StirlingS2

Notations

Traditional name

Stirling number of the second kind

Traditional notation

$\mathcal{S}_n^{(m)}$

Mathematica StandardForm notation

```
StirlingS2[n, m]
```

Primary definition

04.15.02.0001.01

$$\mathcal{S}_n^{(m)} = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n \;/;\; n \in \mathbb{N}^+ \land m \in \mathbb{N}$$

04.15.02.0002.01

$$\mathcal{S}_0^{(m)} = \delta_m \;/;\; m \in \mathbb{N}$$

$\mathcal{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m nonempty subsets.

Examples: 1) The set $\{a, b, c\}$ can be partitioned into three subsets in one way: $\{\{a\}, \{b\}, \{c\}\}$; into two subsets in three ways: $\{\{a, b\}, \{c\}\}$, $\{\{a, c\}, \{b\}\}$, $\{\{b, c\}, \{a\}\}$; and into one subset in one way: $\{\{a, b, c\}\}$. By these reasons $\mathcal{S}_3^{(3)} = 1$, $\mathcal{S}_3^{(2)} = 3$, and $\mathcal{S}_3^{(1)} = 1$.

2) The set $\{a, b, c, d\}$ can be partitioned into four subsets in one way: $\{\{a\}, \{b\}, \{c\}, \{d\}\}$;
into three subsets in six ways:
$\{\{a, b\}, \{c\}, \{d\}\}$, $\{\{a, c\}, \{b\}, \{d\}\}$, $\{\{a, d\}, \{b\}, \{c\}\}$, $\{\{b, c\}, \{a\}, \{d\}\}$, $\{\{b, d\}, \{a\}, \{c\}\}$, $\{\{c, d\}, \{a\}, \{b\}\}$;
into two subsets in seven ways:
$\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}, \{b, d\}\}$, $\{\{a, d\}, \{b, c\}\}$, $\{\{a, b, c\}, \{d\}\}$, $\{\{a, b, d\}, \{c\}\}$, $\{\{a, c, d\}, \{b\}\}$, $\{\{b, c, d\}, \{a\}\}$;
and into one subset in one way: $\{\{a, b, c, d\}\}$. By these reasons $\mathcal{S}_4^{(4)} = 1$, $\mathcal{S}_4^{(3)} = 6$, $\mathcal{S}_4^{(2)} = 7$, and $\mathcal{S}_4^{(1)} = 1$.

Specific values

Specialized values

For fixed *n*

04.15.03.0001.01

$$\mathcal{S}_n^{(0)} = \delta_n \;/;\; n \in \mathbb{N}$$

04.15.03.0002.02

$$\mathcal{S}_n^{(1)} = 1 - \delta_n \;/;\; n \in \mathbb{N}$$

04.15.03.0003.02

$$\mathcal{S}_n^{(2)} = \frac{\delta_n}{2} + 2^{n-1} - 1 \;/;\; n \in \mathbb{N}$$

04.15.03.0028.01

$$\mathcal{S}_n^{(3)} = \frac{1}{6}\left(3 - \delta_n - 3\, 2^n + 3^n\right) /;\; n \in \mathbb{N}$$

04.15.03.0029.01

$$\mathcal{S}_n^{(4)} = \frac{1}{24}\left(\delta_n - 4\, 3^n + 4^n + 3\, 2^{n+1} - 4\right) /;\; n \in \mathbb{N}$$

04.15.03.0030.01

$$\mathcal{S}_n^{(5)} = \frac{5 - \delta_n + 10\, 3^n - 5\, 4^n + 5^n - 5\, 2^{n+1}}{120} \;/;\; n \in \mathbb{N}$$

04.15.03.0031.01

$$\mathcal{S}_n^{(6)} = \frac{\delta_n + 15\, 2^n - 20\, 3^n + 15\, 4^n - 6\, 5^n + 6^n - 6}{720} \;/;\; n \in \mathbb{N}$$

04.15.03.0032.01

$$\mathcal{S}_n^{(7)} = \frac{7 - \delta_n - 21\, 2^n + 35\, 3^n - 35\, 4^n + 21\, 5^n - 7\, 6^n + 7^n}{5040} \;/;\; n \in \mathbb{N}$$

