WeierstrassPPrime

Notations

Traditional name

Derivative of the Weierstrass elliptic function

Traditional notation

$\wp'(z; g_2, g_3)$

Mathematica StandardForm notation

`WeierstrassPPrime[z, {g2, g3}]`

Primary definition

09.14.02.0001.01

$$\wp'(z; g_2, g_3) == -2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2\, m\, \omega_1 - 2\, n\, \omega_3)^3} \;/;\; \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Special notations for this file:

09.14.02.0002.01

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.14.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.14.02.0004.01

$$e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.14.02.0005.01

$$\eta_n == \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.14.02.0006.01

$$q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed *z*

Degenerate case

09.14.03.0001.01

$$\wp'(z;\,0,\,0) = -\frac{2}{z^3}$$

09.14.03.0002.01

$$\wp'(z;\,3,\,1) = -3\sqrt{\frac{3}{2}}\,\cot\left(\sqrt{\frac{3}{2}}\,z\right)\csc^2\left(\sqrt{\frac{3}{2}}\,z\right)$$

For fixed $\{g_2,\,g_3\}$

09.14.03.0003.01

$$\wp'(\omega_1;\,g_2,\,g_3) = 0$$

09.14.03.0004.01

$$\wp'(\omega_2;\,g_2,\,g_3) = 0$$

09.14.03.0005.01

$$\wp'(\omega_3;\,g_2,\,g_3) = 0$$

09.14.03.0006.01

$$\wp'(m\,\omega_1 + n\,\omega_3;\,g_2,\,g_3) = 0\;/;\;\left\{\frac{m-1}{2},\,n\right\} \in \mathbb{Z} \bigvee \left\{m,\,\frac{n-1}{2}\right\} \in \mathbb{Z}$$

Values at poles:

09.14.03.0007.01

$$\wp'(2\,m\,\omega_1 + 2\,n\,\omega_3;\,g_2,\,g_3) = \tilde{\infty}\;/;\;\{m,\,n\} \in \mathbb{Z}$$

Values at infinities

09.14.03.0008.01

$$\wp'(z;\,g_2(\omega_1,\,\tilde{\infty}),\,g_3(\omega_1,\,\tilde{\infty})) = -\frac{\pi^3}{4\,\omega_1^3}\cot\left(\frac{\pi\,z}{2\,\omega_1}\right)\csc^2\left(\frac{\pi\,z}{2\,\omega_1}\right)$$

09.14.03.0009.01

$$\wp'(z;\,g_2(\tilde{\infty},\,\tilde{\infty}),\,g_3(\tilde{\infty},\,\tilde{\infty})) = -\frac{2}{z^3}$$

General characteristics

Domain and analyticity

$\wp'(z;\,g_2,\,g_3)$ is an analytical function of z, g_2, and g_3, which is defined in $\mathbb{C}^3$.

09.14.04.0001.01

$$(z * \{g_2 * g_3\}) \longrightarrow \wp'(z;\,g_2,\,g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\wp'(z;\,g_2,\,g_3)$ is an odd function with respect to z.

09.14.04.0002.01

$\wp'(z; g_2, g_3) = -\wp'(-z; g_2, g_3)$

Mirror symmetry

09.14.04.0003.01

$\wp'(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\wp'(z; g_2, g_3)}$

Periodicity

$\wp'(z; g_2, g_3)$ is a doubly periodic function with respect to z with periods $2\,\omega_1$ and $2\,\omega_3$.

09.14.04.0004.01

$\wp'(z + 2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) = \wp'(z; g_2, g_3) \;/;\; \{m, n\} \in \mathbb{Z}$

Transformation of half-periods

09.14.04.0005.01

$\wp'(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) = \wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) \;/;$
$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c = \pm 1$

Homogeneity

09.14.04.0006.01

$$\wp'(z\,t; g_2, g_3) = \frac{1}{t^3}\,\wp'\left(z; g_2\,t^4, g_3\,t^6\right) \;/;\; t \in \mathbb{R}$$

09.14.04.0007.01

$$\wp'(\lambda\,z; g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)) = \wp'\left(\lambda\,z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

09.14.04.0008.01

$$\wp'(\lambda\,z; g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)) = \frac{1}{\lambda^3}\,\wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

Poles and essential singularities

With respect to *z*

For fixed g_2, g_3, the function $\wp'(z; g_2, g_3)$ has an infinite set of singular points:

a) $z = 2\,m\,\omega_1(g_2, g_3) + 2\,n\,\omega_3(g_2, g_3)$, $\{m, n\} \in \mathbb{Z}$, are the poles of order three with residues 0;

b) $z = \tilde{\infty}$ is an essential singular point.

