

WeierstrassZeta

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Notations

Traditional name

Weierstrass zeta function

Traditional notation

$\zeta(z; g_2, g_3)$

Mathematica StandardForm notation

WeierstrassZeta[z, {g₂, g₃}]

Primary definition

09.17.02.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} + \sum_{\substack{m, n = -\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{z}{(2m\omega_1 + 2n\omega_3)^2} + \frac{1}{2m\omega_1 + 2n\omega_3} + \frac{1}{z - 2m\omega_1 - 2n\omega_3} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Special notations for this file:

09.17.02.0002.01

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.17.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.17.02.0004.01

$$e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.17.02.0005.01

$$\eta_n = \zeta(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.17.02.0006.01

$$q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed z

Degenerate case:

09.17.03.0001.01

$$\zeta(z; 0, 0) = \frac{1}{z}$$

09.17.03.0002.01

$$\zeta(z; 3, 1) = \frac{z}{2} + \sqrt{\frac{3}{2}} \cot\left(\sqrt{\frac{3}{2}} z\right)$$

For fixed $\{g_2, g_3\}$

One-third period values

09.17.03.0003.01

$$\left(\zeta\left(\frac{2\omega_i}{3}; g_2, g_3\right) - \frac{2\eta_i}{3}\right)^2 = \frac{1}{3} \wp\left(\frac{2\omega_i}{3}; g_2, g_3\right); i \in \{1, 2, 3\}$$

Values at half-periods

09.17.03.0004.01

$$\{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$$

Values at poles

09.17.03.0005.01

$$\zeta(2m\omega_1 + 2n\omega_3; g_2, g_3) = \infty; \{m, n\} \in \mathbb{Z}$$

Values at fixed points

Equianharmonic case $\{g_2, g_3\} = \{0, 1\}$

09.17.03.0006.01

$$\zeta(\omega_1; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}}$$

09.17.03.0007.01

$$\zeta(\omega_2; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}} e^{2\pi i/3}$$

09.17.03.0008.01

$$\zeta(\omega_3; 0, 1) = \frac{\pi}{2\omega_1\sqrt{3}} e^{4\pi i/3}$$

Lemniscatic case $\{g_2, g_3\} = \{1, 0\}$

09.17.03.0009.01

$$\zeta(\omega_1; 1, 0) = \frac{\pi}{4\omega_1}$$

09.17.03.0010.01

$$\zeta(\omega_2; 1, 0) = \frac{\pi}{4\omega_1} (1 - i)$$

09.17.03.0011.01

$$\zeta(\omega_3; 1, 0) = -\frac{\pi i}{4\omega_1}$$

Values at infinities

09.17.03.0012.01

$$\zeta(z; g_2(\omega_1, \infty), g_3(\omega, \infty)) = \frac{1}{3} \left(\frac{\pi}{2\omega_1} \right)^2 z + \frac{\pi}{2\omega_1} \cot\left(\frac{\pi z}{2\omega_1}\right)$$

09.17.03.0013.01

$$\zeta(z; g_2(\infty, \infty), g_3(\infty, \infty)) = \frac{1}{z}$$

09.17.03.0014.01

$$\zeta(\infty; g_2(\infty, \infty), g_3(\infty, \infty)) = 0$$

General characteristics

Domain and analyticity

$\zeta(z; g_2, g_3)$ is an analytical function of z , g_2 , and g_3 , which is defined in \mathbb{C}^3 .

09.17.04.0001.01

$$(z * \{g_2 * g_3\}) \rightarrow \zeta(z; g_2, g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \rightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\zeta(z; g_2, g_3)$ is an odd function with respect to z .

09.17.04.0002.01

$$\zeta(-z; g_2, g_3) = -\zeta(z; g_2, g_3)$$

Mirror symmetry

09.17.04.0003.01

$$\zeta(\bar{z}; \bar{g}_2, \bar{g}_3) = \overline{\zeta(z; g_2, g_3)}$$

Periodicity

$\zeta(z; g_2, g_3)$ is a quasi-periodic function with respect to z .

