightarrow 0$ ), the eigenvalues are  $\lambda_{\nu,\mu}(0) = \nu(\nu + 1)$  and  $PS_{\nu,\mu}(0, z) = P_{\nu}^{\mu}(z)$ , where  $P_{\nu}^{\mu}(z)$  is the Legendre function of the first kind. The angular spheroidal functions are normalized according to the Meixner-Schärfke normalization scheme, meaning  $\int_{-1}^1 PS_{n_1,m}(\gamma, \eta) PS_{n_2,m}(\gamma, \eta) d\eta = 2(m + n_1)! / ((2n_1 + 1)(n_1 - m)!) \delta_{n_1,n_2} /;$   $PS_{\nu,\mu}(\gamma, z)$  is an analytical function in the variables  $\nu, \mu, \gamma$  and  $z$ .

$$n_1 \in \mathbb{N} \wedge n_2 \in \mathbb{N} \wedge m \in \mathbb{N} \wedge n_1 \geq m \wedge n_2 \geq m$$

## Specific values

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### Specialized values

For fixed  $\nu, \mu, z$

11.08.03.0001.01

$PS_{\nu,\mu}(0, z) = P_{\nu}^{\mu}(z)$

11.08.03.0002.01

$$PS_{\nu,\mu}(0, z) = \frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}} P_{\nu}^{\mu}(z)$$

**For fixed  $\nu, \gamma, z$**

11.08.03.0003.01

$$PS_{\nu+\frac{1}{2}}^{\frac{1}{2}}(\gamma, z) = \frac{\sqrt{2}}{\sqrt{\pi} \sqrt[4]{1-z^2}} \text{Ce} \left( a_{\nu+\frac{1}{2}} \left( \frac{\gamma^2}{4} \right), \frac{\gamma^2}{4}, \cos^{-1}(z) \right)$$

**For fixed  $\nu, z$**

11.08.03.0004.01

$$PS_{\nu,0}(0, z) = P_{\nu}(z)$$

11.08.03.0005.01

$$PS_{\nu+\frac{1}{2}}^{\frac{1}{2}}(0, z) = \frac{\sqrt{\pi}}{\sqrt{2} \sqrt[4]{1-z^2}} \cos \left( \left( \nu + \frac{1}{2} \right) \cos^{-1}(z) \right)$$

11.08.03.0006.01

$$PS_{\nu+\frac{1}{2}}^{\frac{1}{2}}(0, z) = \frac{\sqrt{\pi}}{\sqrt{2} \sqrt[4]{1-z^2}} T_{\nu+\frac{1}{2}}(z)$$

## General characteristics

### Domain and analyticity

$PS_{\nu,\mu}(\gamma, z)$  is an analytical function of  $\nu, \mu, \gamma, z$  which is defined in  $\mathbb{C}^4$ .

11.08.04.0001.01

$$(\nu * \mu * \gamma * z) \rightarrow PS_{\nu,\mu}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

### Symmetries and periodicities

#### Parity

$PS_{\nu,\mu}(\gamma, z)$  is an even function with respect to  $\gamma$ .

11.08.04.0002.01

$$PS_{\nu,\mu}(-\gamma, z) = PS_{\nu,\mu}(\gamma, z)$$

11.08.04.0003.01

$$PS_{-\nu,\mu}(\gamma, z) = PS_{\nu-1,\mu}(\gamma, z)$$

#### Mirror symmetry

11.08.04.0004.01

$$PS_{\bar{\nu},\bar{\mu}}(\bar{\gamma}, \bar{z}) = \overline{PS_{\nu,\mu}(\gamma, z)}$$

#### Periodicity

No periodicity

## Series representations

### Generalized power series

#### Expansions at $\gamma = 0$

11.08.06.0001.01

$$PS_{\nu,\mu}(\gamma, z) \propto P_{\nu}^{\mu}(z) + \frac{1}{2(2\nu+1)} \left( \frac{(-\mu+\nu+1)(-\mu+\nu+2)}{(2\nu+3)^2} P_{\nu+2}^{\mu}(z) - \frac{(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^2} P_{\nu-2}^{\mu}(z) \right) \gamma^2 +$$

$$\frac{1}{8} \left( \frac{(\mu+\nu-3)(\mu+\nu-2)(\mu+\nu-1)(\mu+\nu)}{(4\nu^2-8\nu-5)(4\nu^2-8\nu+3)^2} P_{\nu-4}^{\mu}(z) + \frac{8(4\mu^2-1)(\mu^2+(2\nu-1)\mu+(\nu-1)\nu)}{(1-2\nu)^4(8\nu^3-4\nu^2-34\nu-15)} P_{\nu-2}^{\mu}(z) - \right.$$

$$\frac{1}{2\nu+1} \left( \frac{(\mu-\nu)(\mu-\nu+1)(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^4(2\nu-3)} + \frac{(-\mu+\nu+1)(-\mu+\nu+2)(\mu+\nu+1)(\mu+\nu+2)}{(2\nu+3)^4(2\nu+5)} \right) P_{\nu}^{\mu}(z) +$$