09.14.04.0009.01

$\mathcal{S}ing_z(\wp'(z; g_2, g_3)) = \{\{\{2\,m\,\omega_1 + 2\,n\,\omega_3, 3\} \;/;\; \{m, n\} \in \mathbb{Z}\}, \{\tilde{\infty}, \infty\}\}$

09.14.04.0010.01

$\mathrm{res}_z(\wp'(z; g_2, g_3))\,(2\,m\,\omega_1 + 2\,n\,\omega_3) = 0 \;/;\; \{m, n\} \in \mathbb{Z}$

Branch points

With respect to *z*

For fixed g_2, g_3, the function $\wp'(z; g_2, g_3)$ does not have branch points.

09.14.04.0011.01

$$\mathcal{BP}_z(\wp'(z; g_2, g_3)) == \{\}$$

Branch cuts

With respect to z

For fixed g_2, g_3, the function $\wp'(z; g_2, g_3)$ does not have branch cuts.

09.14.04.0012.01

$$\mathcal{BC}_z(\wp'(z; g_2, g_3)) == \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

09.14.06.0011.01

$$\wp'(z; g_2, g_3) \propto -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \ldots /; (z \to 0)$$

09.14.06.0012.01

$$\wp'(z; g_2, g_3) \propto -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + + O(z^5)$$

09.14.06.0001.01

$$\wp'(z; g_2, g_3) == -\frac{2}{z^3} + \sum_{k=2}^{\infty} (2k-2) a_k z^{2k-3} /; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

09.14.06.0002.01

$$\wp'(z; g_2, g_3) == -\frac{2}{z^3} + 2 \sum_{k=1}^{\infty} k(2k+1) \sum_{\substack{m,n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2k+2}} z^{2k-1}$$

09.14.06.0013.01

$$\wp'(z; g_2, g_3) \propto -\frac{2}{z^3} (1 + O(z^4))$$

q-series

09.14.06.0003.01

$$\wp'(z; g_2, g_3) == -\frac{\pi^3}{4\omega_1^3} \cot\left(\frac{\pi z}{2\omega_1}\right) \csc^2\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} \frac{k^2 q^{2k}}{1-q^{2k}} \sin\left(\frac{k\pi z}{\omega_1}\right)$$

09.14.06.0004.01

$$\wp'(z+\omega_1; g_2, g_3) == \frac{\pi^3}{4\omega_1^3} \tan\left(\frac{\pi z}{2\omega_1}\right) \sec^2\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} (-1)^k \frac{k^2 q^{2k}}{1-q^{2k}} \sin\left(\frac{k\pi z}{\omega_1}\right)$$

09.14.06.0005.01

$$\wp'(z+\omega_2; g_2, g_3) == \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} (-1)^k \frac{k^2 q^k}{1-q^{2k}} \sin\left(\frac{k\pi z}{\omega_1}\right)$$

09.14.06.0006.01

$$\wp'(z+\omega_3; g_2, g_3) = \frac{2\pi^3}{\omega_1^3} \sum_{k=1}^{\infty} \frac{k^2 q^k}{1-q^{2k}} \sin\left(\frac{k\pi z}{\omega_1}\right)$$

Other series representations

09.14.06.0007.01

$$\wp'(z; g_2, g_3) = -2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-2m\omega_1-2n\omega_3)^3}$$

09.14.06.0008.01

$$\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3) = -2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-2m\omega_1-2n\omega_3)^5}$$

