WeierstrassInvariants

Notations

Traditional name

Weierstrass invariants

Traditional notation

$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$

Mathematica StandardForm notation

`WeierstrassInvariants[{`ω_1`, `ω_3`}]`

Primary definition

09.19.02.0001.02

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{60 \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^4}, 140 \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^6}\right\} /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

Specific values

Values at infinities

09.19.03.0001.01

$$g_2(\omega_1, \tilde{\infty})^3 - 27\, g_3(\omega_1, \tilde{\infty})^2 = 0$$

09.19.03.0002.01

$$\frac{9\, g_3(\omega_1, \tilde{\infty})}{2\, g_2(\omega_1, \tilde{\infty})} = \left(\frac{\pi}{2\, \omega_1}\right)^2$$

09.19.03.0003.01

$$\{g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})\} = \{0, 0\}$$

General characteristics

Domain and analyticity

$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$ is an vector-valued function of ω_1 and ω_3 that is analytic in each component and it is defined over $\mathbb{C}^2$(for $\omega_1 \neq a\, \omega_3$, $a \in \mathbb{R}$).

09.19.04.0001.01

$$\{\omega_1 * \omega_3\} \longrightarrow \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \longrightarrow \{\mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.19.04.0002.01

$$\{g_2(\overline{\omega_1}, \overline{\omega_3}), g_3(\overline{\omega_1}, \overline{\omega_3})\} = \overline{\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}$$

Permutation symmetry

09.19.04.0003.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_3, \omega_1), g_3(\omega_3, \omega_1)\}$$

Periodicity

$\{g_2(1, \omega_3), g_3(1, \omega_3)\}$ is a periodic vector-valued function with period 1 .

09.19.04.0007.01

$$\{g_2(1, \omega_3 + 1), g_3(1, \omega_3 + 1)\} = \{g_2(1, \omega_3), g_3(1, \omega_3)\}$$

09.19.04.0004.01

$$\{g_2(1, \omega_3 + n), g_3(1, \omega_3 + n)\} = \{g_2(1, \omega_3), g_3(1, \omega_3)\} /; n \in \mathbb{Z}$$

09.19.04.0008.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_1, \omega_3 + n\,\omega_1), g_3(\omega_1, \omega_3 + n\,\omega_1)\} /; n \in \mathbb{Z}$$

09.19.04.0009.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{g_2(\omega_1 + n\,\omega_3, \omega_3), g_3(\omega_1 + n\,\omega_3, \omega_3)\} /; n \in \mathbb{Z}$$

Transformation of half-periods

09.19.04.0005.01

$$\{g_2(\alpha\,\omega_1 + \beta\,\omega_3, \gamma\,\omega_1 + \delta\,\omega_3), g_3(\alpha\,\omega_1 + \beta\,\omega_3, \gamma\,\omega_1 + \delta\,\omega_3)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} /; \{\alpha, \beta, \gamma, \delta\} \in \mathbb{Z} \bigwedge \alpha\,\delta - \beta\,\gamma = \pm 1$$

Homogeneity

09.19.04.0006.01

$$\{g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)\} = \left\{\frac{1}{\lambda^4} g_2(\omega_1, \omega_3), \frac{1}{\lambda^6} g_3(\omega_1, \omega_3)\right\} /; \lambda \in \mathbb{C}$$

09.19.04.0010.01

$$\left\{g_2\left(1, \frac{\omega_3}{\omega_1}\right), g_3\left(1, \frac{\omega_3}{\omega_1}\right)\right\} = \left\{\omega_1^4, \omega_1^6\right\} \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}$$