$$\frac{(-\mu+\nu+1)(-\mu+\nu+2)(-\mu+\nu+3)(-\mu+\nu+4)}{(2\nu+1)(2\nu+3)^2(2\nu+5)^2(2\nu+7)} P_{\nu+4}^{\mu}(z) -$$

$$\left. \frac{8(4\mu^2-1)(\mu^2-(2\nu+3)\mu+\nu^2+3\nu+2)}{(2\nu+3)^4(8\nu^3+28\nu^2-2\nu-7)} P_{\nu+2}^{\mu}(z) \right) \gamma^4 + \dots /; (\gamma \rightarrow 0)$$

11.08.06.0002.01

$$PS_{\nu,\mu}(\gamma, z) \propto P_{\nu}^{\mu}(z) (1 + O(\gamma^2))$$

#### Expansions at generic point $z = z_0$

11.08.06.0003.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, z_0) + PS_{\nu,\mu}'(\gamma, z_0) (z - z_0) -$$

$$\frac{1}{2(z_0^2 - 1)^2} \left( 2 PS_{\nu,\mu}'(\gamma, z_0) z_0 (z_0^2 - 1) + PS_{\nu,\mu}(\gamma, z_0) \left( -\mu^2 + \gamma^2 (z_0^2 - 1)^2 - \lambda_{\nu,\mu}(\gamma) (z_0^2 - 1) \right) \right) (z - z_0)^2 -$$

$$\frac{1}{6(z_0^2 - 1)^3} \left( PS_{\nu,\mu}'(\gamma, z_0) (z_0^2 - 1) (\gamma^2 z_0^4 - 2(\gamma^2 + 3) z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma) (z_0^2 - 1) - 2) - \right.$$

$$\left. 2 PS_{\nu,\mu}(\gamma, z_0) z_0 \left( -3\mu^2 + \gamma^2 (z_0^2 - 1)^2 - 2\lambda_{\nu,\mu}(\gamma) (z_0^2 - 1) \right) \right) (z - z_0)^3 + \dots /; (z \rightarrow z_0)$$

11.08.06.0004.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, z_0) (1 + O(z - z_0))$$

#### Expansions at $z = 0$

11.08.06.0005.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, 0) + PS_{\nu,\mu}'(\gamma, 0) z - \frac{1}{2} (\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)) PS_{\nu,\mu}(\gamma, 0) z^2 +$$

$$\frac{1}{6} (2 PS_{\nu,\mu}'(\gamma, 0) - (\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)) PS_{\nu,\mu}'(\gamma, 0)) z^3 + \dots /; (z \rightarrow 0)$$

11.08.06.0006.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, 0) (1 + O(z))$$

### Other series representations

11.08.06.0007.01

$$M_{\nu,\mu}(0) = 0 \ ; \ \operatorname{Re}(\mu) > -\frac{1}{2}$$

## Differential equations

### Ordinary linear differential equations and wronskians

For the direct function itself

11.08.13.0001.01

$$(1 - z^2) w''(z) - 2z w'(z) + \left( (1 - z^2) \gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) w(z) = 0 \ ; \ w(z) = c_1 PS_{\nu,\mu}(\gamma, z) + c_2 QS_{\nu,\mu}(\gamma, z)$$

11.08.13.0002.01

$$W_z(PS_{\nu,\mu}(\gamma, z), QS_{\nu,\mu}(\gamma, z)) = \frac{1}{1 - z^2} (PS_{\nu,\mu}(\gamma, 0) QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0) QS_{\nu,\mu}(\gamma, 0))$$

11.08.13.0003.01

$$(1 - g(z)^2) w''(z) + \left( (g(z)^2 - 1) \frac{g''(z)}{g'(z)} - 2g(z)g'(z) \right) w'(z) + \left( \frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu,\mu}(\gamma)g'(z)^2 \right) w(z) = 0 \ ;$$

$$w(z) = c_1 PS_{\nu,\mu}(\gamma, g(z)) + c_2 QS_{\nu,\mu}(\gamma, g(z))$$

11.08.13.0004.01

$$W_z(PS_{\nu,\mu}(\gamma, g(z)), QS_{\nu,\mu}(\gamma, g(z))) = \frac{g'(z)}{1 - g[z]^2} (PS_{\nu,\mu}(\gamma, 0) QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0) QS_{\nu,\mu}(\gamma, 0))$$