04.15.03.0033.01

$$\mathcal{S}_n^{(8)} = \frac{1}{40\,320}\left(\delta_n + 7\, 2^{n+2}\, 3^n - 56\, 3^n - 56\, 5^n - 8\, 7^n + 8^n + 7\, 2^{n+2} + 35\, 2^{2n+1} - 8\right) /;\; n \in \mathbb{N}$$

04.15.03.0034.01

$$\mathcal{S}_n^{(9)} = \frac{1}{362\,880}\left(9 - \delta_n + 126\, 5^n + 36\, 7^n - 9\, 8^n + 9^n - 7\, 2^{n+2}\, 3^{n+1} + 28\, 3^{n+1} - 9\, 2^{n+2} - 63\, 2^{2n+1}\right) /;\; n \in \mathbb{N}$$

04.15.03.0035.01

$$\mathcal{S}_n^{(10)} = \frac{1}{3\,628\,800}\left(\delta_n + 45\, 2^n - 252\, 5^n - 120\, 7^n + 45\, 8^n - 10\, 9^n + 10^n - 40\, 3^{n+1} + 35\, 6^{n+1} + 105\, 2^{2n+1} - 10\right) /;\; n \in \mathbb{N}$$

04.15.03.0004.01

$$\mathcal{S}_n^{(n-1)} = \binom{n}{2} \;/;\; n \in \mathbb{N}^+$$

04.15.03.0005.01

$$\mathcal{S}_n^{(n)} = 1 \;/;\; n \in \mathbb{N}$$

04.15.03.0006.01

$$\mathcal{S}_n^{(m)} = 0 \;/;\; n \in \mathbb{N} \land m \in \mathbb{N}^+ \land m > n$$

For fixed *m*

04.15.03.0036.01

$$\mathcal{S}_0^{(m)} = 0 \;/;\; m \in \mathbb{N}^+$$

04.15.03.0007.01

$$\mathcal{S}_0^{(m)} \equiv \delta_m \;/;\; m \in \mathbb{N}$$

04.15.03.0037.01

$$\mathcal{S}_1^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \delta_{m-1} \;/;\; m \in \mathbb{N}$$

04.15.03.0038.01

$$\mathcal{S}_2^{(m)} \equiv \frac{(-1)^m}{(m-1)!} ((m-1)\,\delta_{m-2} - \delta_{m-1}) \;/;\; m \in \mathbb{N}$$

04.15.03.0039.01

$$\mathcal{S}_3^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} ((m-2)(m-1)\,\delta_{m-3} + 3(1-m)\,\delta_{m-2} + \delta_{m-1}) \;/;\; m \in \mathbb{N}$$

04.15.03.0040.01

$$\mathcal{S}_4^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} (\delta_{m-1} - (m-1)((m-2)((m-3)\,\delta_{m-4} - 6\,\delta_{m-3}) + 7\,\delta_{m-2})) \;/;\; m \in \mathbb{N}$$

04.15.03.0041.01

$$\mathcal{S}_5^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} (15(1-m)\,\delta_{m-2} + \delta_{m-1} + (m-4)(m-3)(m-2)(m-1)\,\delta_{m-5} - 10(m-3)(m-2)(m-1)\,\delta_{m-4} + 25(m-2)(m-1)\,\delta_{m-3}) \;/;\; m \in \mathbb{N}$$

04.15.03.0042.01

$$\mathcal{S}_6^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \left(31(1-m)\,\delta_{m-2} + \delta_{m-1} - (m-5)_5\,\delta_{m-6} - 65(m-3)(m-2)(m-1)\,\delta_{m-4} + 90(m-2)(m-1)\,\delta_{m-3} + 15\,\delta_{m-5}\,(m-4)_4\right) \;/;\; m \in \mathbb{N}$$

04.15.03.0043.01

$$\mathcal{S}_7^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \left(63(1-m)\,\delta_{m-2} + \delta_{m-1} - 21(m-5)_5\,\delta_{m-6} - 350(m-3)(m-2)(m-1)\,\delta_{m-4} + 301(m-2)(m-1)\,\delta_{m-3} + \delta_{m-7}\,(m-6)_6 + 140\,\delta_{m-5}\,(m-4)_4\right) \;/;\; m \in \mathbb{N}$$