09.14.06.0009.01

$$\wp'(z; g_2, g_3) = -\frac{\pi^3}{4\omega_i^3} \sum_{n=-\infty}^{\infty} \csc^2\left(\frac{\pi\left(z-2n\omega_j\right)}{2\omega_i}\right) \cot\left(\frac{\pi\left(z-2n\omega_j\right)}{2\omega_i}\right) /; \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

09.14.06.0010.01

$$\wp'(z; g_2, g_3) = -\frac{\pi^3}{4\omega_1^3} \sum_{n=-\infty}^{\infty} \csc^2\left(\frac{\pi(z-2n\omega_3)}{2\omega_1}\right) \cot\left(\frac{\pi(z-2n\omega_3)}{2\omega_1}\right)$$

Integral representations

On the real axis

Of the direct function

09.14.07.0001.01

$$\wp'(z; g_2, g_3) = -\frac{2}{z^3} + \frac{1}{8}\int_0^{\infty} t^2 \left(\frac{e^{\frac{i t}{2}\omega_2} \cos\left(\frac{t\omega_2}{2}\right) \sin\left(\frac{t z}{2}\right)}{\sin\left(\frac{1}{2}t(\omega_1-\omega_2)\right) \sin\left(\frac{1}{2}t(\omega_1+\omega_2)\right)} + \frac{\sinh\left(\frac{t z}{2}\right)\left(\cosh(t\omega_2) + e^{-\frac{t}{2}\omega_2} \sinh\left(\frac{t\omega_2}{2}\right)\right)}{\sinh\left(\frac{1}{2}t(\omega_1-\omega_2)\right) \sinh\left(\frac{1}{2}t(\omega_1+\omega_2)\right)}\right) dt$$

Involving related functions

09.14.07.0002.01

$$\wp'(z; g_2, g_3) = \sqrt{4w^3 - g_2 w - g_3} /; z = \int_{\infty}^{w} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}\, dt \bigwedge w \in \mathbb{R}$$

Differential equations

Ordinary nonlinear differential equations

09.14.13.0001.01

$$2w'(z)^3 - 3g_2 w'(z)^2 - 27w(z)^4 - 54g_3 w(z)^2 + g_2^3 - 27g_3^2 = 0 /; w(z) = \wp'(z; g_2, g_3)$$

09.14.13.0002.01

$$w'(z)^3 - w(z)^2 (w(z)-a)^2 = 0 /; w(z) = \frac{a}{2} + \frac{27}{16}\wp'\left(\frac{z}{2}; 0, -\frac{64}{729}a^2\right)$$

09.14.13.0003.01

$$w'(z)^3 - \left(w(z)^3 - 3\,a\,w(z)^2 + 3\,w(z)\right)^2 = 0\ /;\ w(z) = \frac{2}{a - 3\,\wp'\left(z;\,0,\,\frac{1}{27}\left(4 - 3\,a^2\right)\right)}$$

Partial differential equations

09.14.13.0004.01

$$z\,\frac{\partial^2 \wp'(z;\,g_2,\,g_3)}{\partial z^2} + \frac{\partial \wp'(z;\,g_2,\,g_3)}{\partial z} - 4\,g_2\,\frac{\partial \wp'(z;\,g_2,\,g_3)}{\partial g_2} - 6\,g_2\,\frac{\partial \wp'(z;\,g_2,\,g_3)}{\partial g_3} - 2\,\wp'(z;\,g_2,\,g_3) = 0$$

09.14.13.0005.01

$$4\,g_2\,\frac{\partial \wp'(z;\,g_2,\,g_3)}{\partial g_2} + 6\,g_2\,\frac{\partial \wp'(z;\,g_2,\,g_3)}{\partial g_3} - 6\,\wp(z;\,g_2,\,g_3)^2 - 12\,z\,\wp'(z;\,g_2,\,g_3)\,\wp(z;\,g_2,\,g_3) + 2\,\wp'(z;\,g_2,\,g_3) + \frac{g_2}{2} = 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.14.16.0001.01

$$\wp'(i\,z;\,g_2,\,g_3) = i\,\wp'(z;\,g_2,\,-g_3)$$

Addition formulas

09.14.16.0002.01

$$\wp'(z_1 + z_2;\,g_2,\,g_3) = \left(\wp(z_1 + z_2;\,g_2,\,g_3)\left(\wp'(z_1;\,g_2,\,g_3) - \wp'(z_2;\,g_2,\,g_3)\right) + \wp(z_1;\,g_2,\,g_3)\,\wp'(z_2;\,g_2,\,g_3) - \wp'(z_1;\,g_2,\,g_3)\,\wp(z_2;\,g_2,\,g_3)\right) / \left(\wp(z_2;\,g_2,\,g_3) - \wp(z_1;\,g_2,\,g_3)\right)$$