Series representations

Generalized power series

09.19.06.0001.02

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{60 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^4}, 140 \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^6}\right\} /; \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) \neq 0$$

q-series

09.19.06.0002.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{20\left(\frac{\pi}{2\,\omega_1}\right)^4\left(\frac{1}{15}+16\sum_{k=1}^{\infty}\frac{k^3\,q^{2k}}{1-q^{2k}}\right),28\left(\frac{\pi}{2\,\omega_1}\right)^6\left(\frac{2}{189}-\frac{16}{3}\sum_{k=1}^{\infty}\frac{k^5\,q^{2k}}{1-q^{2k}}\right)\right\}\;/;\;q=\exp\left(\pi\,i\,\frac{\omega_3}{\omega_1}\right)$$

Other series representations

09.19.06.0003.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{\frac{\pi^4}{12}\left(\frac{1}{\omega_3^4}+\frac{1}{\omega_1^4}\right)+\frac{15}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{(m\,\omega_1-n\,\omega_3)^4}+\frac{1}{(m\,\omega_1+n\,\omega_3)^4}\right),\right.$$
$$\left.\frac{\pi^6}{216}\left(\frac{1}{\omega_3^6}+\frac{1}{\omega_1^6}\right)+\frac{35}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{(m\,\omega_1-n\,\omega_3)^6}+\frac{1}{(m\,\omega_1+n\,\omega_3)^6}\right)\right\}\;/;\;\mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right)\neq 0$$

09.19.06.0004.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{\frac{\pi^4}{12}\,\frac{1}{\omega_1^4}+\frac{15}{2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m\,\omega_1+n\,\omega_3)^4},\;\frac{\pi^6}{216}\,\frac{1}{\omega_1^6}+\frac{35}{8}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m\,\omega_1+n\,\omega_3)^6}\right\}\;/;\;\mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right)\neq 0$$

09.19.06.0005.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{\frac{\pi^4}{12}\,\frac{1}{\omega_3^4}+\frac{15}{2}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m\,\omega_1+n\,\omega_3)^4},\;\frac{\pi^6}{216}\,\frac{1}{\omega_3^6}+\frac{35}{8}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m\,\omega_1+n\,\omega_3)^6}\right\}\;/;\;\mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right)\neq 0$$

09.19.06.0006.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{\frac{\pi^4}{\omega_1^4}\left(20\sum_{k=1}^{\infty}\sigma_3(k)\,q^{2k}+\frac{1}{12}\right),\;\frac{\pi^6}{\omega_1^6}\left(\frac{1}{216}-\frac{7}{3}\sum_{k=1}^{\infty}\sigma_5(k)\,q^{2k}\right)\right\}\;/;\;q=\exp\left(\frac{i\,\pi\,\omega_2}{\omega_1}\right)\bigwedge\mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right)\neq 0$$

Integral representations

On the real axis

Of the direct function

09.19.07.0001.01

$$\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=\left\{\frac{5}{8}\int_0^{\infty}t^3\,(u(t,\omega_1,\omega_3)+v(t,\omega_1,\omega_3))\,dt,\;\frac{7}{384}\int_0^{\infty}t^5\,(u(t,\omega_1,\omega_3)-v(t,\omega_1,\omega_3))\,dt\right\}\;/;$$
$$u(t,\omega_1,\omega_3)=\frac{\cosh(t\,\omega_3)+e^{-\frac{t}{2}\omega_3}\sinh\left(\frac{t\,\omega_3}{2}\right)}{\sinh\left(\frac{1}{2}\,t\,(\omega_1-\omega_3)\right)\sinh\left(\frac{1}{2}\,t\,(\omega_1+\omega_3)\right)}\bigwedge v(t,\omega_1,\omega_3)=\frac{e^{\frac{i\,t}{2}\omega_3}\cos\left(\frac{t\,\omega_3}{2}\right)}{\sin\left(\frac{1}{2}\,t\,(\omega_1-\omega_3)\right)\sin\left(\frac{1}{2}\,t\,(\omega_1+\omega_3)\right)}$$

Differential equations

Partial differential equations

09.19.13.0001.01

$$\omega_1\,\frac{\partial\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}}{\partial\omega_1}+\omega_3\,\frac{\partial\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}}{\partial\omega_3}+\{4,6\}\,\{g_2(\omega_1,\omega_3),g_3(\omega_1,\omega_3)\}=0$$

