WeierstrassHalfPeriods

Notations

Traditional name

Weierstrass half-periods

Traditional notation

$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$

Mathematica StandardForm notation

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WeierstrassHalfPeriods[{g2, g3}]
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Primary definition

09.18.02.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m,n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m+2nt)^4} \right)^{1/4}, i\, t \left(\frac{60}{g_2} \sum_{\substack{m,n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m+2nt)^4} \right)^{1/4} \right\} /; J(t) = \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

The following abbreviation will be used below:

09.18.02.0002.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

Specific values

Values at fixed points

09.18.03.0001.01

$$\{\omega_1(-1, 0), \omega_3(-1, 0)\} = \left\{ \frac{1+i}{4\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2, \frac{-1+i}{4\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 \right\}$$

09.18.03.0002.01

$$\{\omega_1(1, 0), \omega_3(1, 0)\} = \left\{ \frac{1}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2, \frac{i}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 \right\}$$

09.18.03.0003.01

$$\{\omega_1(0, 1), \omega_3(0, 1)\} = \left\{ \frac{1}{4\pi} \Gamma\left(\frac{1}{3}\right)^3, \frac{1+\sqrt{3}\, i}{8\pi} \Gamma\left(\frac{1}{3}\right)^3 \right\}$$

General characteristics

Domain and analyticity

$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ is an vector-valued function of g_2 and g_3 that is analytic in each component and it is defined over $\mathbb{C}^2$.

09.18.04.0001.01

$$\{g_2 * g_3\} \longrightarrow \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \longrightarrow \{\mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.18.04.0002.01

$$\omega_1(\overline{g_2}, \overline{g_3}) = \overline{\omega_1(g_2, g_3)}$$

09.18.04.0003.01

$$\omega_3(\overline{g_2}, \overline{g_3}) = -\overline{\omega_3(g_2, g_3)}$$

Series representations

Generalized power series

09.18.06.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4} \right)^{1/4}, i\,t \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\,m + 2\,n\,t)^4} \right)^{1/4} \right\} /; J(t) = \frac{g_2^3}{g_2^3 - 27\,g_3^2}$$

Integral representations

On the real axis

Of the direct function

09.18.07.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ \int_{e_1}^{\infty} \frac{1}{\sqrt{4\,t^3 - g_2\,t - g_3}} \,dt,\ i \int_{-\infty}^{e_3} \frac{1}{\sqrt{4\,t^3 - g_2\,t - g_3}} \,dt \right\} /;$$

$$g_2 \in \mathbb{R} \bigwedge g_3 \in \mathbb{R} \bigwedge g_2^3 - 27\,g_3^2 > 0 \bigwedge 4\,t^3 - g_2\,t - g_3 = 4\,(t - e_1)\,(t - e_2)\,(t - e_3) \bigwedge e_1 > e_2 > e_3$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

09.18.13.0001.01

$$\left(g_2^3-27\,g_3^2\right)\frac{\partial\omega_1}{\partial g_2}+\frac{1}{4}\,\omega_1\,g_2^2-\frac{9}{2}\,g_3\,\zeta(\omega_1;\,g_2,\,g_3)=0\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\bigwedge\left(g_2^3-27\,g_3^2\right)\frac{\partial\omega_3}{\partial g_2}+\frac{1}{4}\,\omega_3\,g_2^2-\frac{9}{2}\,g_3\,\zeta(\omega_3;\,g_2,\,g_3)=0\bigwedge\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}$$

09.18.13.0002.01

$$4\left(g_2^3-27\,g_3^2\right)\frac{\partial\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}}{\partial g_2}=18\,g_3\,\eta_1-g_2^2\,\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\;/;\;\{g_2,\,g_3\}=\{g_2(\eta_1,\,\eta_2),\,g_3(\eta_1,\,\eta_2)\}\land\{\eta_1,\,\eta_2\}=\{\zeta(\omega_1;\,g_2,\,g_3),\,\zeta(\omega_3;\,g_2,\,g_3)\}\land\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}$$