Multiple arguments

Argument involving numeric multiples of variable

Double angle formulas

09.14.16.0003.01

$$\wp'(2\,z;\,g_2,\,g_3) = \frac{1}{4\,\wp'(z;\,g_2,\,g_3)^3}\left(-4\,\wp'(z;\,g_2,\,g_3)^4 + 12\,\wp(z;\,g_2,\,g_3)\,\wp'(z;\,g_2,\,g_3)^2\,\wp''(z;\,g_2,\,g_3) - \wp''(z;\,g_2,\,g_3)^3\right)$$

Argument involving symbolic multiples of variable

Multiple angle formula:

09.14.16.0004.01

$$\wp'(n\,z;\,g_2,\,g_3) = \frac{1}{n^3}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\wp'\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right) /;\ n \in \mathbb{N}^+$$

Related transformations

Halving half-period

09.14.16.0005.01

$$\wp'\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \wp'(z + \omega_1; g_2, g_3)$$

Third of half-period

09.14.16.0006.01

$$\wp'\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \wp'\left(z + \frac{2\,\omega_1}{3}; g_2, g_3\right) + \wp'\left(z + \frac{4\,\omega_1}{3}; g_2, g_3\right)$$

General fractions of half-periods

09.14.16.0007.01

$$\wp'\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \wp'(z; g_2, g_3) + \sum_{k=1}^{n-1} \wp'\left(z + \frac{2\,k\,\omega_1}{n}; g_2, g_3\right) /; n \in \mathbb{N}^+$$

Other constructions

09.14.16.0008.01

$$\frac{\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)} = \frac{1}{\wp'(z_1; g_2, g_3)} \left(2\,(\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3))\,(\wp(z_1 + z_2; g_2, g_3) - \wp(z_1; g_2, g_3)) + \wp''(z_1; g_2, g_3)\right)$$

09.14.16.0009.01

$$\begin{vmatrix} \wp'(z_1; g_2, g_3) & \wp(z_1; g_2, g_3) & 1 \\ \wp'(z_2; g_2, g_3) & \wp(z_2; g_2, g_3) & 1 \\ -\wp'(z_1 + z_2; g_2, g_3) & \wp(z_1 + z_2; g_2, g_3) & 1 \end{vmatrix} = 0$$

Identities

Functional identities

Expressions involving $a + b + c = 0 \bmod(2\,\omega_1, 2\,\omega_3)$

09.14.17.0001.01

$$\frac{\wp'(a; g_2, g_3) - \wp'(b; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} = \frac{\wp'(b; g_2, g_3) - \wp'(c; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(c; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

09.14.17.0002.01

$$\frac{\wp'(a; g_2, g_3) - \wp'(b; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} = \frac{\wp'(c; g_2, g_3) - \wp'(a; g_2, g_3)}{\wp(c; g_2, g_3) - \wp(a; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

09.14.17.0003.01

$$\frac{\wp(b; g_2, g_3)\,\wp'(c; g_2, g_3) - \wp(c; g_2, g_3)\,\wp'(b; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(c; g_2, g_3)} = \frac{\wp(c; g_2, g_3)\,\wp'(a; g_2, g_3) - \wp(a; g_2, g_3)\,\wp'(c; g_2, g_3)}{\wp(c; g_2, g_3) - \wp(a; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

09.14.17.0004.01

$$\frac{\wp(b; g_2, g_3)\,\wp'(c; g_2, g_3) - \wp(c; g_2, g_3)\,\wp'(b; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(c; g_2, g_3)} = \frac{\wp(a; g_2, g_3)\,\wp'(b; g_2, g_3) - \wp(b; g_2, g_3)\,\wp'(a; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} /; a + b + c = 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m, n\} \in \mathbb{Z}$$