04.15.03.0044.01

$$\mathcal{S}_8^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \left(127(1-m)\,\delta_{m-2} + \delta_{m-1} - (m-7)_7\,\delta_{m-8} - 266(m-5)_5\,\delta_{m-6} - 1701(m-3)(m-2)(m-1)\,\delta_{m-4} + 966(m-2)(m-1)\,\delta_{m-3} + 28\,\delta_{m-7}\,(m-6)_6 + 1050\,\delta_{m-5}\,(m-4)_4\right) \;/;\; m \in \mathbb{N}$$

04.15.03.0045.01

$$\mathcal{S}_9^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \left(255(1-m)\,\delta_{m-2} + \delta_{m-1} - 36(m-7)_7\,\delta_{m-8} - 2646(m-5)_5\,\delta_{m-6} - 7770(m-3)(m-2)(m-1)\,\delta_{m-4} + 3025(m-2)(m-1)\,\delta_{m-3} + \delta_{m-9}\,(m-8)_8 + 462\,\delta_{m-7}\,(m-6)_6 + 6951\,\delta_{m-5}\,(m-4)_4\right) \;/;\; m \in \mathbb{N}$$

04.15.03.0046.01

$$\mathcal{S}_{10}^{(m)} \equiv \frac{(-1)^{m-1}}{(m-1)!} \left(511(1-m)\,\delta_{m-2} + \delta_{m-1} - (m-9)_9\,\delta_{m-10} - 750(m-7)_7\,\delta_{m-8} - 22\,827(m-5)_5\,\delta_{m-6} - 34\,105(m-3)(m-2)(m-1)\,\delta_{m-4} + 9330(m-2)(m-1)\,\delta_{m-3} + 45\,\delta_{m-9}\,(m-8)_8 + 5880\,\delta_{m-7}\,(m-6)_6 + 42\,525\,\delta_{m-5}\,(m-4)_4\right) \;/;\; m \in \mathbb{N}$$

04.15.03.0047.01

$$\mathcal{S}_{m+1}^{(m)} \equiv \frac{1}{2} m(m+1) \;/;\; m \in \mathbb{N}$$

04.15.03.0048.01

$$\mathcal{S}_{m+2}^{(m)} = \frac{1}{24} m (m+1) (m+2) (3 m+1) /; m \in \mathbb{N}$$

04.15.03.0049.01

$$\mathcal{S}_{m+3}^{(m)} = \frac{1}{48} m^2 (m+1)^2 (m+2) (m+3) /; m \in \mathbb{N}$$

04.15.03.0050.01

$$\mathcal{S}_{m+4}^{(m)} = \frac{1}{5760} \left(m (m+1) (m+2) (m+3) (m+4) \left(15 m^3 + 30 m^2 + 5 m - 2\right)\right) /; m \in \mathbb{N}$$

04.15.03.0051.01

$$\mathcal{S}_{m+5}^{(m)} = \frac{1}{11\,520} \left(m^2 (m+1)^2 (m+2) (m+3) (m+4) (m+5) \left(3 m^2 + 7 m - 2\right)\right) /; m \in \mathbb{N}$$

04.15.03.0052.01

$$\mathcal{S}_{m+6}^{(m)} = \frac{1}{2\,903\,040} \left(m (m+1) (m+2) (m+3) (m+4) (m+5) (m+6) \left(63 m^5 + 315 m^4 + 315 m^3 - 91 m^2 - 42 m + 16\right)\right) /; m \in \mathbb{N}$$

04.15.03.0053.01

$$\mathcal{S}_{m+7}^{(m)} = \frac{1}{5\,806\,080} \left(m^2 (m+1)^2 (m+2) (m+3) (m+4) (m+5) (m+6) (m+7) \left(9 m^4 + 54 m^3 + 51 m^2 - 58 m + 16\right)\right) /; m \in \mathbb{N}$$

04.15.03.0054.01

$$\mathcal{S}_{m+8}^{(m)} = \frac{1}{1\,393\,459\,200} \left(m (m+1) (m+2) (m+3) (m+4) (m+5) (m+6) (m+7) (m+8) \left(135 m^7 + 1260 m^6 + 3150 m^5 + 840 m^4 - 2345 m^3 + 540 m^2 + 404 m - 144\right)\right) /; m \in \mathbb{N}$$

