Zeta

Notations

Traditional name

Riemann zeta function

Traditional notation

$\zeta(s)$

Mathematica StandardForm notation

`Zeta[s]`

Primary definition

10.01.02.0001.01

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \;/;\; \mathrm{Re}(s) > 1$$

Specific values

Specialized values

10.01.03.0001.01

$$\zeta(-n) = \frac{(-1)^n}{n+1} B_{n+1} \;/;\; n \in \mathbb{N}$$

10.01.03.0002.01

$$\zeta(-2n) = 0 \;/;\; n \in \mathbb{N}^+$$

10.01.03.0003.01

$$\zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n}}{(2n)!} B_{2n} \;/;\; n \in \mathbb{N}$$

10.01.03.0053.01

$$\zeta(2n) = \frac{2^{2n-1} \pi^{2n}}{(2n)!} \left| \sum_{k=0}^{2n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r r^{2n} \binom{k}{r} \right| \;/;\; n \in \mathbb{N}$$

10.01.03.0004.01

$$\zeta(2n) = \frac{(-1)^n 2^{2n-2} \pi^{2n}}{\left(2^{2n} - 1\right)(2n-1)!} E_{2n-1}(0) \;/;\; n \in \mathbb{N}^+$$

Values at fixed points

10.01.03.0005.01

$$\zeta(-10) = 0$$

10.01.03.0006.01

$$\zeta(-9) = -\frac{1}{132}$$

10.01.03.0007.01

$$\zeta(-8) = 0$$

10.01.03.0008.01

$$\zeta(-7) = \frac{1}{240}$$

10.01.03.0009.01

$$\zeta(-6) = 0$$

10.01.03.0010.01

$$\zeta(-5) = -\frac{1}{252}$$

10.01.03.0011.01

$$\zeta(-4) = 0$$

10.01.03.0012.01

$$\zeta(-3) = \frac{1}{120}$$

10.01.03.0013.01

$$\zeta(-2) = 0$$

10.01.03.0014.01

$$\zeta(-1) = -\frac{1}{12}$$

10.01.03.0015.01

$$\zeta(0) = -\frac{1}{2}$$

10.01.03.0016.01

$$\zeta(1) = \tilde{\infty}$$

10.01.03.0017.01

$$\zeta(2) = \frac{\pi^2}{6}$$

10.01.03.0018.01

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{k=1}^{\infty} \frac{1}{k^3 \left(e^{2\pi k} - 1\right)}$$

10.01.03.0054.01

$$\zeta(3) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\left(205k^2 - 160k + 32\right)(k!)^2}{k^5\,(2k)!}$$

10.01.03.0055.01

$$\zeta(3) == \frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^k\left(205\,k^2+250\,k+77\right)(k!)^{10}}{((2\,k+1)!)^5}$$

As of 2003, the values of $\zeta(3)$ was computed with an accuracy of approximately 1000000000 decimal digits by using above formula.

10.01.03.0056.01

$$\zeta(3) = -\frac{2}{3}\,\pi^2\left(6\log(A)+\log(2\,\pi)-12\,\psi^{(-4)}(1)\right)$$

10.01.03.0057.01

$$\zeta(3) = \frac{29}{24}\,{}_8F_7\!\left(\frac{1}{2},\frac{1}{2},1,1,1,1,\frac{1}{28}\left(48-i\sqrt{6}\right),\frac{1}{28}\left(48+i\sqrt{6}\right);\frac{4}{3},\frac{5}{3},\frac{3}{2},\frac{3}{2},2,\frac{1}{28}\left(20-i\sqrt{6}\right),\frac{1}{28}\left(20+i\sqrt{6}\right);-\frac{1}{27}\right)$$

Brychkov Yu.A. (2006)

10.01.03.0019.01

$$\zeta(4) == \frac{\pi^4}{90}$$

10.01.03.0020.01

$$\zeta(5) == \frac{\pi^5}{294}-\frac{72}{35}\sum_{k=1}^{\infty}\frac{1}{k^5\left(e^{2\pi k}-1\right)}-\frac{2}{35}\sum_{k=1}^{\infty}\frac{1}{k^5\left(e^{2\pi k}+1\right)}$$

10.01.03.0058.01

$$\zeta(5) = \frac{\pi^4}{45}\left(\log(2\,\pi)-120\left(12\,\psi^{(-6)}(1)-6\,\psi^{(-5)}(1)+\psi^{(-4)}(1)\right)\right)$$

10.01.03.0021.01

$$\zeta(6) == \frac{\pi^6}{945}$$

10.01.03.0022.01

$$\zeta(7) == \frac{19\,\pi^7}{56\,700}-2\sum_{k=1}^{\infty}\frac{1}{k^7\left(e^{2\pi k}-1\right)}$$

10.01.03.0059.01

$$\zeta(7) = \frac{2\,\pi^6}{945}\left(\log(2\,\pi)+84\left(720\,\psi^{(-8)}(1)-360\,\psi^{(-7)}(1)+60\,\psi^{(-6)}(1)-\psi^{(-4)}(1)\right)\right)$$

10.01.03.0023.01

$$\zeta(8) == \frac{\pi^8}{9450}$$

10.01.03.0024.01

$$\zeta(9) == \frac{125\,\pi^9}{3\,704\,778}-\frac{992}{495}\sum_{k=1}^{\infty}\frac{1}{k^9\left(e^{2\pi k}-1\right)}-\frac{2}{495}\sum_{k=1}^{\infty}\frac{1}{k^9\left(e^{2\pi k}+1\right)}$$

10.01.03.0060.01

$$\zeta(9) = \frac{\pi^8}{4725}\left(\log(2\,\pi)-80\left(30\,240\,\psi^{(-10)}(1)-15\,120\,\psi^{(-9)}(1)+2520\,\psi^{(-8)}(1)-42\,\psi^{(-6)}(1)+\psi^{(-4)}(1)\right)\right)$$

10.01.03.0025.01

$$\zeta(10) = \frac{\pi^{10}}{93\,555}$$

10.01.03.0026.01

$$\zeta(11) = \frac{1453\,\pi^{11}}{425\,675\,250} - 2\sum_{k=1}^{\infty} \frac{1}{k^{11}\left(e^{2\pi k}-1\right)}$$

10.01.03.0027.01

$$\zeta(12) = \frac{691\,\pi^{12}}{638\,512\,875}$$

10.01.03.0028.01

$$\zeta(13) = \frac{89\,\pi^{13}}{257\,432\,175} - \frac{16\,512}{8255}\sum_{k=1}^{\infty} \frac{1}{k^{13}\left(e^{2\pi k}-1\right)} - \frac{2}{8255}\sum_{k=1}^{\infty} \frac{1}{k^{13}\left(e^{2\pi k}+1\right)}$$

10.01.03.0029.01

$$\zeta(14) = \frac{2\,\pi^{14}}{18\,243\,225}$$

10.01.03.0030.01

$$\zeta(15) = \frac{13\,687\,\pi^{15}}{390\,769\,879\,500} - 2\sum_{k=1}^{\infty} \frac{1}{k^{15}\left(e^{2\pi k}-1\right)}$$

10.01.03.0031.01

$$\zeta(16) = \frac{3617\,\pi^{16}}{325\,641\,566\,250}$$

10.01.03.0032.01

$$\zeta(17) = \frac{397\,549\,\pi^{17}}{112\,024\,529\,867\,250} - \frac{261\,632}{130\,815}\sum_{k=1}^{\infty} \frac{1}{k^{17}\left(e^{2\pi k}-1\right)} - \frac{2}{130\,815}\sum_{k=1}^{\infty} \frac{1}{k^{17}\left(e^{2\pi k}+1\right)}$$

10.01.03.0033.01

$$\zeta(18) = \frac{43\,867\,\pi^{18}}{38\,979\,295\,480\,125}$$

10.01.03.0034.01

$$\zeta(19) = \frac{7\,708\,537\,\pi^{19}}{21\,438\,612\,514\,068\,750} - 2\sum_{k=1}^{\infty} \frac{1}{k^{19}\left(e^{2\pi k}-1\right)}$$