09.17.04.0004.01

$$\zeta(z + 2m\omega_1 + 2n\omega_2 + 2r\omega_3; g_2, g_3) = \zeta(z; g_2, g_3) + 2m\eta_1 + 2n\eta_2 + 2r\eta_3 \ ; \ \{m, n, r\} \in \mathbb{Z}$$

Transformation of half-periods

09.17.04.0005.01

$$\zeta(z; g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3)) = \zeta(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)) /;$$

$$\{a, b, c, d\} \in \mathbb{Z} \wedge ad - bc = \pm 1$$

Homogeneity

09.17.04.0006.01

$$\zeta(zt; g_2, g_3) = \frac{1}{t} \zeta(z; g_2 t^4, g_3 t^6) /; t \in \mathbb{R}$$

09.17.04.0007.01

$$\zeta(\lambda z; g_2(\lambda\omega_1, \lambda\omega_3), g_3(\lambda\omega_1, \lambda\omega_3)) = \zeta\left(\lambda z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

09.17.04.0008.01

$$\zeta(\lambda z; g_2(\lambda\omega_1, \lambda\omega_3), g_3(\lambda\omega_1, \lambda\omega_3)) = \frac{1}{\lambda} \zeta(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

Poles and essential singularities**With respect to z**

For fixed g_2, g_3 , the function $\zeta(z; g_2, g_3)$ has an infinite set of singular points:

- a) $z = 2m\omega_1(g_2, g_3) + 2n\omega_3(g_2, g_3)$, $\{m, n\} \in \mathbb{Z}$, are the simple poles with residues 1;
 b) $z = \infty$ is an essential singular point.

09.17.04.0009.01

$$\text{Sing}_z(\zeta(z; g_2, g_3)) = \{(2m\omega_1 + 2n\omega_3, 1) /; \{m, n\} \in \mathbb{Z}\}, \{\infty, \infty\}$$

09.17.04.0010.01

$$\text{res}_z(\zeta(z; g_2, g_3))(2m\omega_1 + 2n\omega_3) = 1 /; \{m, n\} \in \mathbb{Z}$$

Branch points**With respect to z**

For fixed g_2, g_3 , the function $\zeta(z; g_2, g_3)$ does not have branch points.

09.17.04.0011.01

$$\mathcal{BP}_z(\zeta(z; g_2, g_3)) = \{\}$$

Branch cuts**With respect to z**

For fixed g_2, g_3 , the function $\zeta(z; g_2, g_3)$ does not have branch cuts.

09.17.04.0012.01

$$\mathcal{BC}_z(\zeta(z; g_2, g_3)) = \{\}$$

Series representations**Generalized power series**

Expansions at $z = 0$

09.17.06.0013.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 - \dots /; (z \rightarrow 0)$$

09.17.06.0014.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} - \frac{g_2}{60} z^3 - \frac{g_3}{140} z^5 - O(z^7)$$

09.17.06.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \sum_{k=2}^{\infty} \frac{a_k z^{2k-1}}{2k-1} /; a_2 = \frac{g_2}{20} \wedge a_3 = \frac{g_3}{28} \wedge a_k = \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

09.17.06.0002.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \sum_{k=1}^{\infty} \sum_{\substack{m, n=-\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2k+2}} z^{2k+1}$$

09.17.06.0015.01

$$\zeta(z; g_2, g_3) \propto \frac{1}{z} (1 + O(z^4))$$

q-series

09.17.06.0003.01

$$\zeta(z; g_2, g_3) = \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2\omega_1} \cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \sin\left(\frac{k\pi z}{\omega_1}\right)$$

09.17.06.0004.01

$$\zeta(z; g_2, g_3) = \frac{\eta_1 z}{\omega_1} + \frac{\pi}{2\omega_1} \cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k} \sin\left(\frac{\pi z}{\omega_1}\right)}{1 - 2 \cos\left(\frac{\pi z}{\omega_1}\right) q^{2k} + q^{4k}}$$

09.17.06.0005.01

$$\eta_1 = \frac{\pi^2}{12\omega_1} - \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{k q^{2k}}{1-q^{2k}}$$

09.17.06.0006.01

$$\eta_1 = \frac{\pi^2}{2\omega_1} \left(4 \sum_{k=1}^{\infty} \frac{q^{2k}}{(q^{2k} + 1)^2} + \frac{1}{2} \right) - e_1 \omega_1$$

09.17.06.0007.01

$$\eta_1 = \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(q^{2k-1} + 1)^2} - e_2 \omega_1$$

09.17.06.0008.01

$$\eta_1 = -e_3 \omega_1 - \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(1-q^{2k-1})^2}$$

