WeierstrassZeta

Notations

Traditional name

Weierstrass zeta function

Traditional notation

$\zeta(z; g_2, g_3)$

Mathematica StandardForm notation

`WeierstrassZeta[z, {g2, g3}]`

Primary definition

09.17.02.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} + \sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{z}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^2} + \frac{1}{2\,m\,\omega_1 + 2\,n\,\omega_3} + \frac{1}{z - 2\,m\,\omega_1 - 2\,n\,\omega_3} \;/;\; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Special notations for this file:

09.17.02.0002.01

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.17.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.17.02.0004.01

$$e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.17.02.0005.01

$$\eta_n = \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.17.02.0006.01

$$q = \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed z

Degenerate case:

09.17.03.0001.01

$$\zeta(z;\,0,\,0) = \frac{1}{z}$$

09.17.03.0002.01

$$\zeta(z;\,3,\,1) = \frac{z}{2} + \sqrt{\frac{3}{2}}\,\cot\!\left(\sqrt{\frac{3}{2}}\,z\right)$$

For fixed $\{g_2, g_3\}$

One-third period values

09.17.03.0003.01

$$\left(\zeta\!\left(\frac{2\,\omega_i}{3};\,g_2,\,g_3\right) - \frac{2\,\eta_i}{3}\right)^2 = \frac{1}{3}\,\wp\!\left(\frac{2\,\omega_i}{3};\,g_2,\,g_3\right) /;\, i \in \{1,\,2,\,3\}$$

Values at half-periods

09.17.03.0004.01

$$\{\zeta(\omega_1;\,g_2,\,g_3),\,\zeta(\omega_2;\,g_2,\,g_3),\,\zeta(\omega_3;\,g_2,\,g_3)\} = \{\eta_1(g_2,\,g_3),\,\eta_2(g_2,\,g_3),\,\eta_3(g_2,\,g_3)\}$$

Values at poles

09.17.03.0005.01

$$\zeta(2\,m\,\omega_1 + 2\,n\,\omega_3;\,g_2,\,g_3) = \tilde{\infty} /;\, \{m,\,n\} \in \mathbb{Z}$$

Values at fixed points

Equianharmonic case $\{g_2, g_3\} = \{0, 1\}$

09.17.03.0006.01

$$\zeta(\omega_1;\,0,\,1) = \frac{\pi}{2\,\omega_1\,\sqrt{3}}$$

09.17.03.0007.01

$$\zeta(\omega_2;\,0,\,1) = \frac{\pi}{2\,\omega_1\,\sqrt{3}}\,e^{2\,\pi\,i/3}$$

09.17.03.0008.01

$$\zeta(\omega_1;\,0,\,1) = \frac{\pi}{2\,\omega_1\,\sqrt{3}}\,e^{4\,\pi\,i/3}$$

Lemniscatic case $\{g_2, g_3\} = \{1, 0\}$

09.17.03.0009.01

$$\zeta(\omega_1;\,1,\,0) = \frac{\pi}{4\,\omega_1}$$

09.17.03.0010.01

$$\zeta(\omega_2; 1, 0) = \frac{\pi}{4\,\omega_1}(1 - i)$$

09.17.03.0011.01

$$\zeta(\omega_3; 1, 0) = -\frac{\pi\, i}{4\,\omega_1}$$

Values at infinities

09.17.03.0012.01

$$\zeta(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega, \tilde{\infty})) = \frac{1}{3}\left(\frac{\pi}{2\,\omega_1}\right)^2 z + \frac{\pi}{2\,\omega_1}\cot\left(\frac{\pi\, z}{2\,\omega_1}\right)$$

09.17.03.0013.01

$$\zeta(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) = \frac{1}{z}$$

09.17.03.0014.01

$$\zeta(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) = 0$$

General characteristics

Domain and analyticity

$\zeta(z; g_2, g_3)$ is an analytical function of z, g_2, and g_3, which is defined in $\mathbb{C}^3$.

09.17.04.0001.01

$$(z * \{g_2 * g_3\}) \longrightarrow \zeta(z; g_2, g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\zeta(z; g_2, g_3)$ is an odd function with respect to z.