04.15.03.0055.01

$$\mathcal{S}_{m+9}^{(m)} = \frac{1}{2\,786\,918\,400} \left(m^2 (m+1)^2 (m+2) (m+3) (m+4) (m+5) (m+6) (m+7) (m+8) (m+9) \left(15 m^6 + 165 m^5 + 465 m^4 - 17 m^3 - 648 m^2 + 548 m - 144\right)\right) /; m \in \mathbb{N}$$

04.15.03.0056.01

$$\mathcal{S}_{m+10}^{(m)} = \frac{1}{367\,873\,228\,800} \left(m (m+1) (m+2) (m+3) (m+4) (m+5) (m+6) (m+7) (m+8) (m+9) (m+10) \left(99 m^9 + 1485 m^8 + 6930 m^7 + 8778 m^6 - 8085 m^5 - 8195 m^4 + 11\,792 m^3 - 2068 m^2 - 2288 m + 768\right)\right) /; m \in \mathbb{N}$$

Values at fixed points

04.15.03.0008.01

$$\mathcal{S}_0^{(0)} = 1$$

04.15.03.0009.01

$$\mathcal{S}_0^{(1)} = 0$$

04.15.03.0010.01

$$\mathcal{S}_1^{(0)} = 0$$

04.15.03.0011.01

$$\mathcal{S}_1^{(1)} = 1$$

04.15.03.0012.01

$$\mathcal{S}_1^{(2)} = 0$$

04.15.03.0013.01

$$\mathcal{S}_2^{(0)} == 0$$

04.15.03.0014.01

$$\mathcal{S}_2^{(1)} == 1$$

04.15.03.0015.01

$$\mathcal{S}_2^{(2)} == 1$$

04.15.03.0016.01

$$\mathcal{S}_2^{(3)} == 0$$

04.15.03.0017.01

$$\mathcal{S}_3^{(0)} == 0$$

04.15.03.0018.01

$$\mathcal{S}_3^{(1)} == 1$$

04.15.03.0019.01

$$\mathcal{S}_3^{(2)} == 3$$

04.15.03.0020.01

$$\mathcal{S}_3^{(3)} == 1$$

04.15.03.0021.01

$$\mathcal{S}_3^{(4)} == 0$$

04.15.03.0022.01

$$\mathcal{S}_4^{(0)} == 0$$

04.15.03.0023.01

$$\mathcal{S}_4^{(1)} == 1$$

04.15.03.0024.01

$$\mathcal{S}_4^{(2)} == 7$$

04.15.03.0025.01

$$\mathcal{S}_4^{(3)} == 6$$

04.15.03.0026.01

$$\mathcal{S}_4^{(4)} == 1$$

04.15.03.0027.01

$$\mathcal{S}_4^{(5)} == 0$$

General characteristics

Domain and analyticity

$\mathcal{S}_n^{(m)}$ is a nonanalytical function which is defined only for nonnegative integers n, m.

04.15.04.0001.01

$$(n * m) \longrightarrow \mathcal{S}_n^{(m)} :: (\mathbb{N} \otimes \mathbb{N}) \longrightarrow \mathbb{N}$$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Generalized power series

04.15.06.0001.01

$$\mathcal{S}_n^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} j^n$$

04.15.06.0002.01

$$\mathcal{S}_n^{(m)} = \sum_{r_1=1}^{m} \ldots \sum_{r_{n-m}=r_{n-m-1}}^{m} \prod_{j=1}^{n-m} r_j$$

04.15.06.0003.01

$$\mathcal{S}_n^{(m)} = \frac{n!}{m!} \sum_{r_1=1}^{n} \ldots \sum_{r_m=1}^{n} \frac{1}{\prod_{j=1}^{m} r_j!} \delta_{n,\sum_{j=1}^{m} r_j}$$

04.15.06.0004.01

$$\mathcal{S}_n^{(m)} = \sum_{r_1=0}^{m} \ldots \sum_{r_m=r_{m-1}}^{m} \delta_{n-m,\sum_{j=1}^{m} r_j} \prod_{j=1}^{m} j^{r_j}$$