09.17.06.0009.01

$$\eta_1 = \frac{\pi^2}{2\omega_1} \left(\frac{1}{6} - 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^{2k})^2} \right)$$

Other series representations

09.17.06.0010.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} + \sum_{\substack{m, n=-\infty \\ (m, n) \neq (0, 0)}}^{\infty} \frac{z}{(2m\omega_1 + 2n\omega_3)^2} + \frac{1}{2m\omega_1 + 2n\omega_3} + \frac{1}{z - 2m\omega_1 - 2n\omega_3}$$

09.17.06.0011.01

$$\zeta(z; g_2, g_3) = \frac{z\eta_1}{\omega_1} + \frac{\pi}{2\omega_1} \sum_{n=-\infty}^{\infty} \cot\left(\frac{(z - 2n\omega_3)\pi}{2\omega_1}\right)$$

09.17.06.0012.01

$$\zeta(z; g_2, g_3) = \frac{\eta_i z}{\omega_i} + \frac{\pi}{2\omega_i} \cot\left(\frac{\pi z}{2\omega_i}\right) + \frac{\pi}{2\omega_i} \sum_{k=-\infty}^{\infty} \cot\left(\pi \frac{k\omega_j}{\omega_i}\right) + \cot\left(\pi \frac{z - 2k\omega_j}{2\omega_i}\right); \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

Integral representations

On the real axis

Of the direct function

09.17.07.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \frac{1}{4} \int_0^{\infty} \left(\frac{(tz - 2 \sin(\frac{tz}{2})) e^{\frac{it}{2}\omega_2} \cos(\frac{t\omega_2}{2})}{\sin(\frac{1}{2}t(\omega_1 - \omega_2)) \sin(\frac{1}{2}t(\omega_1 + \omega_2))} - \frac{(tz - 2 \sinh(\frac{tz}{2})) (\cosh(t\omega_2) + e^{-\frac{t}{2}\omega_2} \sinh(\frac{t\omega_2}{2}))}{\sinh(\frac{1}{2}t(\omega_1 - \omega_2)) \sinh(\frac{1}{2}t(\omega_1 + \omega_2))} \right) dt$$

Involving related functions

09.17.07.0002.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \int_0^z \left(\wp(t; g_2, g_3) - \frac{1}{t^2} \right) dt$$

Differential equations

Ordinary nonlinear differential equations

09.17.13.0001.01

$$\left(\frac{\partial^2 \zeta(z; g_2, g_3)}{\partial z^2} \right)^2 + 4 \left(\frac{\partial \zeta(z; g_2, g_3)}{\partial z} \right)^3 - g_2 \frac{\partial \zeta(z; g_2, g_3)}{\partial z} + g_3 = 0$$

Partial differential equations

09.17.13.0002.01

$$72 g_3 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_2} + 4 g_2^2 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_3} - \omega_1 g_2 = 0$$

09.17.13.0003.01

$$72 g_3 \frac{\partial \zeta(\omega_3; g_2, g_3)}{\partial g_2} + 4 g_2^2 \frac{\partial \zeta(\omega_3; g_2, g_3)}{\partial g_3} - \omega_3 g_2 = 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.17.16.0001.01

$$\zeta(i z; g_2, g_3) = -i \zeta(z; g_2, -g_3)$$

Addition formulas

Translation by half-periods

09.17.16.0002.01

$$\zeta(z \pm \omega_i; g_2, g_3) = \zeta(z; g_2, g_3) \pm \eta_i + \frac{1}{2} \frac{\wp'(z; g_2, g_3)}{\wp(z; g_2, g_3) - \eta_i} \quad ; i \in \{1, 2, 3\}$$

09.17.16.0003.01

$$\zeta(z_1 \pm z_2; g_2, g_3) = \zeta(z_1; g_2, g_3) \pm \zeta(z_2; g_2, g_3) + \frac{1}{2} \frac{\wp'(z_1; g_2, g_3) \mp \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

09.17.16.0004.01

$$\zeta(z_1 + z_2; g_2, g_3) = (1 - z_1) \wp(z_2; g_2, g_3) + \sum_{m,n=-\infty}^{\infty} \left(\frac{1}{z_1 + z_2 - 2m\omega_1 - 2n\omega_3} - \frac{1}{z_2 - 2m\omega_1 - 2n\omega_3} + \frac{z_1}{(z_2 - 2m\omega_1 - 2n\omega_3)^2} \right)$$