09.17.04.0002.01

$$\zeta(-z; g_2, g_3) = -\zeta(z; g_2, g_3)$$

Mirror symmetry

09.17.04.0003.01

$$\zeta(\bar{z}; \overline{g_2}, \overline{g_3}) = \overline{\zeta(z; g_2, g_3)}$$

Periodicity

$\zeta(z; g_2, g_3)$ is a quasi-periodic function with respect to z.

09.17.04.0004.01

$$\zeta(z + 2\,m\,\omega_1 + 2\,n\,\omega_2 + 2\,r\,\omega_3; g_2, g_3) = \zeta(z; g_2, g_3) + 2\,m\,\eta_1 + 2\,n\,\eta_2 + 2\,r\,\eta_3 \,/;\, \{m, n, r\} \in \mathbb{Z}$$

Transformation of half-periods

09.17.04.0005.01

$$\zeta(z;\, g_2(a\,\omega_1 + b\,\omega_3,\, c\,\omega_1 + d\,\omega_3),\, g_3(a\,\omega_1 + b\,\omega_3,\, c\,\omega_1 + d\,\omega_3)) = \zeta(z;\, g_2(\omega_1,\, \omega_3),\, g_3(\omega_1,\, \omega_3)) \;/;\; \{a,\, b,\, c,\, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c = \pm 1$$

Homogeneity

09.17.04.0006.01

$$\zeta(z\,t;\, g_2,\, g_3) = \frac{1}{t}\,\zeta\!\left(z;\, g_2\,t^4,\, g_3\,t^6\right) \;/;\; t \in \mathbb{R}$$

09.17.04.0007.01

$$\zeta(\lambda\,z;\, g_2(\lambda\,\omega_1,\, \lambda\,\omega_3),\, g_3(\lambda\,\omega_1,\, \lambda\,\omega_3)) = \zeta\!\left(\lambda\,z;\, \frac{g_2(\omega_1,\, \omega_3)}{\lambda^4},\, \frac{g_3(\omega_1,\, \omega_3)}{\lambda^6}\right)$$

09.17.04.0008.01

$$\zeta(\lambda\,z;\, g_2(\lambda\,\omega_1,\, \lambda\,\omega_3),\, g_3(\lambda\,\omega_1,\, \lambda\,\omega_3)) = \frac{1}{\lambda}\,\zeta(z;\, g_2(\omega_1,\, \omega_3),\, g_3(\omega_1,\, \omega_3))$$

Poles and essential singularities

With respect to *z*

For fixed g_2, g_3, the function $\zeta(z;\, g_2,\, g_3)$ has an infinite set of singular points:

a) $z = 2\,m\,\omega_1(g_2,\, g_3) + 2\,n\,\omega_3(g_2,\, g_3)$, $\{m,\, n\} \in \mathbb{Z}$, are the simple poles with residues 1;

b) $z = \tilde{\infty}$ is an essential singular point.

09.17.04.0009.01

$$\mathcal{S}ing_z(\zeta(z;\, g_2,\, g_3)) = \{\{\{2\,m\,\omega_1 + 2\,n\,\omega_3,\, 1\} \;/;\; \{m,\, n\} \in \mathbb{Z}\},\, \{\tilde{\infty},\, \infty\}\}$$

09.17.04.0010.01

$$\mathrm{res}_z(\zeta(z;\, g_2,\, g_3))\,(2\,m\,\omega_1 + 2\,n\,\omega_3) = 1 \;/;\; \{m,\, n\} \in \mathbb{Z}$$

Branch points

With respect to *z*

For fixed g_2, g_3, the function $\zeta(z;\, g_2,\, g_3)$ does not have branch points.

09.17.04.0011.01

$$\mathcal{BP}_z(\zeta(z;\, g_2,\, g_3)) = \{\}$$

Branch cuts

With respect to *z*

For fixed g_2, g_3, the function $\zeta(z;\, g_2,\, g_3)$ does not have branch cuts.

