

WeierstrassHalfPeriods

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Notations

Traditional name

Weierstrass half-periods

Traditional notation

$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$

Mathematica StandardForm notation

WeierstrassHalfPeriods[$\{g_2, g_3\}$]

Primary definition

09.18.02.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4}, i t \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4} \right\}; J(t) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

The following abbreviation will be used below:

09.18.02.0002.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

Specific values

Values at fixed points

09.18.03.0001.01

$$\{\omega_1(-1, 0), \omega_3(-1, 0)\} = \left\{ \frac{1+i}{4\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2, \frac{-1+i}{4\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 \right\}$$

09.18.03.0002.01

$$\{\omega_1(1, 0), \omega_3(1, 0)\} = \left\{ \frac{1}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2, \frac{i}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 \right\}$$

09.18.03.0003.01

$$\{\omega_1(0, 1), \omega_3(0, 1)\} = \left\{ \frac{1}{4\pi} \Gamma\left(\frac{1}{3}\right)^3, \frac{1+\sqrt{3}i}{8\pi} \Gamma\left(\frac{1}{3}\right)^3 \right\}$$

General characteristics

Domain and analyticity

$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$ is an vector-valued function of g_2 and g_3 that is analytic in each component and it is defined over \mathbb{C}^2 .

09.18.04.0001.01

$$\{g_2 * g_3\} \rightarrow \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \rightarrow \{\mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.18.04.0002.01

$$\omega_1(\overline{g_2}, \overline{g_3}) = \overline{\omega_1(g_2, g_3)}$$

09.18.04.0003.01

$$\omega_3(\overline{g_2}, \overline{g_3}) = -\overline{\omega_3(g_2, g_3)}$$

Series representations

Generalized power series

09.18.06.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ i \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4}, i t \left(\frac{60}{g_2} \sum_{\substack{m, n=-\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m + 2nt)^4} \right)^{1/4} \right\}; J(t) = \frac{g_2^3}{g_2^3 - 27g_3^2}$$

Integral representations

On the real axis

Of the direct function

09.18.07.0001.01

$$\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} = \left\{ \int_{e_1}^{\infty} \frac{1}{\sqrt{4t^3 - g_2t - g_3}} dt, i \int_{-\infty}^{e_3} \frac{1}{\sqrt{4t^3 - g_2t - g_3}} dt \right\};$$

$$g_2 \in \mathbb{R} \wedge g_3 \in \mathbb{R} \wedge g_2^3 - 27g_3^2 > 0 \wedge 4t^3 - g_2t - g_3 = 4(t - e_1)(t - e_2)(t - e_3) \wedge e_1 > e_2 > e_3$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

09.18.13.0001.01

$$(g_2^3 - 27 g_3^2) \frac{\partial \omega_1}{\partial g_2} + \frac{1}{4} \omega_1 g_2^2 - \frac{9}{2} g_3 \zeta(\omega_1; g_2, g_3) = 0 \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge$$

$$(g_2^3 - 27 g_3^2) \frac{\partial \omega_3}{\partial g_2} + \frac{1}{4} \omega_3 g_2^2 - \frac{9}{2} g_3 \zeta(\omega_3; g_2, g_3) = 0 \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.18.13.0002.01

$$4 (g_2^3 - 27 g_3^2) \frac{\partial \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_2} = 18 g_3 \eta_1 - g_2^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} /;$$

$$\{g_2, g_3\} = \{g_2(\eta_1, \eta_2), g_3(\eta_1, \eta_2)\} \wedge \{\eta_1, \eta_2\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Differentiation

Low-order differentiation

With respect to g_2

09.18.20.0005.01

$$\frac{\partial \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_2} = \frac{18 g_3 \zeta(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3) - g_2^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{4 (g_2^3 - 27 g_3^2)}$$

09.18.20.0001.01

$$\frac{\partial \omega_1}{\partial g_2} = \frac{18 g_3 \eta_1 - g_2^2 \omega_1}{4 (g_2^3 - 27 g_3^2)} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \eta_i = \zeta(\omega_i; g_2, g_3)$$

09.18.20.0002.01

$$\frac{\partial \omega_3}{\partial g_2} = \frac{18 g_3 \eta_3 - g_2^2 \omega_3}{4 (g_2^3 - 27 g_3^2)} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \eta_i = \zeta(\omega_i; g_2, g_3)$$

09.18.20.0006.01

$$\frac{\partial^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_2^2} = \frac{1}{16 (g_2^3 - 27 g_3^2)^2} (5 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} g_2^4 - 216 g_3 \zeta(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3) g_2^2 +$$

$$189 g_3^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} g_2 + 162 g_3^2 \varphi'(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3))$$

With respect to g_3

09.18.20.0007.01

$$\frac{\partial \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_3} = \frac{9 g_3 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} - 6 g_2 \zeta(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3)}{2 (g_2^3 - 27 g_3^2)}$$

09.18.20.0003.01

$$\frac{\partial \omega_1}{\partial g_3} = \frac{9 g_3 \omega_1 - 6 g_2 \eta_1}{2 (g_2^3 - 27 g_3^2)} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \eta_n = \zeta(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.18.20.0004.01

$$\frac{\partial \omega_3}{\partial g_3} = \frac{9 g_3 \omega_3 - 6 g_2 \eta_3}{2 (g_2^3 - 27 g_3^2)} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \eta_n = \zeta(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.18.20.0008.01

$$\frac{\partial^2 \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}}{\partial g_3^2} = \frac{1}{4(g_2^3 - 27g_3^2)^2} \left(3(5\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}g_2^3 + 6\wp'(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3)g_2^2 - 216g_3\zeta(\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}; g_2, g_3)g_2 + 189g_3^2\{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}) \right)$$

Representations through equivalent functions

With inverse function

09.18.27.0001.01

$$\{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

With related functions

Involving elliptic integrals

09.18.27.0002.01

$$\frac{\omega_3}{\omega_1} = \frac{iK(1-m)}{K(m)} /; m = q^{-1} \left(\exp \left(\frac{i\pi\omega_3}{\omega_1} \right) \right) \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.18.27.0003.01

$$\{\omega_1, \omega_3\} = \left\{ \frac{K(m)}{\sqrt{e_1 - e_3}}, \frac{iK(1-m)}{\sqrt{e_1 - e_3}} \right\} /; \{e_1, e_2, e_3\} = \{\wp(\omega_1; g_2, g_3), \wp(\omega_1 + \omega_3; g_2, g_3), \wp(\omega_3; g_2, g_3)\} \wedge$$

$$m = q^{-1} \left(\exp \left(\frac{i\pi\omega_3}{\omega_1} \right) \right) \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Theorems

Transcendentality of one expression with Weierstrass functions

Any $\alpha_1 \omega_1(g_2, g_3) + \alpha_2 \omega_3(g_2, g_3) + \alpha_3 \zeta(\omega_1(g_2, g_3); g_2, g_3) + \alpha_4 \zeta(\omega_3(g_2, g_3); g_2, g_3) + \alpha_5 2\pi i \neq 0$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are algebraic numbers, is transcendental.

History

- C. G. J. Jacobi (1835)
- K. Weierstrass (1862)
- A. Hurwitz (1905)

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