09.17.16.0005.01

$$\zeta(z_1 + z_2; g_2, g_3) + \zeta(z_1 - z_2; g_2, g_3) = 2\zeta(z_1; g_2, g_3) + \frac{\wp'(z_1; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

09.17.16.0006.01

$$\zeta(z_1 + z_2; g_2, g_3) - \zeta(z_1 - z_2; g_2, g_3) = 2\zeta(z_2; g_2, g_3) - \frac{\wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)}$$

Multiple arguments

Argument involving numeric multiples of variable

Double angle formulas

09.17.16.0007.01

$$\zeta(2z; g_2, g_3) = \frac{1}{2} (\zeta(z; g_2, g_3) + \zeta(z - \omega_1; g_2, g_3) + \zeta(z - \omega_2; g_2, g_3) + \zeta(z - \omega_3; g_2, g_3))$$

09.17.16.0008.01

$$\zeta(2z; g_2, g_3) = 2\zeta(z; g_2, g_3) + \frac{\wp''(z; g_2, g_3)}{2\wp'(z; g_2, g_3)}$$

Triple angle formulas

09.17.16.0009.01

$$\zeta(3z; g_2, g_3) = 3\zeta(z; g_2, g_3) + \frac{4\wp'(z; g_2, g_3)^3}{\wp'(z; g_2, g_3)\wp^{(3)}(z; g_2, g_3) - \wp''(z; g_2, g_3)^2}$$

Argument involving symbolic multiples of variable

Multiple angle formula:

09.17.16.0010.01

$$\zeta(nz; g_2, g_3) = -(n-1)\eta_2 + \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \zeta\left(z - \frac{2j\omega_1 + 2k\omega_3}{n}; g_2, g_3\right); n \in \mathbb{N}^+$$

Related transformations

Halving half-period

09.17.16.0011.01

$$\zeta\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = e_1 z + \zeta(z; g_2, g_3) + \zeta(z + \omega_1; g_2, g_3) - \eta_1$$

Third of half-period

09.17.16.0012.01

$$\zeta\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \zeta(z; g_2, g_3) + \zeta\left(z + \frac{2\omega_1}{3}; g_2, g_3\right) + \zeta\left(z + \frac{4\omega_1}{3}; g_2, g_3\right) + 2z\wp\left(\frac{2\omega_1}{3}; g_2, g_3\right) - 2\eta_1$$

General fractions of half-periods

09.17.16.0013.01

$$\zeta\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \zeta(z; g_2, g_3) + \sum_{k=1}^{n-1} \left(\zeta\left(z + \frac{2k\omega_1}{n}; g_2, g_3\right) + z\wp\left(\frac{2k\omega_1}{n}; g_2, g_3\right) - \frac{2k\eta_1}{n} \right); n \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to z

09.17.20.0001.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial z} = -\wp(z; g_2, g_3)$$

09.17.20.0002.01

$$\frac{\partial^2 \zeta(z; g_2, g_3)}{\partial z^2} = -\wp'(z; g_2, g_3)$$

09.17.20.0003.01

$$\frac{\partial^3 \zeta(z; g_2, g_3)}{\partial z^3} = \frac{g_2}{2} - 6\wp(z; g_2, g_3)^2$$

09.17.20.0004.01

$$\frac{\partial^4 \zeta(z; g_2, g_3)}{\partial z^4} = -12 \wp(z; g_2, g_3) \wp'(z; g_2, g_3)$$

With respect to g_2

09.17.20.0005.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_2} = \frac{1}{8(g_2^3 - 27g_3^2)} (2(g_2^2 + 18g_3 \wp(z; g_2, g_3)) \zeta(z; g_2, g_3) - z g_2 (3g_3 + 2g_2 \wp(z; g_2, g_3)) + 18g_3 \wp'(z; g_2, g_3))$$