09.17.04.0012.01

$$\mathcal{BC}_z(\zeta(z;\, g_2,\, g_3)) = \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

09.17.06.0013.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 - \ldots \;/;\; (z \to 0)$$

09.17.06.0014.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 - O(z^7)$$

09.17.06.0001.01

$$\zeta(z; g_2, g_3) == \frac{1}{z} - \sum_{k=2}^{\infty} \frac{a_k z^{2k-1}}{2k-1} \;/;\; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

09.17.06.0002.01

$$\zeta(z; g_2, g_3) == \frac{1}{z} - \sum_{k=1}^{\infty} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2 m \omega_1 + 2 n \omega_3)^{2k+2}} z^{2k+1}$$

09.17.06.0015.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} \left(1 + O(z^4)\right)$$

q-series

09.17.06.0003.01

$$\zeta(z; g_2, g_3) == \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2 \omega_1} \cot\left(\frac{\pi z}{2 \omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} \sin\left(\frac{k \pi z}{\omega_1}\right)$$

09.17.06.0004.01

$$\zeta(z; g_2, g_3) == \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2 \omega_1} \cot\left(\frac{\pi z}{2 \omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k} \sin\left(\frac{\pi z}{\omega_1}\right)}{1 - 2 \cos\left(\frac{\pi z}{\omega_1}\right) q^{2k} + q^{4k}}$$

09.17.06.0005.01

$$\eta_1 == \frac{\pi^2}{12 \omega_1} - \frac{2 \pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{k q^{2k}}{1 - q^{2k}}$$

09.17.06.0006.01

$$\eta_1 == \frac{\pi^2}{2 \omega_1} \left(4 \sum_{k=1}^{\infty} \frac{q^{2k}}{\left(q^{2k} + 1\right)^2} + \frac{1}{2}\right) - e_1 \omega_1$$

09.17.06.0007.01

$$\eta_1 == \frac{2 \pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{\left(q^{2k-1} + 1\right)^2} - e_2 \omega_1$$

09.17.06.0008.01

$$\eta_1 == -e_3 \omega_1 - \frac{2 \pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{\left(1 - q^{2k-1}\right)^2}$$

09.17.06.0009.01

$$\eta_1 == \frac{\pi^2}{2\,\omega_1}\left(\frac{1}{6} - 4\sum_{k=1}^{\infty}\frac{q^{2k}}{\left(1-q^{2k}\right)^2}\right)$$

Other series representations

09.17.06.0010.01

$$\zeta(z;\,g_2,\,g_3) == \frac{1}{z} + \sum_{\substack{m,\,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{z}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^2} + \frac{1}{2\,m\,\omega_1 + 2\,n\,\omega_3} + \frac{1}{z - 2\,m\,\omega_1 - 2\,n\,\omega_3}$$

09.17.06.0011.01

$$\zeta(z;\,g_2,\,g_3) == \frac{z\,\eta_1}{\omega_1} + \frac{\pi}{2\,\omega_1}\sum_{n=-\infty}^{\infty}\cot\left(\frac{(z - 2\,n\,\omega_3)\,\pi}{2\,\omega_1}\right)$$

09.17.06.0012.01

$$\zeta(z;\,g_2,\,g_3) == \frac{\eta_i\,z}{\omega_i} + \frac{\pi}{2\,\omega_i}\cot\left(\frac{\pi\,z}{2\,\omega_i}\right) + \frac{\pi}{2\,\omega_i}\sum_{k=-\infty}^{\infty}\cot\left(\pi\,\frac{k\,\omega_j}{\omega_i}\right) + \cot\left(\pi\,\frac{z - 2\,k\,\omega_j}{2\,\omega_i}\right)\,/;\,\{i,\,j\}\in\{1,\,2,\,3\}\bigwedge i\neq j$$

Integral representations

On the real axis

Of the direct function

09.17.07.0001.01

$$\zeta(z;\,g_2,\,g_3) == \frac{1}{z} - \frac{1}{4}\int_0^{\infty}\left(\frac{\left(t\,z - 2\sin\left(\frac{t\,z}{2}\right)\right)e^{\frac{i\,t}{2}\,\omega_2}\cos\left(\frac{t\,\omega_2}{2}\right)}{\sin\left(\frac{1}{2}\,t\,(\omega_1 - \omega_2)\right)\sin\left(\frac{1}{2}\,t\,(\omega_1 + \omega_2)\right)} - \frac{\left(t\,z - 2\sinh\left(\frac{t\,z}{2}\right)\right)\left(\cosh(t\,\omega_2) + e^{-\frac{t}{2}\,\omega_2}\sinh\left(\frac{t\,\omega_2}{2}\right)\right)}{\sinh\left(\frac{1}{2}\,t\,(\omega_1 - \omega_2)\right)\sinh\left(\frac{1}{2}\,t\,(\omega_1 + \omega_2)\right)}\right)dt$$