Generalization for arbitrary *a*, *b*, *c*, *d*

09.14.17.0005.01

$$\frac{\wp'(a; g_2, g_3) - \wp'(b; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(b; g_2, g_3)} + \frac{\wp'(c; g_2, g_3) - \wp'(d; g_2, g_3)}{\wp(c; g_2, g_3) - \wp(d; g_2, g_3)} + \frac{\wp'(a+b; g_2, g_3) - \wp'(c+d; g_2, g_3)}{\wp(a+b; g_2, g_3) - \wp(c+d; g_2, g_3)} = \frac{\wp'(a; g_2, g_3) - \wp'(c; g_2, g_3)}{\wp(a; g_2, g_3) - \wp(c; g_2, g_3)} + \frac{\wp'(b; g_2, g_3) - \wp'(d; g_2, g_3)}{\wp(b; g_2, g_3) - \wp(d; g_2, g_3)} + \frac{\wp'(a+c; g_2, g_3) - \wp'(b+d; g_2, g_3)}{\wp(a+c; g_2, g_3) - \wp(b+d; g_2, g_3)}$$

Differentiation

Low-order differentiation

With respect to *z*

09.14.20.0001.01

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial z} = 6\,\wp(z; g_2, g_3)^2 - \frac{g_2}{2}$$

09.14.20.0002.01

$$\frac{\partial^2 \wp'(z; g_2, g_3)}{\partial z^2} = 12\,\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3)$$

09.14.20.0003.01

$$\frac{\partial^3 \wp'(z; g_2, g_3)}{\partial z^3} = 120\,\wp(z; g_2, g_3)^3 - 18\,g_2\,\wp(z; g_2, g_3) - 12\,g_3$$

With respect to g_2

09.14.20.0004.01

$$\frac{1}{8\left(g_2^3 - 27\,g_3^2\right)} \left(12\,z\,\wp(z; g_2, g_3)^2\,g_2^2 - z\,g_2^3 + 6\left(g_2^2 - 18\,g_3\,\wp(z; g_2, g_3)\right)\wp'(z; g_2, g_3) + 18\,g_3\left(g_2 - 12\,\wp(z; g_2, g_3)^2\right)\zeta(z; g_2, g_3)\right)$$

09.14.20.0005.01

$$\frac{\partial^2 \wp'(z; g_2, g_3)}{\partial g_2^2} = \frac{3}{32\left(g_2^3 - 27\,g_3^2\right)^2} \Big(-z\,g_2^5 + 2\,g_2^4\left(4\,\wp(z; g_2, g_3)\,\wp'(z; g_2, g_3)\,z^2 + 6\,\wp(z; g_2, g_3)^2\,z - \wp'(z; g_2, g_3)\right) + 12\,g_2^3\,g_3\,(5\,z\,\wp(z; g_2, g_3) - 2\,\zeta(z; g_2, g_3)) + 9\,g_2^2\,g_3\left(11\,z\,g_3 - 4\left(12\,z\,\wp(z; g_2, g_3)^3 - 8\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3)^2 + 4\,\wp'(z; g_2, g_3)\,(2\,z\,\zeta(z; g_2, g_3) - 1)\,\wp(z; g_2, g_3) + z\,\wp'(z; g_2, g_3)^2\right)\right) - 54\,g_2\,g_3^2\left(14\,z\,\wp(z; g_2, g_3)^2 + 20\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3) + 11\,\wp'(z; g_2, g_3)\right) - 648\,g_3^2\left(-12\,\zeta(z; g_2, g_3)\,\wp(z; g_2, g_3)^3 - 4\,\wp'(z; g_2, g_3)\,\wp(z; g_2, g_3)^2 - 4\,\wp'(z; g_2, g_3)\,\zeta(z; g_2, g_3)^2\,\wp(z; g_2, g_3) + \left(g_3 - \wp'(z; g_2, g_3)^2\right)\zeta(z; g_2, g_3)\right)\Big)$$