09.19.13.0002.01

$$\omega_1 \frac{\partial g_2(\omega_1, \omega_3)}{\partial \omega_1} + \omega_3 \frac{\partial g_2(\omega_1, \omega_3)}{\partial \omega_3} + 4\, g_2(\omega_1, \omega_3) = 0$$

09.19.13.0003.01

$$\omega_1 \frac{\partial g_3(\omega_1, \omega_3)}{\partial \omega_1} + \omega_3 \frac{\partial g_3(\omega_1, \omega_3)}{\partial \omega_3} + 6\, g_3(\omega_1, \omega_3) = 0$$

Transformations

Transformations and argument simplifications

09.19.16.0001.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{\omega_1^{-4}, \omega_1^{-6}\right\} \left\{g_2\left(1, \frac{\omega_3}{\omega_1}\right), g_3\left(1, \frac{\omega_3}{\omega_1}\right)\right\}$$

09.19.16.0002.01

$$\{g_2(\omega_1, \omega_3 + n\,\omega_1), g_3(\omega_1, \omega_3 + n\,\omega_1)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \;/;\; n \in \mathbb{Z}$$

09.19.16.0003.01

$$\{g_2(\omega_1 + n\,\omega_3, \omega_3), g_3(\omega_1 + n\,\omega_3, \omega_3)\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \;/;\; n \in \mathbb{Z}$$

Identities

Functional identities

09.19.17.0001.01

$$\Delta(\tau)^6 = \frac{1}{16\,777\,216} \left(\Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^4 + 2\,\Delta\left(\frac{\tau}{2}\right)^3 \Delta\left(\frac{\tau+1}{2}\right)^3 + \Delta\left(\frac{\tau}{2}\right)^4 \Delta\left(\frac{\tau+1}{2}\right)^2 - 2304\,\Delta(\tau)^2\,\Delta\left(\frac{\tau}{2}\right)^2 \Delta\left(\frac{\tau+1}{2}\right)^2 + 393\,216\,\Delta\left(\frac{\tau}{2}\right) \Delta(\tau)^4\,\Delta\left(\frac{\tau+1}{2}\right)\right) \;/;\; \Delta(\tau) = g_2^3 - 27\,g_3^2 \bigwedge \{g_2, g_3\} = \{g_2(1, \tau), g_3(1, \tau)\}$$

Differentiation

Low-order differentiation

With respect to ω_1

09.19.20.0015.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} = -\frac{\omega_1}{\pi\,\omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\left\{12, \frac{2}{3}\right\} \mathrm{Reverse}\left[\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}^{\{2,1\}}\right] \omega_3 - \{8, 12\}\,\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}\,\zeta(\omega_3; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\right)$$

09.19.20.0001.01

$$\frac{\partial \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\}}{\partial \omega_1} = \left\{-30 \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m\,\omega_1 + n\,\omega_3)^5}, -\frac{105}{4} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{m}{(m\,\omega_1 + n\,\omega_3)^7}\right\}$$

09.19.20.0002.01

$$\frac{\partial\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_1} = \left\{-\frac{4\,g_2}{\omega_1}-\frac{40\,i\,\pi^5\,\omega_3}{\omega_1^6}\sum_{k=1}^{\infty}\frac{k^4\,q^{2k}}{\left(1-q^{2k}\right)^2},\ \frac{14\,i\,\pi^7\,\omega_3}{3\,\omega_1^8}\sum_{k=1}^{\infty}\frac{k^6\,q^{2k}}{\left(1-q^{2k}\right)^2}-\frac{6\,g_3}{\omega_1}\right\} /;\ q = \exp\left(\pi\,i\,\frac{\omega_3}{\omega_1}\right)$$