Involving related functions

09.17.07.0002.01

$$\zeta(z;\,g_2,\,g_3) == \frac{1}{z} - \int_0^{z}\left(\wp(t;\,g_2,\,g_3) - \frac{1}{t^2}\right)dt$$

Differential equations

Ordinary nonlinear differential equations

09.17.13.0001.01

$$\left(\frac{\partial^2\zeta(z;\,g_2,\,g_3)}{\partial z^2}\right)^2 + 4\left(\frac{\partial\zeta(z;\,g_2,\,g_3)}{\partial z}\right)^3 - g_2\,\frac{\partial\zeta(z;\,g_2,\,g_3)}{\partial z} + g_3 == 0$$

Partial differential equations

09.17.13.0002.01

$$72\,g_3\,\frac{\partial\zeta(\omega_1;\,g_2,\,g_3)}{\partial g_2} + 4\,g_2^2\,\frac{\partial\zeta(\omega_1;\,g_2,\,g_3)}{\partial g_3} - \omega_1\,g_2 == 0$$

09.17.13.0003.01

$$72\, g_3 \frac{\partial \zeta(\omega_3; g_2, g_3)}{\partial g_2} + 4\, g_2^2 \frac{\partial \zeta(\omega_3; g_2, g_3)}{\partial g_3} - \omega_3\, g_2 = 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.17.16.0001.01

$$\zeta(i\, z; g_2, g_3) = -i\, \zeta(z; g_2, -g_3)$$

Addition formulas

Translation by half-periods

09.17.16.0002.01

$$\zeta(z \pm \omega_i; g_2, g_3) = \zeta(z; g_2, g_3) \pm \eta_i + \frac{1}{2} \frac{\wp'(z; g_2, g_3)}{\wp(z; g_2, g_3) - \eta_i} \;/;\; i \in \{1, 2, 3\}$$

09.17.16.0003.01

$$\zeta(z_1 \pm z_2; g_2, g_3) = \zeta(z_1; g_2, g_3) \pm \zeta(z_2; g_2, g_3) + \frac{1}{2} \frac{\wp'(z_1; g_2, g_3) \mp \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

09.17.16.0004.01

$$\zeta(z_1 + z_2; g_2, g_3) = (1 - z_1)\, \wp(z_2; g_2, g_3) + \sum_{m,n=-\infty}^{\infty} \left(\frac{1}{z_1 + z_2 - 2\, m\, \omega_1 - 2\, n\, \omega_3} - \frac{1}{z_2 - 2\, m\, \omega_1 - 2\, n\, \omega_3} + \frac{z_1}{(z_2 - 2\, m\, \omega_1 - 2\, n\, \omega_3)^2} \right)$$

09.17.16.0005.01

$$\zeta(z_1 + z_2; g_2, g_3) + \zeta(z_1 - z_2; g_2, g_3) = 2\, \zeta(z_1; g_2, g_3) + \frac{\wp'(z_1; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

09.17.16.0006.01

$$\zeta(z_1 + z_2; g_2, g_3) - \zeta(z_1 - z_2; g_2, g_3) = 2\, \zeta(z_2; g_2, g_3) - \frac{\wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

Multiple arguments

Argument involving numeric multiples of variable

Double angle formulas

09.17.16.0007.01

$$\zeta(2\, z; g_2, g_3) = \frac{1}{2} (\zeta(z; g_2, g_3) + \zeta(z - \omega_1; g_2, g_3) + \zeta(z - \omega_2; g_2, g_3) + \zeta(z - \omega_3; g_2, g_3))$$

09.17.16.0008.01

$$\zeta(2\, z; g_2, g_3) = 2\, \zeta(z; g_2, g_3) + \frac{\wp''(z; g_2, g_3)}{2\, \wp'(z; g_2, g_3)}$$