11.08.13.0005.01

$$(1 - g(z)^2) w''(z) + \left( (g(z)^2 - 1) \left( \frac{2h'(z)}{h(z)} + \frac{g''(z)}{g'(z)} \right) - 2g(z)g'(z) \right) w'(z) +$$

$$\left( \frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu,\mu}(\gamma)g'(z)^2 + \frac{2g(z)h'(z)g'(z)}{h(z)} + \frac{(g(z)^2 - 1)(h(z)h''(z) - 2h'(z)^2)}{h(z)^2} - \frac{(g(z)^2 - 1)h'(z)g''(z)}{h(z)g'(z)} \right) w(z) = 0 \ ; \ w(z) = c_1 h(z) PS_{\nu,\mu}(\gamma, g(z)) + c_2 h(z) QS_{\nu,\mu}(\gamma, g(z))$$

11.08.13.0006.01

$$W_z(h(z) PS_{\nu,\mu}(\gamma, g(z)), h(z) QS_{\nu,\mu}(\gamma, g(z))) = \frac{h(z)^2 g'(z)}{1 - g[z]^2} (PS_{\nu,\mu}(\gamma, 0) QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0) QS_{\nu,\mu}(\gamma, 0))$$

11.08.13.0007.01

$$(1 - a^2 z^{2r}) w''(z) - \frac{a^2(r - 2s + 1)z^{2r} + r + 2s - 1}{z} w'(z) +$$

$$\left( a^2 r^2 \lambda_{\nu,\mu}(\gamma) z^{2r-2} + \frac{-a^2 r^2 \left( (a^2 z^{2r} - 1)^2 \gamma^2 - \mu^2 \right) z^{2r} - s^2 (a^2 z^{2r} - 1)^2 + r s (a^4 z^{4r} - 1)}{z^2 (a^2 z^{2r} - 1)} \right) w(z) =$$

$$0 \ ; \ w(z) = c_1 z^s PS_{\nu,\mu}(\gamma, a z^r) + c_2 z^s QS_{\nu,\mu}(\gamma, a z^r)$$

11.08.13.0008.01

$$W_z(z^s PS_{v,\mu}(\gamma, a z^r), z^s QS_{v,\mu}(\gamma, a z^r)) = \frac{a r z^{r+2s-1}}{1 - a^2 z^{2r}} (PS_{v,\mu}(\gamma, 0) QS_{v,\mu}'(\gamma, 0) - PS_{v,\mu}'(\gamma, 0) QS_{v,\mu}(\gamma, 0))$$

11.08.13.0009.01

$$(1 - a^2 r^{2z}) w''(z) + (-a^2 \log(r) r^{2z} + 2 a^2 \log(s) r^{2z} - \log(r) - 2 \log(s)) w'(z) + \left( a^2 \log^2(r) \lambda_{v,\mu}(\gamma) r^{2z} + \frac{1}{a^2 r^{2z} - 1} \left( -a^2 \left( (a^2 r^{2z} - 1)^2 \gamma^2 - \mu^2 \right) \log^2(r) r^{2z} - (a^2 r^{2z} - 1)^2 \log^2(s) + (a^4 r^{4z} - 1) \log(r) \log(s) \right) \right) w(z) = 0; w(z) = c_1 s^z PS_{v,\mu}(\gamma, a r^z) + c_2 s^z QS_{v,\mu}(\gamma, a r^z)$$

11.08.13.0010.01

$$W_z(s^z PS_{v,\mu}(\gamma, a r^z), s^z QS_{v,\mu}(\gamma, a r^z)) = \frac{a r^z s^{2z} \log(r)}{1 - a^2 r^{2z}} (PS_{v,\mu}(\gamma, 0) QS_{v,\mu}'(\gamma, 0) - PS_{v,\mu}'(\gamma, 0) QS_{v,\mu}(\gamma, 0))$$

## Differentiation

### Low-order differentiation

With respect to  $z$

11.08.20.0001.01

$$\frac{\partial PS_{v,\mu}(\gamma, z)}{\partial z} = PS_{v,\mu}'(\gamma, z)$$

11.08.20.0002.01

$$\frac{\partial^2 PS_{v,\mu}(\gamma, z)}{\partial z^2} = \frac{1}{1 - z^2} \left( 2z PS_{v,\mu}'(\gamma, z) - \left( (1 - z^2) \gamma^2 + \lambda_{v,\mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) PS_{v,\mu}(\gamma, z) \right)$$

## Integration

### Definite integration

Involving the direct function

11.08.21.0001.01

$$\int_{-1}^1 PS_{n,m}(\gamma, t)^2 dt = \frac{2(m+n)!}{(2n+1)(n-m)!} /; n \in \mathbb{N} \wedge m \in \mathbb{Z} \wedge |m| \leq n$$

11.08.21.0002.01

$$\int_{-1}^1 PS_{n_1,m}(\gamma, t) PS_{n_2,m}(\gamma, t) dt = \frac{(2(m+n_1)!) \delta_{n_1,n_2}}{(2n_1+1)(n_1-m)!} /; n_1 \in \mathbb{Z} \wedge n_2 \in \mathbb{Z} \wedge m \in \mathbb{Z} \wedge n_1 \geq m \wedge n_2 \geq m$$

## Representations through equivalent functions

### Theorems

### History

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