09.19.20.0003.01

$$\frac{\partial^2\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_1^2} = \left\{150\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{m^2}{(m\,\omega_1+n\,\omega_3)^6},\ \frac{735}{4}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{m^2}{(m\,\omega_1+n\,\omega_3)^8}\right\}$$

09.19.20.0011.02

$$\frac{\partial g_2}{\partial\omega_1} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\,(12\,g_3\,\omega_3-8\,g_2\,\eta_3) /;\ \{g_2,g_3\} = \{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}\bigwedge\eta_3 = \zeta(\omega_3;\,g_2,\,g_3)$$

09.19.20.0012.02

$$\frac{\partial g_3}{\partial\omega_1} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\frac{2}{3}\,g_2^2\,\omega_3-12\,g_3\,\eta_3\right) /;\ \{g_2,g_3\} = \{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}\bigwedge\eta_3 = \zeta(\omega_3;\,g_2,\,g_3)$$

With respect to ω_3

09.19.20.0016.01

$$\frac{\partial\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_3} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}$$

$$\left(\left\{12,\,\frac{2}{3}\right\}\text{Reverse}\left[\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}^{\{2,1\}}\right]\omega_1-\{8,\,12\}\,\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}\,\zeta(\omega_1;\,g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3))\right)$$

09.19.20.0004.01

$$\frac{\partial\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_3} = \left\{-30\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{n}{(m\,\omega_1+n\,\omega_3)^5},\ -\frac{105}{4}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{n}{(m\,\omega_1+n\,\omega_3)^7}\right\}$$

09.19.20.0005.01

$$\frac{\partial\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_3} = \left\{\frac{40\,i\,\pi^5}{\omega_1^5}\sum_{k=1}^{\infty}\frac{q^{2k}\,k^4}{\left(1-q^{2k}\right)^2},\ -\frac{14\,i\,\pi^7}{3\,\omega_1^7}\sum_{k=1}^{\infty}\frac{q^{2k}\,k^6}{\left(1-q^{2k}\right)^2}\right\} /;\ q = \exp\left(\pi\,i\,\frac{\omega_3}{\omega_1}\right)$$

09.19.20.0006.01

$$\frac{\partial^2\{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}}{\partial\omega_3^2} = \left\{150\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{n^2}{(m\,\omega_1+n\,\omega_3)^6},\ \frac{735}{4}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{n^2}{(m\,\omega_1+n\,\omega_3)^8}\right\}$$

09.19.20.0013.02

$$\frac{\partial g_2}{\partial\omega_3} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\,(12\,g_3\,\omega_1-8\,g_2\,\eta_1) /;\ \{g_2,g_3\} = \{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}\bigwedge\eta_1 = \zeta(\omega_1;\,g_2,\,g_3)$$

09.19.20.0014.02

$$\frac{\partial g_3}{\partial\omega_3} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}\left(\frac{2}{3}\,g_2^2\,\omega_1-12\,g_3\,\eta_1\right) /;\ \{g_2,g_3\} = \{g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)\}\bigwedge\eta_1 = \zeta(\omega_1;\,g_2,\,g_3)$$

Symbolic differentiation

With respect to ω_1

09.19.20.0007.01

$$\frac{\partial^k \{g_2(\omega_1,\omega_3),\, g_3(\omega_1,\omega_3)\}}{\partial \omega_1^k} = \left\{\frac{5}{4}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^k\,(k+3)!\,m^k}{(m\,\omega_1+n\,\omega_3)^{4+k}},\ \frac{7}{192}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^k\,(k+5)!\,m^k}{(m\,\omega_1+n\,\omega_3)^{6+k}}\right\} /;\ k\in\mathbb{N}^+$$