04.15.06.0005.01

$$\mathcal{S}_n^{(m)} = \sum_{k_1=0}^{\max(m,n)} \sum_{k_2=0}^{\max(m,n)} \ldots \sum_{k_n=0}^{\max(m,n)} \frac{n!}{\prod_{j=1}^{n} j!^{k_j} k_j!} \delta_{m,\sum_{j=1}^{n} k_j} \delta_{n,\sum_{j=1}^{n} j k_j}$$

Asymptotic series expansions

04.15.06.0007.01

$$\mathcal{S}_n^{(m)} \propto \frac{(e^u - 1)^m u^{-n} n!}{\sqrt{2\pi} \sqrt{(n+1)\left(1 - \frac{u}{e^u - 1}\right)} m!} /; (n \to \infty) \bigwedge u = \frac{n+1}{m} + W\left(\frac{e^{-\frac{n+1}{m}} (-n-1)}{m}\right)$$

Residue representations

04.15.06.0006.01

$$\mathcal{S}_n^{(m)} = \frac{n!}{m!} \operatorname{res}_z\left((e^z - 1)^m z^{-n-1}\right)(0)$$

Integral representations

Contour integral representations

04.15.07.0001.01

$$\mathcal{S}_n^{(m)} = \frac{n!}{2\pi i\, m!} \int_{|z|=1} (e^z - 1)^m z^{-n-1} \, dz$$

Limit representations

04.15.09.0001.01

$$\mathcal{S}_n^{(m)} = \frac{(-1)^{m-1}}{\Gamma(m)} \lim_{z\to 1} {}_nF_{n-1}(1-m, a_1+1, a_2+1, \ldots, a_{n-1}+1; a_1, a_2, \ldots, a_{n-1}; z) \,/;\, a_1 = a_2 = \ldots = a_{n-1} = 1 \bigwedge n \in \mathbb{N}^+$$

Generating functions

04.15.11.0001.01

$$\mathcal{S}_n^{(m)} = \frac{n!}{m!} \left([t^n] \left(e^t - 1\right)^m\right) /;\, m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.11.0002.01

$$\mathcal{S}_n^{(m)} = \frac{n!}{m!} \left([t^{n-m}] \left(\frac{e^t - 1}{t}\right)^m\right) /;\, m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.11.0003.01

$$\mathcal{S}_n^{(m)} = \left([t^n] \frac{t^m}{\prod_{k=1}^m (1 - k\,t)}\right) /;\, m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.11.0004.01

$$\mathcal{S}_n^{(m)} = n! \left([z^n, w^m]\, e^{w(e^z - 1)}\right) /;\, m \in \mathbb{N} \land n \in \mathbb{N}$$

Identities

Recurrence identities

04.15.17.0001.01

$$\mathcal{S}_{n+1}^{(m)} = \mathcal{S}_n^{(m-1)} + m\, \mathcal{S}_n^{(m)} /;\, m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge m \leq n$$

Functional identities

04.15.17.0002.01

$$\mathcal{S}_n^{(m)} = \sum_{k=m}^{n} m^{n-k}\, \mathcal{S}_{k-1}^{(m-1)} /;\, m \in \mathbb{N}^+$$

04.15.17.0003.01

$$\mathcal{S}_n^{(m)} = \mathcal{S}_{k+n}^{(k+m)} - \sum_{j=0}^{k-1} (j + m + 1)\, \mathcal{S}_{j+n}^{(j+m+1)} /;\, n \in \mathbb{N} \land m \in \mathbb{N}$$

Identities involving determinants

04.15.17.0004.01

$$\left| \left(\frac{(x\,k)!\, \mathcal{S}_{l+k\,x}^{(x\,k)}}{(l+k\,x)!} \right)_{\substack{0\le k\le n\\ 0\le l\le n}} \right| = \left(\frac{x}{2}\right)^{\binom{n+1}{2}} /;\, n \in \mathbb{N}^+$$

Complex characteristics

Real part

04.15.19.0001.01

$$\mathrm{Re}\left(\mathcal{S}_n^{(m)}\right) = \mathcal{S}_n^{(m)}$$

Imaginary part

04.15.19.0002.01

$$\mathrm{Im}\left(\mathcal{S}_n^{(m)}\right) = 0$$

Absolute value

04.15.19.0003.01

$$\left|\mathcal{S}_n^{(m)}\right| = \mathcal{S}_n^{(m)}$$

Argument

04.15.19.0004.01

$$\arg\left(\mathcal{S}_n^{(m)}\right) = 0$$

Conjugate value

04.15.19.0005.01

$$\overline{\mathcal{S}_n^{(m)}} = \mathcal{S}_n^{(m)}$$

Signum value

04.15.19.0006.01

$$\mathrm{sgn}\left(\mathcal{S}_n^{(m)}\right) = \begin{cases} 1 & 1 \le m \le n \lor m = n = 0 \\ 0 & \text{True} \end{cases}$$