Triple angle formulas

09.17.16.0009.01

$$\zeta(3\,z;\,g_2,\,g_3) = 3\,\zeta(z;\,g_2,\,g_3) + \frac{4\,\wp'(z;\,g_2,\,g_3)^3}{\wp'(z;\,g_2,\,g_3)\,\wp^{(3)}(z;\,g_2,\,g_3) - \wp''(z;\,g_2,\,g_3)^2}$$

Argument involving symbolic multiples of variable

Multiple angle formula:

09.17.16.0010.01

$$\zeta(n\,z;\,g_2,\,g_3) = -(n-1)\,\eta_2 + \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\zeta\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right) /;\, n \in \mathbb{N}^+$$

Related transformations

Halving half-period

09.17.16.0011.01

$$\zeta\left(z;\,g_2\left(\frac{\omega_1}{2},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{2},\,\omega_3\right)\right) = e_1\,z + \zeta(z;\,g_2,\,g_3) + \zeta(z+\omega_1;\,g_2,\,g_3) - \eta_1$$

Third of half-period

09.17.16.0012.01

$$\zeta\left(z;\,g_2\left(\frac{\omega_1}{3},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{3},\,\omega_3\right)\right) = \zeta\,(z;\,g_2,\,g_3) + \zeta\left(z + \frac{2\,\omega_1}{3};\,g_2,\,g_3\right) + \zeta\left(z + \frac{4\,\omega_1}{3};\,g_2,\,g_3\right) + 2\,z\,\wp\left(\frac{2\,\omega_1}{3};\,g_2,\,g_3\right) - 2\,\eta_1$$

General fractions of half-periods

09.17.16.0013.01

$$\zeta\left(z;\,g_2\left(\frac{\omega_1}{n},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{n},\,\omega_3\right)\right) = \zeta(z;\,g_2,\,g_3) + \sum_{k=1}^{n-1}\left(\zeta\left(z + \frac{2\,k\,\omega_1}{n};\,g_2,\,g_3\right) + z\,\wp\left(\frac{2\,k\,\omega_1}{n};\,g_2,\,g_3\right) - \frac{2\,k\,\eta_1}{n}\right) /;\, n \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to *z*

09.17.20.0001.01

$$\frac{\partial\zeta(z;\,g_2,\,g_3)}{\partial z} = -\wp(z;\,g_2,\,g_3)$$

09.17.20.0002.01

$$\frac{\partial^2\zeta(z;\,g_2,\,g_3)}{\partial z^2} = -\wp'(z;\,g_2,\,g_3)$$

09.17.20.0003.01

$$\frac{\partial^3\zeta(z;\,g_2,\,g_3)}{\partial z^3} = \frac{g_2}{2} - 6\,\wp(z;\,g_2,\,g_3)^2$$

09.17.20.0004.01

$$\frac{\partial^4 \zeta(z; g_2, g_3)}{\partial z^4} = -12\, \wp(z; g_2, g_3)\, \wp'(z; g_2, g_3)$$

With respect to g_2

09.17.20.0005.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27\, g_3^2\right)} \left(2\left(g_2^2 + 18\, g_3\, \wp(z; g_2, g_3)\right) \zeta(z; g_2, g_3) - z\, g_2\, (3\, g_3 + 2\, g_2\, \wp(z; g_2, g_3)) + 18\, g_3\, \wp'(z; g_2, g_3)\right)$$

09.17.20.0006.01

$$\frac{\partial^2 \zeta(z; g_2, g_3)}{\partial g_2^2} = \frac{1}{16\left(g_2^3 - 27\, g_3^2\right)^2} \Big(g_2^4\left(-\wp'(z; g_2, g_3)\, z^2 + \wp(z; g_2, g_3)\, z - 3\, \zeta(z; g_2, g_3)\right) + 36\, g_3\, g_2^2 \left(2\, z\, \wp(z; g_2, g_3)^2 - 4\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3) + \wp'(z; g_2, g_3)\, (z\, \zeta(z; g_2, g_3) - 2)\right) + 162\, z\, g_3^3 + 27\, g_3^2 \Big(g_2\, (7\, z\, \wp(z; g_2, g_3) - \zeta(z; g_2, g_3)) - 12\left(4\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3)^2 + \wp'(z; g_2, g_3)\, \wp(z; g_2, g_3) + \wp'(z; g_2, g_3)\, \zeta(z; g_2, g_3)^2\right)\Big)\Big)$$