With respect to ω_3

09.19.20.0008.01

$$\frac{\partial^k \{g_2(\omega_1,\omega_3),\, g_3(\omega_1,\omega_3)\}}{\partial \omega_3^k} = \left\{\frac{5}{4}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^k\,(k+3)!\,n^k}{(m\,\omega_1+n\,\omega_3)^{k+4}},\ \frac{7}{192}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^k\,(k+5)!\,n^k}{(m\,\omega_1+n\,\omega_3)^{k+6}}\right\} /;\ k\in\mathbb{N}^+$$

Fractional integro-differentiation

With respect to ω_1

09.19.20.0009.01

$$\left\{\frac{\partial^\alpha g_2(\omega_1,\omega_3)}{\partial \omega_1^\alpha},\ \frac{\partial^\alpha g_3(\omega_1,\omega_3)}{\partial \omega_1^\alpha}\right\} = \left\{\frac{\pi^4}{12}\mathcal{FC}_{\exp}^{(\alpha)}(\omega_1,-4)\,\omega_1^{-4-\alpha}+\frac{15\,\omega_1^{-\alpha}}{2\,\omega_3^4}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^4}\,{}_2\tilde{F}_1\!\left(1,4;1-\alpha;-\frac{m\,\omega_1}{n\,\omega_3}\right),\right.$$
$$\left.\frac{\pi^6}{216}\mathcal{FC}_{\exp}^{(\alpha)}(\omega_1,-6)\,\omega_1^{-6-\alpha}+\frac{35\,\omega_1^{-\alpha}}{8\,\omega_3^6}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^6}\,{}_2\tilde{F}_1\!\left(1,6;1-\alpha;-\frac{m\,\omega_1}{n\,\omega_3}\right)\right\}$$

With respect to ω_3

09.19.20.0010.01

$$\left\{\frac{\partial^\alpha g_2(\omega_1,\omega_3)}{\partial \omega_3^\alpha},\ \frac{\partial^\alpha g_3(\omega_1,\omega_3)}{\partial \omega_3^\alpha}\right\} = \left\{\frac{\pi^4}{12}\mathcal{FC}_{\exp}^{(\alpha)}(\omega_3,-4)\,\omega_3^{-4-\alpha}+\frac{15\,\omega_3^{-\alpha}}{2\,\omega_1^4}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^4}\,{}_2\tilde{F}_1\!\left(1,4;1-\alpha;-\frac{n\,\omega_3}{m\,\omega_1}\right),\right.$$
$$\left.\frac{\pi^6}{216}\mathcal{FC}_{\exp}^{(\alpha)}(\omega_3,-6)\,\omega_3^{-6-\alpha}+\frac{35\,\omega_3^{-\alpha}}{8\,\omega_1^6}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^6}\,{}_2\tilde{F}_1\!\left(1,6;1-\alpha;-\frac{n\,\omega_3}{m\,\omega_1}\right)\right\}$$

Integration

Indefinite integration

Involving only one direct function with respect to ω_1

09.19.21.0001.01

$$\left\{\int g_2(\omega_1,\omega_3)\,d\omega_1,\ \int g_3(\omega_1,\omega_3)\,d\omega_1\right\} = \left\{\frac{5\,\omega_1}{2\,\omega_3^3}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{m^2\,\omega_1^2+3\,m\,n\,\omega_3\,\omega_1+3\,n^2\,\omega_3^2}{n^3\,(m\,\omega_1+n\,\omega_3)^3}-\frac{\pi^4}{36\,\omega_1^3},\right.$$
$$\left.\frac{7\,\omega_1}{8\,\omega_3^5}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left(m^4\,\omega_1^4+5\,m^3\,n\,\omega_3\,\omega_1^3+10\,m^2\,n^2\,\omega_3^2\,\omega_1^2+10\,m\,n^3\,\omega_3^3\,\omega_1+5\,n^4\,\omega_3^4\right)\big/\left(n^5\,(m\,\omega_1+n\,\omega_3)^5\right)-\frac{\pi^6}{1080\,\omega_1^5}\right\}$$