With respect to g_3

09.14.20.0006.01

$$\frac{\partial \wp'(z; g_2, g_3)}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27\,g_3^2\right)} \left(9\,z\,g_2\,g_3 - 108\,z\,g_3\,\wp(z; g_2, g_3)^2 + (36\,g_2\,\wp(z; g_2, g_3) - 54\,g_3)\,\wp'(z; g_2, g_3) + \left(72\,g_2\,\wp(z; g_2, g_3)^2 - 6\,g_2^2\right)\zeta(z; g_2, g_3)\right)$$

09.14.20.0007.01

$$\frac{\partial^2 \wp'(z;g_2,g_3)}{\partial g_3^2} = \frac{3}{8\left(g_2^3-27g_3^2\right)^2}\left(5zg_2^4-6g_2^3\left(10z\wp(z;g_2,g_3)^2+20\zeta(z;g_2,g_3)\wp(z;g_2,g_3)+9\wp'(z;g_2,g_3)\right)+\right.$$
$$36g_2^2\left(24\zeta(z;g_2,g_3)\wp(z;g_2,g_3)^3+8\wp'(z;g_2,g_3)\wp(z;g_2,g_3)^2+8\wp'(z;g_2,g_3)\zeta(z;g_2,g_3)^2\wp(z;g_2,g_3)+\right.$$
$$\left.2\wp'(z;g_2,g_3)^2\zeta(z;g_2,g_3)+g_3(5z\wp(z;g_2,g_3)-4\zeta(z;g_2,g_3))\right)+$$
$$27g_2g_3\left(3zg_3-4\left(12z\wp(z;g_2,g_3)^3-8\zeta(z;g_2,g_3)\wp(z;g_2,g_3)^2+4\wp'(z;g_2,g_3)(2z\zeta(z;g_2,g_3)-1)\wp(z;g_2,g_3)+\right.\right.$$
$$\left.\left.\left.z\wp'(z;g_2,g_3)^2\right)\right)+162g_3^2\left(4\wp(z;g_2,g_3)\wp'(z;g_2,g_3)z^2+2\wp(z;g_2,g_3)^2z-3\wp'(z;g_2,g_3)\right)\right)$$

With respect to ω_1

09.14.20.0010.02

$$\frac{\partial \wp'(z;g_2,g_3)}{\partial \omega_1} = -\frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}$$
$$\left(\omega_3\left(6\wp(z;g_2,g_3)\wp'(z;g_2,g_3)+12\zeta(z;g_2,g_3)\wp(z;g_2,g_3)^2-g_2\zeta(z;g_2,g_3)\right)-\eta_3\left(6\wp'(z;g_2,g_3)+12z\wp(z;g_2,g_3)^2-g_2z\right)\right)$$

With respect to ω_3

09.14.20.0011.02

$$\frac{\partial \wp'(z;g_2,g_3)}{\partial \omega_3} = \frac{\omega_1}{\pi\,\omega_3}\sqrt{-\frac{\omega_3^2}{\omega_1^2}}$$
$$\left(\omega_1\left(6\wp(z;g_2,g_3)\wp'(z;g_2,g_3)+12\zeta(z;g_2,g_3)\wp(z;g_2,g_3)^2-g_2\zeta(z;g_2,g_3)\right)-\eta_1\left(6\wp'(z;g_2,g_3)+12z\wp(z;g_2,g_3)^2-g_2z\right)\right)$$

Symbolic differentiation

With respect to z

09.14.20.0008.01

$$\frac{\partial^k \wp'(z;g_2,g_3)}{\partial z^k} = (-1)^{k-1}(k+2)!\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(z-2m\omega_1-2n\omega_3)^{k+3}}\ /;\ k\in\mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

09.14.20.0009.01

$$\frac{\partial^\alpha \wp'(z;g_2,g_3)}{\partial z^\alpha} = \frac{1}{4}z^{-\alpha}\sum_{\substack{m,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{1}{(m\omega_1+n\omega_3)^3}\times$$
$${}_2\tilde{F}_1\left(1,3;1-\alpha;\frac{z}{2m\omega_1+2n\omega_3}\right)$$