09.17.20.0006.01

$$\begin{aligned} \frac{\partial^2 \zeta(z; g_2, g_3)}{\partial g_2^2} = & \frac{1}{16(g_2^3 - 27g_3^2)^2} (g_2^4 (-\wp'(z; g_2, g_3) z^2 + \wp(z; g_2, g_3) z - 3 \zeta(z; g_2, g_3)) + \\ & 36g_3 g_2^2 (2z \wp(z; g_2, g_3)^2 - 4 \zeta(z; g_2, g_3) \wp(z; g_2, g_3) + \wp'(z; g_2, g_3) (z \zeta(z; g_2, g_3) - 2)) + \\ & 162z g_3^3 + 27g_3^2 (g_2 (7z \wp(z; g_2, g_3) - \zeta(z; g_2, g_3)) - \\ & 12(4 \zeta(z; g_2, g_3) \wp(z; g_2, g_3)^2 + \wp'(z; g_2, g_3) \wp(z; g_2, g_3) + \wp'(z; g_2, g_3) \zeta(z; g_2, g_3)^2))) \end{aligned}$$

With respect to g_3

09.17.20.0007.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial g_3} = \frac{1}{4(g_2^3 - 27g_3^2)} (z(g_2^2 + 18g_3 \wp(z; g_2, g_3)) - 6g_2 \wp'(z; g_2, g_3) - 6(3g_3 + 2g_2 \wp(z; g_2, g_3)) \zeta(z; g_2, g_3))$$

09.17.20.0008.01

$$\begin{aligned} \frac{\partial^2 \zeta(z; g_2, g_3)}{\partial g_3^2} = & \frac{3}{4(g_2^3 - 27g_3^2)^2} ((5z \wp(z; g_2, g_3) + \zeta(z; g_2, g_3)) g_2^3 + \\ & 6g_2^2 (z g_3 - 2(4 \zeta(z; g_2, g_3) \wp(z; g_2, g_3)^2 + \wp'(z; g_2, g_3) \wp(z; g_2, g_3) + \wp'(z; g_2, g_3) \zeta(z; g_2, g_3)^2)) + \\ & 36g_3 g_2 (2z \wp(z; g_2, g_3)^2 - 4 \zeta(z; g_2, g_3) \wp(z; g_2, g_3) + \wp'(z; g_2, g_3) (z \zeta(z; g_2, g_3) - 2)) - \\ & 27g_3^2 (\wp'(z; g_2, g_3) z^2 - 3 \wp(z; g_2, g_3) z + 5 \zeta(z; g_2, g_3))) \end{aligned}$$

With respect to ω_1

09.17.20.0011.02

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_3 \left(\wp'(z; g_2, g_3) + 2 \zeta(z; g_2, g_3) \wp(z; g_2, g_3) - \frac{1}{6} g_2 z \right) + 2 \eta_3 (\zeta(z; g_2, g_3) - z \wp(z; g_2, g_3)) \right)$$

With respect to ω_3

09.17.20.0012.02

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_1 \left(\wp'(z; g_2, g_3) + 2 \zeta(z; g_2, g_3) \wp(z; g_2, g_3) - \frac{1}{6} g_2 z \right) + 2 \eta_1 (\zeta(z; g_2, g_3) - z \wp(z; g_2, g_3)) \right)$$

09.17.20.0013.01

$$\frac{\partial \zeta(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_3}{6\pi \sqrt{-\omega_3^2}} (g_2(1, \omega_3) - 6 \wp'(1; g_2(1, \omega_3), g_3(1, \omega_3)) - 12 \zeta(1; g_2(1, \omega_3), g_3(1, \omega_3))^2)$$

Symbolic differentiation

With respect to z

09.17.20.0009.01

$$\frac{\partial^n \zeta(z; g_2, g_3)}{\partial z^n} = -\frac{n}{2} \left(\frac{\pi}{\omega_1}\right)^{n+1} \left(\sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^j 2^{-2k}}{k+1} \binom{n-1}{k} \sin^{-2k-2} \left(\frac{\pi z}{2\omega_1}\right) \binom{2k}{j} (k-j)^{n-1} \sin \left(\frac{\pi z(k-j)}{\omega_1} + \frac{n\pi}{2}\right) \right) - \frac{\pi^2 \delta_{n-1}}{4 \omega_1^2} \operatorname{csc}^2 \left(\frac{\pi z}{2\omega_1}\right) + \frac{z^{1-n} \eta_1}{\omega_1 \Gamma(2-n)} + \frac{2 \pi^{n+1}}{\omega_1^{n+1}} \sum_{k=1}^{\infty} \frac{q^{2k} k^n}{1-q^{2k}} \sin \left(\frac{\pi n}{2} + \frac{k \pi z}{\omega_1}\right); n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