10.01.03.0035.01

$$\zeta(20) = \frac{174\,611\,\pi^{20}}{1\,531\,329\,465\,290\,625}$$

10.01.03.0036.01

$$\zeta(21) = \frac{68\,529\,640\,373\,\pi^{21}}{1\,881\,063\,815\,762\,259\,253\,125} - \frac{4\,196\,352}{2\,098\,175}\sum_{k=1}^{\infty} \frac{1}{k^{21}\left(e^{2\pi k}-1\right)} - \frac{2}{2\,098\,175}\sum_{k=1}^{\infty} \frac{1}{k^{21}\left(e^{2\pi k}+1\right)}$$

10.01.03.0037.01

$$\zeta(22) = \frac{155\,366\,\pi^{22}}{13\,447\,856\,940\,643\,125}$$

10.01.03.0038.01

$$\zeta(24) = \frac{236\,364\,091\,\pi^{24}}{201\,919\,571\,963\,756\,521\,875}$$

10.01.03.0039.01

$$\zeta(26) = \frac{1\,315\,862\,\pi^{26}}{11\,094\,481\,976\,030\,578\,125}$$

10.01.03.0040.01

$$\zeta(28) = \frac{6\,785\,560\,294\,\pi^{28}}{564\,653\,660\,170\,076\,273\,671\,875}$$

10.01.03.0041.01

$$\zeta(30) = \frac{6\,892\,673\,020\,804\,\pi^{30}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}$$

10.01.03.0042.01

$$\zeta(32) = \frac{7\,709\,321\,041\,217\,\pi^{32}}{62\,490\,220\,571\,022\,341\,207\,266\,406\,250}$$

10.01.03.0043.01

$$\zeta(34) = \frac{151\,628\,697\,551\,\pi^{34}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}$$

10.01.03.0044.01

$$\zeta(36) = \frac{26\,315\,271\,553\,053\,477\,373\,\pi^{36}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}$$

10.01.03.0045.01

$$\zeta(38) = \frac{308\,420\,411\,983\,322\,\pi^{38}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}$$

10.01.03.0046.01

$$\zeta(40) = \frac{261\,082\,718\,496\,449\,122\,051\,\pi^{40}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}$$

10.01.03.0047.01

$$\zeta(42) = \frac{3\,040\,195\,287\,836\,141\,605\,382\,\pi^{42}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}$$

10.01.03.0048.01

$$\zeta(44) = \frac{5\,060\,594\,468\,963\,822\,588\,186\,\pi^{44}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}$$

10.01.03.0049.01

$$\zeta(46) = \frac{103\,730\,628\,103\,289\,071\,874\,428\,\pi^{46}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625}$$

10.01.03.0050.01

$$\zeta(48) = \frac{5\,609\,403\,368\,997\,817\,686\,249\,127\,547\,\pi^{48}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875}$$

10.01.03.0051.01

$$\zeta(50) = \frac{39\,604\,576\,419\,286\,371\,856\,998\,202\,\pi^{50}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375}$$

Values at infinities

10.01.03.0052.01

$$\zeta(\infty) = 1$$

General characteristics

Domain and analyticity

$\zeta(s)$ is an analytical function of s, which is defined over the whole complex s-plane.

10.01.04.0001.01

$$s \longrightarrow \zeta(s) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

10.01.04.0002.01

$$\zeta(\bar{s}) = \overline{\zeta(s)}$$

Periodicity

No periodicity

Poles and essential singularities

The function $\zeta(s)$ has only two singular points:

a) $s = 1$ is the simple pole with residue 1;

b) $s = \tilde{\infty}$ is an essential singular point.

10.01.04.0003.01

$$\mathcal{S}ing_s(\zeta(s)) = \{\{1, 1\}, \{\tilde{\infty}, \infty\}\}$$

10.01.04.0004.01

$$\operatorname{res}_s(\zeta(s))(1) = 1$$

Branch points

The function $\zeta(s)$ does not have branch points.

10.01.04.0005.01

$$\mathcal{BP}_s(\zeta(s)) = \{\}$$

Branch cuts

The function $\zeta(s)$ does not have branch cuts.

10.01.04.0006.01

$\mathcal{BC}_s(\zeta(s)) == \{\}$

Series representations

Generalized power series

Expansions at $s == s_0$ /; $s_0 \neq 1$

10.01.06.0017.01

$$\zeta(s) \propto \zeta(s_0) + \zeta'(s_0)\,(s-s_0) + \frac{1}{2}\,\zeta''(s_0)\,(s-s_0)^2 + \ldots \;/;\; (s \to s_0) \bigwedge s_0 \neq 1$$

10.01.06.0018.01

$$\zeta(s) \propto \zeta(s_0) + \zeta'(s_0)\,(s-s_0) + \frac{1}{2}\,\zeta''(s_0)\,(s-s_0)^2 + O\big((s-s_0)^3\big) \;/;\; s_0 \neq 1$$

10.01.06.0019.01

$$\zeta(s) == \sum_{k=0}^{\infty} \frac{\zeta^{(k)}(s_0)\,(s-s_0)^k}{k!} \;/;\; s_0 \neq 1$$

10.01.06.0020.01

$$\zeta(s) \propto \zeta(s_0)\,(1 + O(s-s_0)) \;/;\; s_0 \neq 1$$

Expansions at $s == 0$

10.01.06.0021.01

$$\zeta(s) \propto -\frac{1}{2} - \frac{1}{2}\log(2\,\pi)\,s + \frac{1}{2}\left(-\frac{1}{2}\,(\log(2)+\log(\pi))^2 + \frac{\gamma^2}{2} + \gamma_1 - \frac{\pi^2}{24}\right)s^2 + \ldots \;/;\; (s \to 0)$$

10.01.06.0013.02

$$\zeta(s) \propto -\frac{1}{2} - \frac{1}{2}\log(2\,\pi)\,s + \frac{1}{2}\left(-\frac{1}{2}\,(\log(2)+\log(\pi))^2 + \frac{\gamma^2}{2} + \gamma_1 - \frac{\pi^2}{24}\right)s^2 + O\big(s^3\big)$$

10.01.06.0022.01

$$\zeta(s) \propto -\frac{1}{2}\,(1 + O(s))$$

Expansions at $s == 1$

10.01.06.0023.01

$$\zeta(s) \propto \frac{1}{s-1} + \gamma - \gamma_1\,(s-1) + \frac{1}{2}\,\gamma_2\,(s-1)^2 + \ldots \;/;\; (s \to 1)$$

10.01.06.0024.01

$$\zeta(s) \propto \frac{1}{s-1} + \gamma - \gamma_1\,(s-1) + \frac{1}{2}\,\gamma_2\,(s-1)^2 + O\big((s-1)^3\big)$$

10.01.06.0001.01

$$\zeta(s) == \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k\,\gamma_k}{k!}\,(s-1)^k$$

10.01.06.0002.02

$$\zeta(s) \propto \frac{1}{s-1} + \gamma\,(1 + O(s-1))$$

10.01.06.0025.01

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{k=0}^{\infty} \eta_k\,(s-1)^k \;/;\; \left(\eta_k = \frac{(-1)^k}{k!} \left(\lim_{x\to\infty} \left(\sum_{j=1}^{x} \frac{\Lambda(j)\log^k(j)}{j} - \frac{\log^{k+1}(x)}{k+1}\right)\right) /;\right.$$

$$\left.(\Lambda(p^e) = 1 \;/;\; p \in \mathbb{P} \land e \in \mathbb{Z} \land e \geq 1) \land (\Lambda(a) = 0 \;/;\; \neg\, a \in \mathbb{P} \lor \exists_{e, e\in\mathbb{Z}\land e\geq 1}\, a = p^e)\right)$$

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10.01.06.0026.01

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{k=0}^{\infty} \eta_k\,(s-1)^k \;/;$$

$$\left(\eta_k = (k+1)\sum_{j=0}^{k} \frac{(-1)^{j+1}}{j+1}\, c_{k-j,j+1} \;/;\; \left(c_{0,k} = \gamma^k \bigwedge c_{m,k} = \frac{1}{m\,\gamma}\sum_{i=0}^{m-1} \frac{(k\,m - (k+1)\,i)\,(-1)^{m-i}}{(m-i)!}\,\gamma_{m-i}\, c_{i,k}\right)\right)$$