With respect to g_3

09.17.20.0007.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27\, g_3^2\right)} \left(z\left(g_2^2 + 18\, g_3\, \wp(z; g_2, g_3)\right) - 6\, g_2\, \wp'(z; g_2, g_3) - 6\, (3\, g_3 + 2\, g_2\, \wp(z; g_2, g_3))\, \zeta(z; g_2, g_3)\right)$$

09.17.20.0008.01

$$\frac{\partial^2 \zeta(z; g_2, g_3)}{\partial g_3^2} = \frac{3}{4\left(g_2^3 - 27\, g_3^2\right)^2} \Big((5\, z\, \wp(z; g_2, g_3) + \zeta(z; g_2, g_3))\, g_2^3 + 6\, g_2^2 \left(z\, g_3 - 2\left(4\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3)^2 + \wp'(z; g_2, g_3)\, \wp(z; g_2, g_3) + \wp'(z; g_2, g_3)\, \zeta(z; g_2, g_3)^2\right)\right) + 36\, g_3\, g_2\left(2\, z\, \wp(z; g_2, g_3)^2 - 4\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3) + \wp'(z; g_2, g_3)\, (z\, \zeta(z; g_2, g_3) - 2)\right) - 27\, g_3^2 \left(\wp'(z; g_2, g_3)\, z^2 - 3\, \wp(z; g_2, g_3)\, z + 5\, \zeta(z; g_2, g_3)\right)\Big)$$

With respect to ω_1

09.17.20.0011.02

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_1} = \frac{\omega_1}{\pi\, \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_3 \left(\wp'(z; g_2, g_3) + 2\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3) - \frac{1}{6}\, g_2\, z\right) + 2\, \eta_3\, (\zeta(z; g_2, g_3) - z\, \wp(z; g_2, g_3))\right)$$

With respect to ω_3

09.17.20.0012.02

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_1}{\pi\, \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_1 \left(\wp'(z; g_2, g_3) + 2\, \zeta(z; g_2, g_3)\, \wp(z; g_2, g_3) - \frac{1}{6}\, g_2\, z\right) + 2\, \eta_1\, (\zeta(z; g_2, g_3) - z\, \wp(z; g_2, g_3))\right)$$

09.17.20.0013.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_3}{6\, \pi \sqrt{-\omega_3^2}} \left(g_2(1, \omega_3) - 6\, \wp'(1; g_2(1, \omega_3), g_3(1, \omega_3)) - 12\, \zeta(1; g_2(1, \omega_3), g_3(1, \omega_3))^2\right)$$

Symbolic differentiation

With respect to *z*

09.17.20.0009.01

$$\frac{\partial^n \zeta(z; g_2, g_3)}{\partial z^n} = -\frac{n}{2}\left(\frac{\pi}{\omega_1}\right)^{n+1}\left(\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\frac{(-1)^j\, 2^{-2k}}{k+1}\binom{n-1}{k}\sin^{-2k-2}\left(\frac{\pi z}{2\,\omega_1}\right)\binom{2k}{j}(k-j)^{n-1}\sin\left(\frac{\pi z\,(k-j)}{\omega_1}+\frac{n\,\pi}{2}\right)\right)- \frac{\pi^2\,\delta_{n-1}}{4\,\omega_1^2}\csc^2\left(\frac{\pi z}{2\,\omega_1}\right)+\frac{z^{1-n}\,\eta_1}{\omega_1\,\Gamma(2-n)}+\frac{2\,\pi^{n+1}}{\omega_1^{n+1}}\sum_{k=1}^{\infty}\frac{q^{2k}\,k^n}{1-q^{2k}}\sin\left(\frac{\pi n}{2}+\frac{k\,\pi z}{\omega_1}\right)\;/;\; n\in\mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.17.20.0010.01

$$\frac{\partial^\alpha \zeta(z; g_2, g_3)}{\partial z^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(z, -1)\, z^{-\alpha-1} - 2\, z^{2-\alpha}\sum_{\substack{m,\,n=-\infty\\ \{m,n\}\neq\{0,0\}}}^{\infty}\frac{1}{(2\,m\,\omega_1+2\,n\,\omega_3)^3}\times {}_2\tilde{F}_1\left(1, 3; 3-\alpha; \frac{z}{2\,m\,\omega_1+2\,n\,\omega_3}\right)$$