Differentiation

Low-order differentiation

With respect to g_2

09.18.20.0005.01

$$\frac{\partial\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}}{\partial g_2}=\frac{18\,g_3\,\zeta(\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\};\,g_2,\,g_3)-g_2^2\,\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}}{4\left(g_2^3-27\,g_3^2\right)}$$

09.18.20.0001.01

$$\frac{\partial\omega_1}{\partial g_2}=\frac{18\,g_3\,\eta_1-g_2^2\,\omega_1}{4\left(g_2^3-27\,g_3^2\right)}\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\land\eta_i=\zeta(\omega_i;\,g_2,\,g_3)$$

09.18.20.0002.01

$$\frac{\partial\omega_3}{\partial g_2}=\frac{18\,g_3\,\eta_3-g_2^2\,\omega_3}{4\left(g_2^3-27\,g_3^2\right)}\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\land\eta_i=\zeta(\omega_i;\,g_2,\,g_3)$$

09.18.20.0006.01

$$\frac{\partial^2\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}}{\partial g_2^2}=\frac{1}{16\left(g_2^3-27\,g_3^2\right)^2}\Big(5\,\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\,g_2^4-216\,g_3\,\zeta(\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\};\,g_2,\,g_3)\,g_2^2+189\,g_3^2\,\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\,g_2+162\,g_3^2\,\wp'(\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\};\,g_2,\,g_3)\Big)$$

With respect to g_3

09.18.20.0007.01

$$\frac{\partial\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}}{\partial g_3}=\frac{9\,g_3\,\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}-6\,g_2\,\zeta(\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\};\,g_2,\,g_3)}{2\left(g_2^3-27\,g_3^2\right)}$$

09.18.20.0003.01

$$\frac{\partial\omega_1}{\partial g_3}=\frac{9\,g_3\,\omega_1-6\,g_2\,\eta_1}{2\left(g_2^3-27\,g_3^2\right)}\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\land\eta_n=\zeta(\omega_n;\,g_2,\,g_3)\land n\in\{1,\,2,\,3\}$$

09.18.20.0004.01

$$\frac{\partial\omega_3}{\partial g_3}=\frac{9\,g_3\,\omega_3-6\,g_2\,\eta_3}{2\left(g_2^3-27\,g_3^2\right)}\;/;\;\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\land\eta_n=\zeta(\omega_n;\,g_2,\,g_3)\land n\in\{1,\,2,\,3\}$$

09.18.20.0008.01

$$\frac{\partial^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_3^2} = \frac{1}{4\left(g_2^3 - 27 g_3^2\right)^2} \left(3\left(5\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} g_2^3 + 6\wp'(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3) g_2^2 - 216 g_3 \zeta(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3) g_2 + 189 g_3^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}\right)\right)$$

Representations through equivalent functions

With inverse function

09.18.27.0001.01

$$\{g_2, g_3\} = \left\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\right\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

With related functions

Involving elliptic integrals

09.18.27.0002.01

$$\frac{\omega_3}{\omega_1} = \frac{i K(1-m)}{K(m)} /; m = q^{-1}\left(\exp\left(\frac{i \pi \omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.18.27.0003.01

$$\{\omega_1, \omega_3\} = \left\{\frac{K(m)}{\sqrt{e_1 - e_3}}, \frac{i K(1-m)}{\sqrt{e_1 - e_3}}\right\} /; \{e_1, e_2, e_3\} = \{\wp(\omega_1; g_2, g_3), \wp(\omega_1 + \omega_3; g_2, g_3), \wp(\omega_3; g_2, g_3)\} \bigwedge m = q^{-1}\left(\exp\left(\frac{i \pi \omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Theorems

Transcendentality of one expression with Weierstrass functions

Any $\alpha_1 \omega_1(g_2, g_3) + \alpha_2 \omega_3(g_2, g_3) + \alpha_3 \zeta(\omega_1(g_2, g_3); g_2, g_3) + \alpha_4 \zeta(\omega_3(g_2, g_3); g_2, g_3) + \alpha_5 2 \pi i \neq 0$, where α_1, α_2, α_3, α_4, α_5 are algebraic numbers, is transcendental.

History

– C. G. J. Jacobi (1835)
– K. Weierstrass (1862)
– A. Hurwitz (1905)

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