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Expansions at $s = -2n$

10.01.06.0003.01

$$\zeta(s) \propto \frac{(-1)^n\,(2n)!}{4\,(2\pi)^{2n}} \left(2\,(2n+s)\,\zeta(2n+1) + \left(\left(\log\left(4\pi^2\right) - 2\,\psi(2n+1)\right)\zeta(2n+1) - 2\,\zeta'(2n+1) + O\left((2n+s)^3\right)\right)\right) /;$$
$$(s \to -2n) \land n \in \mathbb{N}^+$$

10.01.06.0004.01

$$\zeta(s) \propto \frac{(-1)^n\,(2n)!\,\zeta(2n+1)}{2\,(2\pi)^{2n}}\,(2n+s)\,(1 + O(s+2n)) \;/;\; (s \to -2n) \land n \in \mathbb{N}^+$$

Exponential Fourier series

10.01.06.0005.01

$$\zeta(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s) \left(\sin\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi k)}{k^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi k)}{k^{1-s}}\right) /;\; \mathrm{Re}(s) < 1$$

Asymptotic series expansions

10.01.06.0006.01

$$\zeta(s) \propto \zeta(s) \;/;\; (|s| \to \infty)$$

This means it cannot be represented through other functions.

Other series representations

10.01.06.0007.01

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \;/;\; \mathrm{Re}(s) > 1$$

10.01.06.0008.01

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} \;/;\; \operatorname{Re}(s) > 0 \land s \neq 1$$

10.01.06.0009.01

$$\zeta(s) = \frac{1}{1-2^{-s}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} \;/;\; \operatorname{Re}(s) > 1$$

10.01.06.0010.01

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^s}$$

10.01.06.0027.01

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)^{s-1}} \binom{n}{k}$$

10.01.06.0011.01

$$\zeta(s)^2 = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \int_0^{\infty} \frac{t^{s-1} e^{-(k+1)t}}{1-e^{-(k+1)t}} \, dt \;/;\; \operatorname{Re}(s) > 1$$

10.01.06.0012.01

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{\left(1-\frac{s}{2}\right)_n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2k+1)\, \zeta(2k+2)$$

Krzysztof Maslanka: Hypergeometric-like Representation of the Zeta-Function of Riemann math-ph/0105007 (2001)
http://arXiv.org/abs/math-ph/0105007

Krzysztof Maslanka: {}[[1,1]] Acta Cosmologica XIII-1, {}[[1,1]] (1997)

For specialized values

10.01.06.0016.01

$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^{n-1}} \sum_{k=1}^{m} \frac{1}{k} - \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n-1}} \;/;\; n \in \mathbb{N} \land n \geq 2$$

10.01.06.0028.01

$$\zeta(5) = \frac{369}{62\,651} \sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k} \left(\frac{4}{(4k+3)^5} + \frac{1}{(4k+4)^5} - \frac{128}{(4k+1)^5} \right) +$$
$$\frac{9}{250\,604} \sum_{k=0}^{\infty} \frac{1}{4096^k} \left(-\frac{6\,610\,944}{(24k+2)^5} + \frac{33\,418\,240}{(24k+3)^5} - \frac{12\,722\,176}{(24k+4)^5} - \frac{31\,744}{(24k+5)^5} + \frac{25\,829\,376}{(24k+6)^5} + \frac{15\,872}{(24k+7)^5} + \right.$$
$$\frac{38\,170\,624}{(24k+8)^5} - \frac{4\,177\,280}{(24k+9)^5} - \frac{413\,184}{(24k+10)^5} - \frac{3968}{(24k+11)^5} - \frac{6\,323\,008}{(24k+12)^5} - \frac{1984}{(24k+13)^5} -$$
$$\frac{103\,296}{(24k+14)^5} - \frac{522\,160}{(24k+15)^5} + \frac{2\,385\,664}{(24k+16)^5} + \frac{496}{(24k+17)^5} + \frac{403\,584}{(24k+18)^5} - \frac{248}{(24k+19)^5} -$$
$$\left. \frac{49\,696}{(24k+20)^5} + \frac{65\,270}{(24k+21)^5} - \frac{6456}{(24k+22)^5} + \frac{62}{(24k+23)^5} + \frac{128\,125}{(24k+24)^5} + \frac{126\,976}{(24k+1)^5} \right)$$

G.Huvent (2006)

Integral representations

On the real axis

Of the direct function

10.01.07.0001.01

$$\zeta(s) = \frac{1}{\left(1-2^{1-s}\right)\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t+1}\, dt \,/;\, \operatorname{Re}(s) > 0$$

10.01.07.0002.01

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t-1}\, dt \,/;\, \operatorname{Re}(s) > 1$$

10.01.07.0003.01

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, e^{-t}}{1-e^{-t}}\, dt \,/;\, \operatorname{Re}(s) > 1$$

10.01.07.0004.01

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-t} \operatorname{csch}(t)\, dt \,/;\, \operatorname{Re}(s) > 1$$

10.01.07.0005.01

$$\zeta(s) = \frac{2^{s-1}}{\left(1-2^{1-s}\right)\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-t} \operatorname{sech}(t)\, dt \,/;\, \operatorname{Re}(s) > 0$$

10.01.07.0006.01

$$\zeta(s) = \frac{1}{2\,(1-2^{-s})\,\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{csch}(t)\, dt \,/;\, \operatorname{Re}(s) > 1$$

10.01.07.0007.01

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^\infty t^s \operatorname{csch}^2(t)\, dt \,/;\, \operatorname{Re}(s) > 1$$

10.01.07.0008.01

$$\zeta(s) = \frac{2^{s-1}}{\left(1-2^{1-s}\right)\Gamma(s+1)} \int_0^\infty t^s \operatorname{sech}^2(t)\, dt \,/;\, \operatorname{Re}(s) > -1$$

10.01.07.0009.01

$$\zeta(s) = 2\int_0^\infty \frac{\sin\left(s \tan^{-1}(t)\right)}{\left(t^2+1\right)^{s/2}\left(e^{2\pi t}-1\right)}\, dt + \frac{1}{2} + \frac{1}{s-1}$$

10.01.07.0010.01

$$\zeta(s) = \frac{2^{s-1}}{1-2^{1-s}} \int_0^\infty \frac{\cos\left(s \tan^{-1}(t)\right)}{\left(t^2+1\right)^{s/2} \cosh\left(\frac{\pi t}{2}\right)}\, dt$$

10.01.07.0021.01

$$\zeta(s) = -s \int_0^\infty t^{s-1} \operatorname{frac}\left(\frac{1}{t}\right) dt \,/;\, 0 < \operatorname{Re}(s) < 1$$

10.01.07.0011.02

$$\zeta(s) = \frac{n^{1-s}}{s-1} - s\int_n^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}}\, dt + \sum_{k=1}^{n} k^{-s} \;/;\; n \in \mathbb{N}^+ \bigwedge \operatorname{Re}(s) > 0$$

10.01.07.0012.02

$$\zeta(s) = \frac{s}{s-1} - \binom{n+s}{n+1}\int_1^\infty B_{n+1}(t - \lfloor t \rfloor)\, t^{-n-s-1}\, dt + \sum_{k=0}^{n} \binom{k+s-1}{k} \frac{B_{k+1}}{k+1} \;/;\; (n \in \mathbb{N} \land \operatorname{Re}(s) > 1) \bigvee (n \in \mathbb{N} \bigwedge \operatorname{Re}(s) > -n \bigwedge s \notin \mathbb{N})$$

10.01.07.0013.01

$$\zeta(s) = \frac{\pi^{s/2}}{2\,\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \left(\vartheta_3\left(0, e^{-\pi t}\right) - 1\right) t^{\frac{s}{2}-1}\, dt \;/;\; \operatorname{Re}(s) > 1$$

10.01.07.0014.01

$$\log(\zeta(s)) = s\int_2^\infty \frac{\pi(t)}{t\,(t^s - 1)}\, dt \;/;\; \operatorname{Re}(s) > 1$$