Integration

Indefinite integration

For the direct function itself

09.17.21.0001.01

$$\int \zeta(z; g_2, g_3)\, dz = \log(\sigma(z; g_2, g_3))$$

Summation

Finite summation

09.17.23.0001.01

$$\sum_{k=1}^{n-1}\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) = \frac{n}{\omega_1}\left(\zeta\left(\frac{\omega_1}{n}; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right)-\eta_1\right)$$

09.17.23.0002.01

$$\sum_{k=1}^{n-1}\wp\left(\frac{2\,k\,\omega_1}{n}; g_2, g_3\right) = \frac{1}{\omega_3}\left(\zeta\left(\frac{\omega_1}{n}; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right)-n\,\eta_3\right)$$

09.17.23.0003.01

$$\sum_{\substack{j,\,k=0\\ \{j,k\}\neq\{0,0\}}}^{n-1}\zeta\left(\frac{2\,j\,\omega_1+2\,k\,\omega_3}{n}; g_2, g_3\right) = -n\,(n-1)\,\eta_2\;/;\; n-1\in\mathbb{N}^+$$

Representations through equivalent functions

With related functions

Involving other Weierstrass functions

09.17.27.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \int_0^z \left(\wp(t; g_2, g_3) - \frac{1}{t^2}\right) dt$$

09.17.27.0002.01

$$\zeta(z; g_2, g_3) = \frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z}$$

09.17.27.0003.01

$$\zeta(z; g_2, g_3) = \frac{\sigma'(z; g_2, g_3)}{\sigma(z; g_2, g_3)}$$

Involving theta functions

09.17.27.0004.01

$$\zeta(z; g_2, g_3) = \frac{z\,\eta_1}{\omega_1} + \frac{\pi\,\vartheta_1'\left(\frac{\pi z}{2\,\omega_1}, q\right)}{2\,\omega_1\,\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)}$$

09.17.27.0005.01

$$\zeta(z + \omega_i; g_2, g_3) = \eta_i + \frac{z\,\eta_1}{\omega_1} + \frac{\pi\,\vartheta_{i+1}'\left(\frac{\pi z}{2\,\omega_1}, q\right)}{2\,\omega_1\,\vartheta_{i+1}\left(\frac{\pi z}{2\,\omega_1}, q\right)} \;/;\; i \in \{1, 2, 3\}$$

09.17.27.0006.01

$$\eta_1 = -\frac{\pi^2}{12\,\omega_1}\,\frac{\vartheta_1^{(3)}(0, q)}{\vartheta_1'(0, q)}$$

09.17.27.0007.01

$$\eta_1 = -e_i\,\omega_1 - \frac{\pi^2}{4\,\omega_1}\,\frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} \;/;\; i \in \{1, 2, 3\}$$

09.17.27.0008.01

$$\eta_1^2 = \left(\frac{g_2}{6} - e_i^2\right)\omega_1^2 - e_i - \frac{\pi^2\,\eta_1}{2\,\omega_1}\,\frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} - \frac{\pi^4}{48\,\omega_1^2}\,\frac{\vartheta_{i+1}^{(4)}(0, q)}{\vartheta_{i+1}(0, q)} \;/;\; i \in \{1, 2, 3\}$$

Involving elliptic integrals and modular functions

09.17.27.0009.01

$$\{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} = \left\{\sqrt{e_1 - e_3}\left(E(m) - \frac{e_1}{e_1 - e_3}\,K(m)\right), -i\,\sqrt{e_1 - e_3}\left(E(1-m) + \frac{e_3}{e_1 - e_3}\,K(1-m)\right)\right\} /;$$
$$m = q^{-1}(q) \bigwedge \{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$$

History

– K. Weierstrass (1862)

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