09.17.20.0010.01

$$\frac{\partial^\alpha \zeta(z; g_2, g_3)}{\partial z^\alpha} = \mathcal{F}C_{\exp}^{(\alpha)}(z, -1) z^{-\alpha-1} - 2 z^{2-\alpha} \sum_{\substack{m, n=-\infty \\ \{m, n\} \neq \{0, 0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^3} \times {}_2\tilde{F}_1 \left(1, 3; 3-\alpha; \frac{z}{2m\omega_1 + 2n\omega_3} \right)$$

Integration

Indefinite integration

For the direct function itself

09.17.21.0001.01

$$\int \zeta(z; g_2, g_3) dz = \log(\sigma(z; g_2, g_3))$$

Summation

Finite summation

09.17.23.0001.01

$$\sum_{k=1}^{n-1} \wp \left(\frac{2k\omega_1}{n}; g_2, g_3 \right) = \frac{n}{\omega_1} \left(\zeta \left(\frac{\omega_1}{n}; g_2 \left(\frac{\omega_1}{n}, \omega_3 \right), g_3 \left(\frac{\omega_1}{n}, \omega_3 \right) \right) - \eta_1 \right)$$

09.17.23.0002.01

$$\sum_{k=1}^{n-1} \wp \left(\frac{2k\omega_1}{n}; g_2, g_3 \right) = \frac{1}{\omega_3} \left(\zeta \left(\frac{\omega_1}{n}; g_2 \left(\frac{\omega_1}{n}, \omega_3 \right), g_3 \left(\frac{\omega_1}{n}, \omega_3 \right) \right) - n \eta_3 \right)$$

09.17.23.0003.01

$$\sum_{\substack{j, k=0 \\ \{j, k\} \neq \{0, 0\}}}^{n-1} \zeta \left(\frac{2j\omega_1 + 2k\omega_3}{n}; g_2, g_3 \right) = -n(n-1) \eta_2; n-1 \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

Involving other Weierstrass functions

09.17.27.0001.01

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \int_0^z \left(\wp(t; g_2, g_3) - \frac{1}{t^2} \right) dt$$

09.17.27.0002.01

$$\zeta(z; g_2, g_3) = \frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z}$$

09.17.27.0003.01

$$\zeta(z; g_2, g_3) = \frac{\sigma'(z; g_2, g_3)}{\sigma(z; g_2, g_3)}$$

Involving theta functions

09.17.27.0004.01

$$\zeta(z; g_2, g_3) = \frac{z \eta_1}{\omega_1} + \frac{\pi \vartheta_1' \left(\frac{\pi z}{2\omega_1}, q \right)}{2 \omega_1 \vartheta_1 \left(\frac{\pi z}{2\omega_1}, q \right)}$$

09.17.27.0005.01

$$\zeta(z + \omega_i; g_2, g_3) = \eta_i + \frac{z \eta_1}{\omega_1} + \frac{\pi \vartheta_{i+1}' \left(\frac{\pi z}{2\omega_1}, q \right)}{2 \omega_1 \vartheta_{i+1} \left(\frac{\pi z}{2\omega_1}, q \right)} \quad ; i \in \{1, 2, 3\}$$

09.17.27.0006.01

$$\eta_1 = -\frac{\pi^2}{12 \omega_1} \frac{\vartheta_1^{(3)}(0, q)}{\vartheta_1'(0, q)}$$

09.17.27.0007.01

$$\eta_i = -e_i \omega_1 - \frac{\pi^2}{4 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}'(0, q)} \quad ; i \in \{1, 2, 3\}$$

09.17.27.0008.01

$$\eta_i^2 = \left(\frac{g_2}{6} - e_i^2 \right) \omega_1^2 - e_i - \frac{\pi^2 \eta_1}{2 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}'(0, q)} - \frac{\pi^4}{48 \omega_1^2} \frac{\vartheta_{i+1}^{(4)}(0, q)}{\vartheta_{i+1}'(0, q)} \quad ; i \in \{1, 2, 3\}$$

Involving elliptic integrals and modular functions

09.17.27.0009.01

$$\{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} = \left\{ \sqrt{e_1 - e_3} \left(E(m) - \frac{e_1}{e_1 - e_3} K(m) \right), -i \sqrt{e_1 - e_3} \left(E(1 - m) + \frac{e_3}{e_1 - e_3} K(1 - m) \right) \right\} /;$$

$$m = q^{-1}(q) \wedge \{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$$

History

–K. Weierstrass (1862)

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