For specific values

10.01.07.0016.01

$$\zeta(2n) = \frac{(-1)^{n+1}\, 2^{2n-3}\, \pi^{2n}}{\left(2^{2n} - 1\right)(2n-2)!} \int_0^1 E_{2n-2}(x)\, dx \;/;\; n - 1 \in \mathbb{N}^+$$

10.01.07.0017.01

$$\zeta(2n+1) = \frac{(-1)^n\, 2^{2n-1}\, \pi^{2n+1}}{\left(2^{2n+1} - 1\right)(2n)!} \int_0^1 E_{2n}(x) \tan\left(\frac{\pi x}{2}\right) dx \;/;\; n \in \mathbb{N}^+$$

10.01.07.0018.01

$$\zeta(2n+1) = \frac{(-1)^n\, 2^{2n}\, \pi^{2n+1}}{(2n+1)!} \int_0^1 B_{2n+1}(x) \tan\left(\frac{\pi x}{2}\right) dx \;/;\; n \in \mathbb{N}^+$$

10.01.07.0019.01

$$\zeta(2n+1) = \frac{(-1)^n\, 2^{2n-1}\, \pi^{2n+1}}{\left(2^{2n+1} - 1\right)(2n)!} \int_0^1 E_{2n}(x) \cot\left(\frac{\pi x}{2}\right) dx \;/;\; n \in \mathbb{N}^+$$

10.01.07.0020.02

$$\zeta(2n+1) = \frac{(-1)^{n-1}\, 2^{2n}\, \pi^{2n+1}}{(2n+1)!} \int_0^1 B_{2n+1}(x) \cot\left(\frac{\pi x}{2}\right) dx \;/;\; n \in \mathbb{N}^+$$

Multiple integral representations

10.01.07.0022.01

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \int_0^1 \frac{(-\log(t\,\tau))^{s-2}}{1 - t\,\tau}\, dt\, d\tau \;/;\; \operatorname{Re}(s) > 3$$

10.01.07.0015.01

$$\zeta(n) = \int_0^{m_1} \int_0^{m_2} \ldots \int_0^{m_n} \frac{1}{1 - \prod_{k=1}^{n} t_k}\, dt_1\, dt_{2\,\ldots}\, dt_n \;/;\; m_1 = m_2 = \ldots = m_n = 1 \bigwedge n - 1 \in \mathbb{N}^+$$

Product representations

10.01.08.0001.01

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}} \;/;\; \operatorname{Re}(s) > 1 \bigwedge p_k = \operatorname{prime}(k)$$

10.01.08.0002.01

$$\zeta(s) = \frac{1}{2\,(s-1)\,\Gamma\left(\frac{s}{2}-1\right)} \exp\left(\log(2\,\pi) - \frac{\gamma}{2} - 1\right) \prod_{k=1}^{\infty} \left(1 - \frac{s}{\rho_k}\right) e^{\frac{s}{\rho_k}} \;/;\; \zeta(\rho_k) = 0 \bigwedge \operatorname{Im}(\rho_k) \neq 0$$

Limit representations

10.01.09.0001.01

$$\zeta(s) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} k^{-s} - \frac{n^{1-s} - 1}{1 - s}\right) - \frac{1}{1-s} \;/;\; \operatorname{Re}(s) > 0$$

10.01.09.0002.01

$$\zeta(s) = \lim_{n \to \infty} \frac{1}{2^{1-s} - 1} \sum_{k=1}^{n} \binom{2\,n}{n-k} \frac{(-1)^k\, k^{-s}}{\binom{2\,n}{n}}$$

10.01.09.0003.01

$$\zeta(2\,n+1) = \lim_{m \to \infty} \frac{1}{(2\,m+1)^{2\,n+1}} \sum_{k=1}^{m} \cot^{2\,n+1}\left(\frac{k}{2\,m+1}\right) \;/;\; n \in \mathbb{N}^+$$

10.01.09.0004.01

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \lim_{n \to \infty} \frac{n\,!}{2^n} \sum_{k=0}^{2\,n+1} \frac{(-1)^k}{(k+1)^s\, \Gamma(2\,n-k)}\, {}_2\tilde{F}_1(1,\, k-2\,n+1;\, k-n+2;\, -1)$$

10.01.09.0005.01

$$\zeta(s) = \lim_{z \to 1} \frac{1}{2^{1-s} - 1} \sum_{k=1}^{\infty} \frac{(-z)^k}{k^s} \;/;\; z < 1$$

This means the classical series is Abel summable for all $s \in \mathbb{C}$.

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

The zeta function does not satisfy any algebraic differential equation (D. Hilbert, 1900).

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

10.01.16.0001.01

$$\zeta(1-s) = \frac{\pi^{\frac{1}{2}-s}\,\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\,\zeta(s)$$

10.01.16.0002.01

$$\zeta(1-s) = \frac{2}{(2\,\pi)^s}\cos\left(\frac{\pi\,s}{2}\right)\Gamma(s)\,\zeta(s)$$

10.01.16.0003.01

$$\zeta(1-s) = \frac{(2\,\pi)^{1-s}}{2\,\Gamma(1-s)\,\sin\left(\frac{\pi\,s}{2}\right)}\,\zeta(s)$$

Identities

Functional identities

For the function itself

10.01.17.0006.01

$$\zeta(s) = \Gamma(1-s)\,2^s\,\pi^{s-1}\,\sin\left(\frac{\pi\,s}{2}\right)\zeta(1-s)$$

10.01.17.0001.01

$$\zeta(s) = \frac{\pi^{s-\frac{1}{2}}\,\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\,\zeta(1-s)$$

10.01.17.0003.01

$$\zeta(s) = \frac{1}{(s-1)\,\Gamma\left(1-\frac{s}{2}\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(k-\frac{s}{2}+1\right)}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(2\,j+1)\,\zeta(2\,j+2)$$

Krzysztof Maslanka: Hypergeometric-like Representation of the Zeta-Function of Riemann math-ph/0105007 (2001)
http://arXiv.org/abs/math-ph/0105007

10.01.17.0005.01

$$\sum_{k=1}^{n-1}\zeta(2\,k)\,\zeta(2\,n-2\,k) = \left(n+\frac{1}{2}\right)\zeta(2\,n)\;/;\;n\in\mathbb{N}\land n>1$$

Including derivatives of the function

10.01.17.0002.01

$$\zeta^{(n)}(1-s) = (-1)^n\sum_{k=1}^{\infty}\binom{n}{k}\left(\exp\left(s\left(-\log(2\,\pi)+\frac{i\,\pi}{2}\right)\right)\left(\frac{i\,\pi}{2}-\log(2\,\pi)\right)^{n-k}+\exp\left(s\left(-\frac{\pi\,i}{2}-\log(2\,\pi)\right)\right)\left(-\frac{\pi\,i}{2}-\log(2\,\pi)\right)^{n-k}\right)\frac{\partial^k(\Gamma(s)\,\zeta(s))}{\partial s^k}\;/;\;n\in\mathbb{N}$$

10.01.17.0004.01

$$\zeta^{(n)}(1-z) = (-1)^n \sum_{k=0}^{n} \left(\binom{n}{k} \left(e^{z\left(\frac{i\pi}{2}-\log(2\pi)\right)} \left(\frac{i\pi}{2} - \log(2\pi)\right)^{n-k} + e^{z\left(-\frac{1}{2}(i\pi)-\log(2\pi)\right)} \left(-\frac{1}{2}(i\pi) - \log(2\pi)\right)^{n-k} \right) \frac{\partial^k (\Gamma(z)\,\zeta(z))}{\partial z^k} \right) /;\ n \in \mathbb{N}$$

Differentiation

Low-order differentiation

General case

10.01.20.0001.01

$$\frac{\partial \zeta(s)}{\partial s} = -\sum_{k=2}^{\infty} \frac{\log(k)}{k^s} /;\ \mathrm{Re}(s) > 1$$