Summation

Finite summation

Not involving Stirling numbers of the first kind

04.15.23.0001.01

$$\sum_{k=0}^{n} (k+m+1)\, \mathcal{S}_{h+k}^{(k+m+1)} = \mathcal{S}_{h+n+1}^{(m+n+1)} - \mathcal{S}_h^{(m)} /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land h \in \mathbb{N}$$

04.15.23.0002.01

$$\sum_{k=m}^{n} \mathcal{S}_{k-1}^{(m-1)} m^{n-k} == \mathcal{S}_n^{(m)} /; m \in \mathbb{N}^+$$

04.15.23.0003.01

$$\sum_{k=1}^{n} (-1)^k (k-1)! \, \mathcal{S}_n^{(k)} == 0 /; n-1 \in \mathbb{N}^+$$

04.15.23.0004.01

$$\sum_{k=1}^{n} (k-1)! \, \mathcal{S}_n^{(k)} z^k == (-1)^n \operatorname{Li}_{1-n}\left(1 + \frac{1}{z}\right) /; n-1 \in \mathbb{N}^+$$

04.15.23.0005.01

$$\sum_{k=0}^{n} \frac{(-1)^k k! \, \mathcal{S}_n^{(k)}}{k+1} == B_n /; n \in \mathbb{N}$$

04.15.23.0006.01

$$\sum_{k=0}^{n} k! \binom{m+1}{k+1} \mathcal{S}_n^{(k)} == \sum_{k=0}^{m} k^n /; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

04.15.23.0007.01

$$\sum_{k=1}^{m} k! \binom{m}{k} \mathcal{S}_n^{(k)} == m^n /; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

04.15.23.0008.01

$$\sum_{k=0}^{n} (z-k+1)_k \, \mathcal{S}_n^{(k)} == z^n /; n \in \mathbb{N}$$

04.15.23.0023.01

$$\sum_{k=0}^{n} (k+1)_m \, \mathcal{S}_{m+n}^{(k+m)} z^k = \frac{\partial^m B_{m+n}(z)}{\partial z^m} /; n \in \mathbb{N} \land m \in \mathbb{N}$$

04.15.23.0009.01

$$\sum_{k=1}^{n} \frac{(-1)^k (-z)_k}{k} \mathcal{S}_{n-1}^{(k-1)} == \frac{B_n(z) - B_n}{n} /; n \in \mathbb{N}^+$$

04.15.23.0010.01

$$\sum_{k=0}^{n} (-1)^k \binom{k+m+n-1}{k+n} \binom{m+2n}{n-k} \mathcal{S}_{k+n}^{(k)} == S_{m+n}^{(m)} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.23.0011.01

$$\sum_{k=0}^{n} (-1)^k (-z)_k \, {}_2F_1(1, z+1; -k+z+1; -1) \, \mathcal{S}_n^{(k)} == \frac{1}{2} E_n(z) /; n \in \mathbb{N}$$

04.15.23.0012.01

$$\sum_{k=0}^{n} (-1)^k \, \Gamma\left(k - \frac{1}{2}\right) {}_2F_1\left(1, \frac{3}{2}; \frac{3}{2} - k; -1\right) \mathcal{S}_n^{(k)} == -2^{-n} \sqrt{\pi} \, E_n /; n \in \mathbb{N}$$

04.15.23.0013.01

$$\sum_{k=m}^{n} \binom{n+r}{k} \mathcal{S}_{n+r-k}^{(r)} \mathcal{S}_k^{(m)} == \binom{m+r}{r} \mathcal{S}_{n+r}^{(m+r)} /; n \in \mathbb{N} \land m \in \mathbb{N} \land r \in \mathbb{N} \land m \leq n$$