Representations through equivalent functions

With inverse function

09.19.27.0001.01

$$\{g_2,g_3\} = \{g_2(\omega_1,\omega_3),\ g_3(\omega_1,\omega_3)\} /;\ \{\omega_1,\omega_3\} = \{\omega_1(g_2,g_3),\ \omega_3(g_2,g_3)\}$$

With related functions

Involving Weierstrass functions

09.19.27.0002.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \{-4(e_1 e_2 + e_3 e_2 + e_1 e_3), 4 e_1 e_2 e_3\} /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.19.27.0003.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{2\left(e_1^2 + e_2^2 + e_3^2\right), 4 e_1 e_2 e_3\right\} /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.19.27.0004.01

$$\{g_2, g_3\} = \{-4(e_1 e_3 + e_2(e_1 + e_3)), 4 e_1 e_3 e_2\} /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.19.27.0005.01

$$\frac{g_2^2}{8} = e_1^4 + e_2^4 + e_3^4 /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.19.27.0006.01

$$4 e_k^3 - g_2 e_k - g_3 = 0 /; k \in \{1, 2, 3\} \bigwedge \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.19.27.0007.01

$$g_2^3 - 27 g_3^2 = 16(e_1 - e_2)^2 (e_3 - e_1)^2 (e_3 - e_2)^2 /; \{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

Involving theta functions

09.19.27.0008.01

$$g_2(\omega_1, \omega_3) = \frac{1}{12}\left(\frac{\pi}{\omega_1}\right)^4 \left(\vartheta_2(0, q)^8 - \vartheta_3(0, q)^4 \vartheta_2(0, q)^4 + \vartheta_3(0, q)^8\right) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0009.01

$$g_2(\omega_1, \omega_3) = \frac{2}{3}\left(\frac{\pi}{2\omega_1}\right)^4 \left(\vartheta_2(0, q)^8 + \vartheta_3(0, q)^8 + \vartheta_4(0, q)^8\right) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0010.01

$$g_3(\omega_1, \omega_3) = \left(\frac{\pi}{2\omega_1}\right)^6 \left(\frac{8}{27}\left(\vartheta_2(0, q)^{12} + \vartheta_3(0, q)^{12}\right) - \frac{4}{9}\left(\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4\right) \vartheta_2(0, q)^4 \vartheta_3(0, q)^4\right) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

09.19.27.0011.01

$$g_3(\omega_1, \omega_3) = \frac{4}{27}\left(\frac{\pi}{2\omega_1}\right)^6 \left(\vartheta_2(0, q)^4 + \vartheta_3(0, q)^4\right)\left(\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4\right)\left(\vartheta_4(0, q)^4 - \vartheta_2(0, q)^4\right) /; \tau = \frac{\omega_3}{\omega_1} \bigwedge q = e^{\tau \pi i}$$

Involving elliptic integrals and modular functions

09.19.27.0012.01

$$\{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} = \left\{\frac{4\left(m^2 - m + 1\right) K(m)^4}{3 \omega_1^4}, \frac{4(m-2)(2m-1)(m+1) K(m)^6}{27 \omega_1^6}\right\} /; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

09.19.27.0013.01

$$g_2^3 \left(2 m^3 - 3 m^2 - 3 m + 2\right)^2 - 108 g_3^2 \left(m^2 - m + 1\right)^3 1 == 10 \text{ /; } m == q^{-1}\left(\exp\left(\frac{i \pi \omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Theorems

Transcendentality of one expression with Weierstrass functions

Any $\alpha_1 \omega_1(g_2, g_3) + \alpha_2 \omega_3(g_2, g_3) + \alpha_3 \zeta(\omega_1(g_2, g_3); g_2, g_3) + \alpha_4 \zeta(\omega_3(g_2, g_3); g_2, g_3) + \alpha_5 2 \pi i \neq 0$, where α_1, α_2, α_3, α_4, α_5 are algebraic numbers, is transcendental.

History

- A. Cayley
– G. Boole (1845)

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