10.01.20.0004.01

$$\frac{\partial^2 \zeta(s)}{\partial s^2} = \sum_{k=2}^{\infty} \frac{\log^2(k)}{k^s} /;\ \mathrm{Re}(s) > 1$$

Derivatives at zero

10.01.20.0002.01

$$\zeta'(0) = -\frac{1}{2}\log(2\pi)$$

10.01.20.0007.01

$$\zeta''(0) = \gamma_1 - \frac{1}{2}\log^2(2\pi) - \frac{\pi^2}{24} + \frac{\gamma^2}{2}$$

10.01.20.0008.01

$$\zeta^{(3)}(0) = 3\log(2\pi)\,\gamma_1 + 3\,\gamma\,\gamma_1 + \frac{3\,\gamma_2}{2} - \zeta(3) - \frac{1}{2}\log^3(2\pi) - \frac{1}{8}\pi^2\log(2\pi) + \frac{3}{2}\gamma^2\log(2\pi) + \gamma^3$$

10.01.20.0009.01

$$\zeta^{(4)}(0) = -\frac{1}{4}\pi^2\left(\log^2(2\pi) - 2\,\gamma_1\right) + 6\log^2(2\pi)\,\gamma_1 + \gamma^2\left(6\,\gamma_1 + 3\log^2(2\pi) + \frac{\pi^2}{4}\right) + 6\log(2\pi)\,\gamma_2 + 6\,\gamma\,(2\log(2\pi)\,\gamma_1 + \gamma_2) + 2\,\gamma_3 - 4\log(2\pi)\,\zeta(3) - \frac{1}{2}\log^4(2\pi) + 4\,\gamma^3\log(2\pi) - \frac{19\,\pi^4}{480} + \frac{3\,\gamma^4}{2}$$

10.01.20.0010.01

$$\zeta^{(5)}(0) = \frac{1}{96}\Big(80\,\gamma^3\left(12\left(\gamma_1 + \log^2(2\pi)\right) + \pi^2\right) + 240\,\gamma\left(\pi^2\,\gamma_1 + 4\left(3\log^2(2\pi)\,\gamma_1 + 3\log(2\pi)\,\gamma_2 + \gamma_3\right)\right) - 40\,\pi^2\left(-6\log(2\pi)\,\gamma_1 - 3\,\gamma_2 + 2\,\zeta(3) + \log^3(2\pi)\right) + 120\,\gamma^2\left(4\left(6\log(2\pi)\,\gamma_1 + 3\,\gamma_2 + 2\,\zeta(3) + \log^3(2\pi)\right) + \pi^2\log(2\pi)\right) - 48\left(-20\left(\log^3(2\pi) + 2\,\zeta(3)\right)\gamma_1 - 30\log^2(2\pi)\,\gamma_2 - 20\log(2\pi)\,\gamma_3 - 5\,\gamma_4 + 24\,\zeta(5) + 20\log^2(2\pi)\,\zeta(3) + \log^5(2\pi)\right) - 19\,\pi^4\log(2\pi) + 720\,\gamma^4\log(2\pi) + 192\,\gamma^5\Big)$$

10.01.20.0011.01

$$\zeta^{(6)}(0) = -\frac{19}{32}\pi^4\left(\log^2(2\pi)-2\gamma_1\right)+15\log^4(2\pi)\gamma_1+\frac{15}{8}\gamma^4\left(8\gamma_1+12\log^2(2\pi)+\pi^2\right)+30\log^3(2\pi)\gamma_2+ 30\log^2(2\pi)\gamma_3+15\log(2\pi)\gamma_4+3\gamma_5+5\gamma^3\left(12\log(2\pi)\gamma_1+6\gamma_2+8\zeta(3)+4\log^3(2\pi)+\pi^2\log(2\pi)\right)- \frac{5}{8}\pi^2\left(-12\log^2(2\pi)\gamma_1-12\log(2\pi)\gamma_2-4\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+ \frac{1}{32}\gamma^2\left(120\pi^2\left(2\gamma_1+\log^2(2\pi)\right)+240\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+19\pi^4\right)+ \frac{15}{2}\gamma\left(\pi^2\left(2\log(2\pi)\gamma_1+\gamma_2\right)+2\left(4\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+6\log^2(2\pi)\gamma_2+4\log(2\pi)\gamma_3+\gamma_4\right)\right)-72\log(2\pi)\zeta(5)- 20\zeta(3)^2+120\log(2\pi)\gamma_1\zeta(3)+60\gamma_2\zeta(3)-20\log^3(2\pi)\zeta(3)-\frac{1}{2}\log^6(2\pi)+12\gamma^5\log(2\pi)-\frac{275\pi^6}{2688}+\frac{5\gamma^6}{2}$$

10.01.20.0012.01

$$\zeta^{(7)}(0) = \frac{1}{384}\left(1344\gamma^5\left(6\left(\gamma_1+2\log^2(2\pi)\right)+\pi^2\right)- 532\pi^4\left(-6\log(2\pi)\gamma_1-3\gamma_2+2\zeta(3)+\log^3(2\pi)\right)+5040\gamma^4\left(4\left(2\log(2\pi)\gamma_1+\gamma_2+2\zeta(3)+\log^3(2\pi)\right)+\pi^2\log(2\pi)\right)+ 56\gamma^3\left(120\pi^2\left(\gamma_1+\log^2(2\pi)\right)+240\left(6\log^2(2\pi)\gamma_1+6\log(2\pi)\gamma_2+2\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+19\pi^4\right)+ 168\gamma\left(19\pi^4\gamma_1+40\pi^2\left(3\log^2(2\pi)\gamma_1+3\log(2\pi)\gamma_2+\gamma_3\right)+ 48\left(5\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1+10\log^2(2\pi)\gamma_3+5\log(2\pi)\gamma_4+\gamma_5+10\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)\right)\right)- 336\pi^2\left(-20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1-30\log^2(2\pi)\gamma_2-20\log(2\pi)\gamma_3-5\gamma_4+24\zeta(5)+20\log^2(2\pi)\zeta(3)+\log^5(2\pi)\right)+ 84\gamma^2\left(40\pi^2\left(6\log(2\pi)\gamma_1+3\gamma_2+2\zeta(3)+\log^3(2\pi)\right)+48\left(20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+30\log^2(2\pi)\gamma_2+ 20\log(2\pi)\gamma_3+5\gamma_4+24\zeta(5)+20\log^2(2\pi)\zeta(3)+\log^5(2\pi)\right)+19\pi^4\log(2\pi)\right)- 192\left(-42\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)\gamma_1-140\log^3(2\pi)\gamma_3-105\log^2(2\pi)\gamma_4-42\log(2\pi)\gamma_5- 7\gamma_6+720\zeta(7)+504\log^2(2\pi)\zeta(5)-105\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)+280\log(2\pi)\zeta(3)^2- 280\gamma_3\zeta(3)+70\log^4(2\pi)\zeta(3)+\log^7(2\pi)\right)-275\pi^6\log(2\pi)+6720\gamma^6\log(2\pi)+1152\gamma^7\right)$$