04.15.23.0014.01

$$\sum_{k=0}^{n} \mathcal{S}_n^{(k)} z^k \frac{\partial^k f(z)}{\partial z^k} == \underbrace{z \frac{\partial}{\partial z}\left(\ldots\left(z \frac{\partial}{\partial z}\right.\right.}_{n \text{ times}} (f(z))\Big)\Big) /; n \in \mathbb{N}$$

04.15.23.0015.01

$$\sum_{k=0}^{n} \mathcal{S}_n^{(k)} (z-a)^k \frac{\partial^k f(z)}{\partial z^k} == \underbrace{(z-a) \frac{\partial}{\partial z}\left(\ldots\left((z-a) \frac{\partial}{\partial z}\right.\right.}_{n \text{ times}} (f(z))\Big)\Big) /; n \in \mathbb{N}$$

Involving Stirling numbers of the first kind

04.15.23.0016.01

$$\sum_{l=0}^{\max(m,n)+1} S_l^{(m)} \mathcal{S}_n^{(l)} == \delta_{m,n} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.23.0017.01

$$\sum_{l=0}^{\max(m,n)+1} S_m^{(l)} \mathcal{S}_l^{(n)} == \delta_{m,n} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.23.0018.01

$$\sum_{k=m}^{n} \sum_{j=0}^{k} \mathcal{S}_k^{(j)} S_n^{(k)} S_j^{(m)} == \mathcal{S}_n^{(m)} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.23.0019.01

$$\sum_{k=m}^{n} \sum_{j=0}^{k} S_k^{(j)} \mathcal{S}_n^{(k)} \mathcal{S}_j^{(m)} == S_n^{(m)} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

Infinite summation

04.15.23.0020.01

$$\sum_{k=m}^{\infty} \mathcal{S}_k^{(m)} z^k == \frac{z^m}{\prod_{k=1}^{m} (1 - k z)} /; |z| < \frac{1}{m} \bigwedge m \in \mathbb{N}^+$$

04.15.23.0021.01

$$\sum_{k=m}^{\infty} \frac{1}{k!} \mathcal{S}_k^{(m)} z^k == \frac{(e^z - 1)^m}{m!} /; m \in \mathbb{N}$$

04.15.23.0022.01

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!} \mathcal{S}_n^{(m)} z^n w^m == e^{w (e^z - 1)}$$

Operations

Limit operation

04.15.25.0001.01

$$\lim_{n \to \infty} \frac{\mathcal{S}_n^{(m)}}{m^n} == \frac{1}{m!}$$

Representations through equivalent functions

With related functions

04.15.27.0001.01

$$\mathcal{S}_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{k+n-1}{k-m+n} \binom{2n-m}{n-k-m} S_{k-m+n}^{(k)} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.27.0002.01

$$\mathcal{S}_n^{(m)} = \sum_{k=m}^{n} \sum_{j=0}^{k} \mathcal{S}_k^{(j)} \mathcal{S}_n^{(k)} S_j^{(m)} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.15.27.0003.01

$$\mathcal{S}_n^{(m)} = \binom{n}{m} B_{n-m}^{(-m)} /; n \in \mathbb{N} \land m \in \mathbb{N} \land m \leq n$$

Theorems

Number of surjective functions

$n!\,\mathcal{S}_n^{(m)}$ is the number of surjective functions from a set with m elements to a set with n elements.

Derivative of composition with exponential function

$$\frac{\partial^n f(e^z)}{\partial z^n} = \sum_{k=1}^{n} e^{k z} \mathcal{S}_n^{(k)} f^{(k)}(e^z) /; n \in \mathbb{N}^+ .$$

Converting finite differences to derivatives

With the finite difference operator $\Delta_{x,h}^k$ defined by $\Delta_{x,h}^k f(x) = \Delta_{x,h}^{k-1}(\Delta_{x,h} f(x))$, $\Delta_{x,h} f(x) = f(x+1) - f(x)$ the kth finite difference of a function $f(x)$ can be expressed through derivatives as

$$\Delta_{x,h}^k f(x) = \sum_{j=0}^{\infty} \frac{h^{j+k}}{(k+1)_j} \mathcal{S}_{j+k}^{(k)} \frac{\partial^{j+k} f(x)}{\partial x^{j+k}} .$$

History

–J. Stirling (1730)

Copyright