Integration

Indefinite integration

Involving only one direct function

09.14.21.0001.01

$$\int \wp'(z; g_2, g_3)\, dz = \wp(z; g_2, g_3)$$

Representations through equivalent functions

With inverse function

09.14.27.0001.01

$$\wp'\left(\wp^{-1}(z_1, z_2; g_2, g_3); g_2, g_3\right) = z_2 \;/;\; z_2 = \sqrt{4 z_1^3 - g_2 z_1 - g_3}$$

With related functions

Involving other Weierstrass functions

09.14.27.0002.01

$$\wp'(z; g_2, g_3) = \frac{\partial \wp(z; g_2, g_3)}{\partial z}$$

09.14.27.0003.01

$$\frac{\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)} = 2\,(\zeta(z_1 + z_2; g_2, g_3) - \zeta(z_1; g_2, g_3) - \zeta(z_2; g_2, g_3))$$

09.14.27.0004.01

$$\wp'(z; g_2, g_3) = -\frac{\sigma(2 z; g_2, g_3)}{\sigma(z; g_2, g_3)^4}$$

09.14.27.0005.01

$$\wp'(z; g_2, g_3) = \frac{2\,\sigma(z - \omega_1; g_2, g_3)\,\sigma(z - \omega_2; g_2, g_3)\,\sigma(z - \omega_3; g_2, g_3)}{\sigma(z; g_2, g_3)^3\,\sigma(\omega_1; g_2, g_3)\,\sigma(\omega_2; g_2, g_3)\,\sigma(\omega_3; g_2, g_3)}$$

09.14.27.0006.01

$$\wp'(z; g_2, g_3) = -\frac{2\,\sigma_1(z; g_2, g_3)\,\sigma_2(z; g_2, g_3)\,\sigma_3(z; g_2, g_3)}{\sigma(z; g_2, g_3)^3}$$

Involving Jacobi functions

09.14.27.0007.01

$$\wp'(z; g_2, g_3) = -2\,(e_1 - e_3)^{3/2}\,\frac{\operatorname{cn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right)\operatorname{dn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right)}{\operatorname{sn}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, m\right)^3} \;/;\; m = \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

Involving theta functions

09.14.27.0008.01

$$\wp'(z; g_2, g_3) = -\frac{\pi^3}{4\,\omega_1^3}\,\frac{\vartheta_2\left(\frac{\pi z}{2\omega_1}, q\right)\vartheta_3\left(\frac{\pi z}{2\omega_1}, q\right)\vartheta_4\left(\frac{\pi z}{2\omega_1}, q\right)\vartheta_1'(0, q)^3}{\vartheta_2(0, q)\,\vartheta_3(0, q)\,\vartheta_4(0, q)\,\vartheta_1\left(\frac{\pi z}{2\omega_1}, q\right)^3}$$

Zeros

09.14.30.0001.01

$$\wp'(m\,\omega_1 + n\,\omega_3;\, g_2,\, g_3) == 0 \,/;\, \left\{\frac{m-1}{2},\, n\right\} \in \mathbb{Z} \bigvee \left\{m,\, \frac{n-1}{2}\right\} \in \mathbb{Z}$$

Theorems

The conformal mapping from the triangle to the half plane

The conformal map from the triangle in the z-plane with coordinates $\{0, 0\}$, $\{1, 0\}$, $\left\{1/2, \frac{\sqrt{3}}{2}\right\}$ to the upper half w-plane is given by $w(z) == \frac{1}{2} + \frac{27}{2\,\mathrm{B}\left(\frac{1}{3}, \frac{1}{3}\right)^3}\, \wp'\left(z - e^{\frac{\pi i}{3}};\, 0,\, -\frac{1}{729}\,\mathrm{B}\left(\frac{1}{3},\, \frac{1}{3}\right)^6\right)$.

The P^2-equation

The $\mathbf{P}^2$ -equation can be cast in the form $\frac{\partial^2 z(\tau)}{\partial \tau^2} == -\frac{1}{8\pi^2}\, \wp'(z(\tau);\, \{1,\, \tau\})$.

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