10.01.20.0013.01

$$\zeta^{(8)}(0) == -\frac{275}{96}\pi^6\left(\log^2(2\pi)-2\gamma_1\right)+28\log^6(2\pi)\gamma_1+\frac{7}{6}\gamma^6\left(24\gamma_1+60\log^2(2\pi)+5\pi^2\right)+84\log^5(2\pi)\gamma_2+140\log^4(2\pi)\gamma_3+$$
$$140\log^3(2\pi)\gamma_4+84\log^2(2\pi)\gamma_5+28\log(2\pi)\gamma_6+4\gamma_7+28\gamma^5\left(6\log(2\pi)\gamma_1+3\gamma_2+8\zeta(3)+4\log^3(2\pi)+\pi^2\log(2\pi)\right)-$$
$$\frac{133}{48}\pi^4\left(-12\log^2(2\pi)\gamma_1-12\log(2\pi)\gamma_2-4\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+$$
$$\frac{7}{16}\gamma^4\left(40\pi^2\left(2\gamma_1+3\log^2(2\pi)\right)+80\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+24\log(2\pi)\zeta(3)+3\log^4(2\pi)\right)+19\pi^4\right)-$$
$$\frac{7}{6}\pi^2\left(-30\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1-60\log^2(2\pi)\gamma_3-30\log(2\pi)\gamma_4-6\gamma_5+\right.$$
$$\left.144\log(2\pi)\zeta(5)-60\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)+40\zeta(3)^2+40\log^3(2\pi)\zeta(3)+\log^6(2\pi)\right)+$$
$$\frac{7}{6}\gamma^3\left(20\pi^2\left(6\log(2\pi)\gamma_1+3\gamma_2+4\zeta(3)+2\log^3(2\pi)\right)+24\left(20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+30\log^2(2\pi)\gamma_2+\right.\right.$$
$$\left.\left.20\log(2\pi)\gamma_3+5\gamma_4+48\zeta(5)+40\log^2(2\pi)\zeta(3)+2\log^5(2\pi)\right)+19\pi^4\log(2\pi)\right)+$$
$$\frac{1}{96}\gamma^2\left(1596\pi^4\left(2\gamma_1+\log^2(2\pi)\right)+1680\pi^2\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+\right.$$
$$1344\left(30\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1+60\log^2(2\pi)\gamma_3+30\log(2\pi)\gamma_4+6\gamma_5+144\log(2\pi)\zeta(5)+\right.$$
$$\left.\left.60\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)+40\zeta(3)^2+40\log^3(2\pi)\zeta(3)+\log^6(2\pi)\right)+275\pi^6\right)+$$
$$\frac{7}{4}\gamma\left(19\pi^4(2\log(2\pi)\gamma_1+\gamma_2)+20\pi^2\left(4\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+6\log^2(2\pi)\gamma_2+4\log(2\pi)\gamma_3+\gamma_4\right)+\right.$$
$$16\left(6\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)\gamma_1+20\log^3(2\pi)\gamma_3+15\log^2(2\pi)\gamma_4+\right.$$
$$\left.\left.6\log(2\pi)\gamma_5+\gamma_6+15\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)+40\gamma_3\zeta(3)\right)\right)-$$
$$2880\log(2\pi)\zeta(7)+4032\log(2\pi)\gamma_1\zeta(5)+2016\gamma_2\zeta(5)-1344\zeta(3)\zeta(5)-672\log^3(2\pi)\zeta(5)+$$
$$1120\gamma_1\zeta(3)^2-560\log^2(2\pi)\zeta(3)^2+1120\log^3(2\pi)\gamma_1\zeta(3)+$$
$$1680\log^2(2\pi)\gamma_2\zeta(3)+1120\log(2\pi)\gamma_3\zeta(3)+280\gamma_4\zeta(3)-$$
$$56\log^5(2\pi)\zeta(3)-\frac{1}{2}\log^8(2\pi)+24\gamma^7\log(2\pi)-\frac{11813\pi^8}{23040}+\frac{7\gamma^8}{2}$$

10.01.20.0014.01

$$\zeta^{(9)}(0) = 25\,920\,\zeta(7)\,\gamma_1 + 18\,144\log^2(2\pi)\,\zeta(5)\,\gamma_1 + 10\,080\log(2\pi)\,\zeta(3)^2\,\gamma_1 +$$
$$2520\log^4(2\pi)\,\zeta(3)\,\gamma_1 + 36\log^7(2\pi)\,\gamma_1 + 9\,\gamma^7\left(4\left(\gamma_1 + 3\log^2(2\pi)\right) + \pi^2\right) + 126\log^6(2\pi)\,\gamma_2 +$$
$$252\log^5(2\pi)\,\gamma_3 + 315\log^4(2\pi)\,\gamma_4 + 252\log^3(2\pi)\,\gamma_5 + 126\log^2(2\pi)\,\gamma_6 + 36\log(2\pi)\,\gamma_7 + \frac{9\,\gamma_8}{2} -$$
$$\frac{275}{32}\pi^6\left(-6\log(2\pi)\,\gamma_1 - 3\,\gamma_2 + 2\,\zeta(3) + \log^3(2\pi)\right) + \frac{21}{2}\gamma^6\left(4\left(6\log(2\pi)\,\gamma_1 + 3\,\gamma_2 + 10\,\zeta(3) + 5\log^3(2\pi)\right) + 5\,\pi^2\log(2\pi)\right) +$$
$$\frac{21}{20}\gamma^5\left(60\,\pi^2\left(\gamma_1 + 2\log^2(2\pi)\right) + 240\left(3\log^2(2\pi)\,\gamma_1 + 3\log(2\pi)\,\gamma_2 + \gamma_3 + 8\log(2\pi)\,\zeta(3) + \log^4(2\pi)\right) + 19\,\pi^4\right) -$$
$$\frac{399}{80}\pi^4\left(-20\left(\log^3(2\pi) + 2\,\zeta(3)\right)\gamma_1 - 30\log^2(2\pi)\,\gamma_2 - 20\log(2\pi)\,\gamma_3 - 5\,\gamma_4 + 24\,\zeta(5) + 20\log^2(2\pi)\,\zeta(3) + \log^5(2\pi)\right) +$$
$$\frac{63}{16}\gamma^4\left(40\,\pi^2\left(2\log(2\pi)\,\gamma_1 + \gamma_2 + 2\,\zeta(3) + \log^3(2\pi)\right) + 16\left(20\left(\log^3(2\pi) + 2\,\zeta(3)\right)\gamma_1 + 30\log^2(2\pi)\,\gamma_2 +\right.\right.$$
$$\left.\left.20\log(2\pi)\,\gamma_3 + 5\,\gamma_4 + 72\,\zeta(5) + 60\log^2(2\pi)\,\zeta(3) + 3\log^5(2\pi)\right) + 19\,\pi^4\log(2\pi)\right) +$$
$$\frac{1}{16}\gamma^3\left(1596\,\pi^4\left(\gamma_1 + \log^2(2\pi)\right) + 1680\,\pi^2\left(6\log^2(2\pi)\,\gamma_1 + 6\log(2\pi)\,\gamma_2 + 2\,\gamma_3 + 8\log(2\pi)\,\zeta(3) + \log^4(2\pi)\right) +\right.$$
$$1344\left(15\left(\log^4(2\pi) + 8\log(2\pi)\,\zeta(3)\right)\gamma_1 + 30\log^2(2\pi)\,\gamma_3 + 15\log(2\pi)\,\gamma_4 + 3\,\gamma_5 + 144\log(2\pi)\,\zeta(5) +\right.$$
$$\left.\left.30\,\gamma_2\left(\log^3(2\pi) + 2\,\zeta(3)\right) + 40\,\zeta(3)^2 + 40\log^3(2\pi)\,\zeta(3) + \log^6(2\pi)\right) + 275\,\pi^6\right) +$$
$$\frac{3}{16}\gamma\left(275\,\pi^6\,\gamma_1 + 532\,\pi^4\left(3\log^2(2\pi)\,\gamma_1 + 3\log(2\pi)\,\gamma_2 + \gamma_3\right) + 336\,\pi^2\right.$$
$$\left(5\left(\log^4(2\pi) + 8\log(2\pi)\,\zeta(3)\right)\gamma_1 + 10\log^2(2\pi)\,\gamma_3 + 5\log(2\pi)\,\gamma_4 + \gamma_5 + 10\,\gamma_2\left(\log^3(2\pi) + 2\,\zeta(3)\right)\right) +$$
$$192\left(7\left(\log^6(2\pi) + 40\log^3(2\pi)\,\zeta(3) + 40\,\zeta(3)^2 + 144\log(2\pi)\,\zeta(5)\right)\gamma_1 + 35\log^4(2\pi)\,\gamma_3 + 35\log^3(2\pi)\,\gamma_4 + 21\log^2(2\pi)\right.$$
$$\left.\left.\gamma_5 + 7\log(2\pi)\,\gamma_6 + \gamma_7 + 21\,\gamma_2\left(\log^5(2\pi) + 20\log^2(2\pi)\,\zeta(3) + 24\,\zeta(5)\right) + 280\log(2\pi)\,\gamma_3\,\zeta(3) + 70\,\gamma_4\,\zeta(3)\right)\right) -$$
$$\frac{3}{2}\pi^2\left(-42\left(\log^5(2\pi) + 20\log^2(2\pi)\,\zeta(3) + 24\,\zeta(5)\right)\gamma_1 - 140\log^3(2\pi)\,\gamma_3 - 105\log^2(2\pi)\,\gamma_4 - 42\log(2\pi)\,\gamma_5 -\right.$$
$$7\,\gamma_6 + 720\,\zeta(7) + 504\log^2(2\pi)\,\zeta(5) - 105\,\gamma_2\left(\log^4(2\pi) + 8\log(2\pi)\,\zeta(3)\right) + 280\log(2\pi)\,\zeta(3)^2 -$$
$$\left.280\,\gamma_3\,\zeta(3) + 70\log^4(2\pi)\,\zeta(3) + \log^7(2\pi)\right) + \frac{3}{32}\gamma^2\left(532\,\pi^4\left(6\log(2\pi)\,\gamma_1 + 3\,\gamma_2 + 2\,\zeta(3) + \log^3(2\pi)\right) +\right.$$
$$336\,\pi^2\left(20\left(\log^3(2\pi) + 2\,\zeta(3)\right)\gamma_1 + 30\log^2(2\pi)\,\gamma_2 + 20\log(2\pi)\,\gamma_3 + 5\,\gamma_4 + 24\,\zeta(5) + 20\log^2(2\pi)\,\zeta(3) + \log^5(2\pi)\right) +$$
$$192\left(42\left(\log^5(2\pi) + 20\log^2(2\pi)\,\zeta(3) + 24\,\zeta(5)\right)\gamma_1 + 140\log^3(2\pi)\,\gamma_3 + 105\log^2(2\pi)\,\gamma_4 +\right.$$
$$42\log(2\pi)\,\gamma_5 + 7\,\gamma_6 + 720\,\zeta(7) + 504\log^2(2\pi)\,\zeta(5) + 105\,\gamma_2\left(\log^4(2\pi) + 8\log(2\pi)\,\zeta(3)\right) +$$
$$\left.\left.280\log(2\pi)\,\zeta(3)^2 + 280\,\gamma_3\,\zeta(3) + 70\log^4(2\pi)\,\zeta(3) + \log^7(2\pi)\right) + 275\,\pi^6\log(2\pi)\right) -$$
$$20\,160\,\zeta(9) - 12\,960\log^2(2\pi)\,\zeta(7) + 18\,144\log(2\pi)\,\gamma_2\,\zeta(5) + 6048\,\gamma_3\,\zeta(5) - 12\,096\log(2\pi)\,\zeta(3)\,\zeta(5) -$$
$$1512\log^4(2\pi)\,\zeta(5) - 1120\,\zeta(3)^3 + 5040\,\gamma_2\,\zeta(3)^2 -$$
$$1680\log^3(2\pi)\,\zeta(3)^2 + 5040\log^3(2\pi)\,\gamma_2\,\zeta(3) +$$
$$5040\log^2(2\pi)\,\gamma_3\,\zeta(3) + 2520\log(2\pi)\,\gamma_4\,\zeta(3) +$$
$$504\,\gamma_5\,\zeta(3) - 84\log^6(2\pi)\,\zeta(3) - \frac{1}{2}\log^9(2\pi) -$$
$$\frac{11\,813\,\pi^8\log(2\pi)}{2560} + \frac{63}{2}\gamma^8\log(2\pi) + 4\,\gamma^9$$

10.01.20.0015.01

$$\zeta^{(10)}(0) = -\frac{11\,813}{512}\pi^8\left(\log^2(2\pi) - 2\gamma_1\right) + 45\log^8(2\pi)\gamma_1 + \frac{15}{8}\gamma^8\left(24\gamma_1 + 84\log^2(2\pi) + 7\pi^2\right) +$$
$$180\log^7(2\pi)\gamma_2 + 420\log^6(2\pi)\gamma_3 + 630\log^5(2\pi)\gamma_4 + 630\log^4(2\pi)\gamma_5 + 420\log^3(2\pi)\gamma_6 +$$
$$180\log^2(2\pi)\gamma_7 + 45\log(2\pi)\gamma_8 + 5\gamma_9 + 90\gamma^7\left(4\log(2\pi)\gamma_1 + 2\gamma_2 + 8\zeta(3) + 4\log^3(2\pi) + \pi^2\log(2\pi)\right) -$$
$$\frac{1375}{64}\pi^6\left(-12\log^2(2\pi)\gamma_1 - 12\log(2\pi)\gamma_2 - 4\gamma_3 + 8\log(2\pi)\zeta(3) + \log^4(2\pi)\right) +$$
$$\frac{35}{16}\gamma^6\left(24\pi^2\left(2\gamma_1 + 5\log^2(2\pi)\right) + 48\left(12\log^2(2\pi)\gamma_1 + 12\log(2\pi)\gamma_2 + 4\gamma_3 + 40\log(2\pi)\zeta(3) + 5\log^4(2\pi)\right) + 19\pi^4\right) -$$
$$\frac{133}{16}\pi^4\left(-30\left(\log^4(2\pi) + 8\log(2\pi)\zeta(3)\right)\gamma_1 - 60\log^2(2\pi)\gamma_3 - 30\log(2\pi)\gamma_4 - 6\gamma_5 +\right.$$
$$\left.144\log(2\pi)\zeta(5) - 60\gamma_2\left(\log^3(2\pi) + 2\zeta(3)\right) + 40\zeta(3)^2 + 40\log^3(2\pi)\zeta(3) + \log^6(2\pi)\right) +$$
$$\frac{21}{2}\gamma^5\left(10\pi^2\left(6\log(2\pi)\gamma_1 + 3\gamma_2 + 8\zeta(3) + 4\log^3(2\pi)\right) + 12\left(20\left(\log^3(2\pi) + 2\zeta(3)\right)\gamma_1 + 30\log^2(2\pi)\gamma_2 +\right.\right.$$
$$\left.\left.20\log(2\pi)\gamma_3 + 5\gamma_4 + 96\zeta(5) + 80\log^2(2\pi)\zeta(3) + 4\log^5(2\pi)\right) + 19\pi^4\log(2\pi)\right) +$$
$$\frac{15}{64}\gamma^4\left(532\pi^4\left(2\gamma_1 + 3\log^2(2\pi)\right) + 560\pi^2\left(12\log^2(2\pi)\gamma_1 + 12\log(2\pi)\gamma_2 + 4\gamma_3 + 24\log(2\pi)\zeta(3) + 3\log^4(2\pi)\right) +\right.$$
$$1344\left(10\left(\log^4(2\pi) + 8\log(2\pi)\zeta(3)\right)\gamma_1 + 20\log^2(2\pi)\gamma_3 + 10\log(2\pi)\gamma_4 + 2\gamma_5 + 144\log(2\pi)\zeta(5) +\right.$$
$$\left.\left.20\gamma_2\left(\log^3(2\pi) + 2\zeta(3)\right) + 40\zeta(3)^2 + 40\log^3(2\pi)\zeta(3) + \log^6(2\pi)\right) + 275\pi^6\right) -$$
$$\frac{15}{8}\pi^2\left(-56\left(\log^6(2\pi) + 40\log^3(2\pi)\zeta(3) + 40\zeta(3)^2 + 144\log(2\pi)\zeta(5)\right)\gamma_1 - 280\log^4(2\pi)\gamma_3 - 280\log^3(2\pi)\gamma_4 -\right.$$
$$168\log^2(2\pi)\gamma_5 - 56\log(2\pi)\gamma_6 - 8\gamma_7 + 5760\log(2\pi)\zeta(7) - 168\gamma_2\left(\log^5(2\pi) + 20\log^2(2\pi)\zeta(3) + 24\zeta(5)\right) +$$
$$2688\zeta(3)\zeta(5) + 1344\log^3(2\pi)\zeta(5) + 1120\log^2(2\pi)\zeta(3)^2 - 2240\log(2\pi)\gamma_3\zeta(3) - 560\gamma_4\zeta(3) +$$
$$\left.112\log^5(2\pi)\zeta(3) + \log^8(2\pi)\right) + \frac{5}{8}\gamma^3\left(266\pi^4\left(6\log(2\pi)\gamma_1 + 3\gamma_2 + 4\zeta(3) + 2\log^3(2\pi)\right) +\right.$$
$$168\pi^2\left(20\left(\log^3(2\pi) + 2\zeta(3)\right)\gamma_1 + 30\log^2(2\pi)\gamma_2 + 20\log(2\pi)\gamma_3 + 5\gamma_4 + 48\zeta(5) + 40\log^2(2\pi)\zeta(3) + 2\log^5(2\pi)\right) +$$
$$96\left(42\left(\log^5(2\pi) + 20\log^2(2\pi)\zeta(3) + 24\zeta(5)\right)\gamma_1 + 140\log^3(2\pi)\gamma_3 + 105\log^2(2\pi)\gamma_4 +\right.$$
$$42\log(2\pi)\gamma_5 + 7\gamma_6 + 1440\zeta(7) + 1008\log^2(2\pi)\zeta(5) + 105\gamma_2\left(\log^4(2\pi) + 8\log(2\pi)\zeta(3)\right) +$$
$$\left.\left.560\log(2\pi)\zeta(3)^2 + 280\gamma_3\zeta(3) + 140\log^4(2\pi)\zeta(3) + 2\log^7(2\pi)\right) + 275\pi^6\log(2\pi)\right) +$$
$$\frac{1}{512}\gamma^2\left(66\,000\pi^6\left(2\gamma_1 + \log^2(2\pi)\right) + 63\,840\pi^4\left(12\log^2(2\pi)\gamma_1 + 12\log(2\pi)\gamma_2 + 4\gamma_3 + 8\log(2\pi)\zeta(3) + \log^4(2\pi)\right) +\right.$$
$$26\,880\pi^2\left(30\left(\log^4(2\pi) + 8\log(2\pi)\zeta(3)\right)\gamma_1 + 60\log^2(2\pi)\gamma_3 + 30\log(2\pi)\gamma_4 + 6\gamma_5 +\right.$$
$$\left.144\log(2\pi)\zeta(5) + 60\gamma_2\left(\log^3(2\pi) + 2\zeta(3)\right) + 40\zeta(3)^2 + 40\log^3(2\pi)\zeta(3) + \log^6(2\pi)\right) +$$
$$11\,520\left(56\left(\log^6(2\pi) + 40\log^3(2\pi)\zeta(3) + 40\zeta(3)^2 + 144\log(2\pi)\zeta(5)\right)\gamma_1 + 280\log^4(2\pi)\gamma_3 +\right.$$
$$280\log^3(2\pi)\gamma_4 + 168\log^2(2\pi)\gamma_5 + 56\log(2\pi)\gamma_6 + 8\gamma_7 + 5760\log(2\pi)\zeta(7) +$$
$$168\gamma_2\left(\log^5(2\pi) + 20\log^2(2\pi)\zeta(3) + 24\zeta(5)\right) + 2688\zeta(3)\zeta(5) + 1344\log^3(2\pi)\zeta(5) +$$
$$\left.\left.1120\log^2(2\pi)\zeta(3)^2 + 2240\log(2\pi)\gamma_3\zeta(3) + 560\gamma_4\zeta(3) + 112\log^5(2\pi)\zeta(3) + \log^8(2\pi)\right) + 11\,813\pi^8\right) +$$
$$\frac{15}{16}\gamma\left(275\pi^6\left(2\log(2\pi)\gamma_1 + \gamma_2\right) + 266\pi^4\left(4\left(\log^3(2\pi) + 2\zeta(3)\right)\gamma_1 + 6\log^2(2\pi)\gamma_2 + 4\log(2\pi)\gamma_3 + \gamma_4\right) +\right.$$
$$112\pi^2\left(6\left(\log^5(2\pi) + 20\log^2(2\pi)\zeta(3) + 24\zeta(5)\right)\gamma_1 + 20\log^3(2\pi)\gamma_3 +\right.$$
$$\left.15\log^2(2\pi)\gamma_4 + 6\log(2\pi)\gamma_5 + \gamma_6 + 15\gamma_2\left(\log^4(2\pi) + 8\log(2\pi)\zeta(3)\right) + 40\gamma_3\zeta(3)\right) +$$

$$48 \left(8 \left(\log^7(2\pi) + 70 \log^4(2\pi) \zeta(3) + 280 \log(2\pi) \zeta(3)^2 + 504 \log^2(2\pi) \zeta(5) + 720 \zeta(7)\right) \gamma_1 + 56 \log^5(2\pi) \gamma_3 + 70 \log^4(2\pi) \gamma_4 + 56 \log^3(2\pi) \gamma_5 + 28 \log^2(2\pi) \gamma_6 + 8 \log(2\pi) \gamma_7 + \gamma_8 + 28 \gamma_2 \left(\log^6(2\pi) + 40 \log^3(2\pi) \zeta(3) + 40 \zeta(3)^2 + 144 \log(2\pi) \zeta(5)\right) + 1344 \gamma_3 \zeta(5) + 1120 \log^2(2\pi) \gamma_3 \zeta(3) + 560 \log(2\pi) \gamma_4 \zeta(3) + 112 \gamma_5 \zeta(3)\right)\Big) -$$
$$201\,600 \log(2\pi) \zeta(9) + 259\,200 \log(2\pi) \gamma_1 \zeta(7) + 129\,600 \gamma_2 \zeta(7) - 86\,400 \zeta(3) \zeta(7) -$$
$$43\,200 \log^3(2\pi) \zeta(7) -$$
$$36\,288 \zeta(5)^2 +$$
$$60\,480 \log^3(2\pi) \gamma_1 \zeta(5) +$$
$$90\,720 \log^2(2\pi) \gamma_2 \zeta(5) +$$
$$60\,480 \log(2\pi) \gamma_3 \zeta(5) +$$
$$15\,120 \gamma_4 \zeta(5) +$$
$$120\,960 \gamma_1 \zeta(3) \zeta(5) -$$
$$60\,480 \log^2(2\pi) \zeta(3) \zeta(5) -$$
$$3024 \log^5(2\pi) \zeta(5) -$$
$$11\,200 \log(2\pi) \zeta(3)^3 +$$
$$50\,400 \log^2(2\pi) \gamma_1 \zeta(3)^2 +$$
$$50\,400 \log(2\pi) \gamma_2 \zeta(3)^2 + 16\,800 \gamma_3 \zeta(3)^2 -$$
$$4200 \log^4(2\pi) \zeta(3)^2 + 5040 \log^5(2\pi) \gamma_1 \zeta(3) +$$
$$12\,600 \log^4(2\pi) \gamma_2 \zeta(3) + 16\,800 \log^3(2\pi) \gamma_3 \zeta(3) +$$
$$12\,600 \log^2(2\pi) \gamma_4 \zeta(3) + 5040 \log(2\pi) \gamma_5 \zeta(3) +$$
$$840 \gamma_6 \zeta(3) - 120 \log^7(2\pi) \zeta(3) - \frac{1}{2} \log^{10}(2\pi) +$$
$$40 \gamma^9 \log(2\pi) - \frac{95\,265 \pi^{10}}{22\,528} + \frac{9 \gamma^{10}}{2}$$

10.01.20.0016.01

$$\zeta^{(11)}(0) = 2\,217\,600 \zeta(9) \gamma_1 + 1\,425\,600 \log^2(2\pi) \zeta(7) \gamma_1 + 1\,330\,560 \log(2\pi) \zeta(3) \zeta(5) \gamma_1 + 166\,320 \log^4(2\pi) \zeta(5) \gamma_1 +$$
$$123\,200 \zeta(3)^3 \gamma_1 + 184\,800 \log^3(2\pi) \zeta(3)^2 \gamma_1 + 9240 \log^6(2\pi) \zeta(3) \gamma_1 + 55 \log^9(2\pi) \gamma_1 + \frac{55}{3} \gamma^9 \left(3 \left(\gamma_1 + 4 \log^2(2\pi)\right) + \pi^2\right) +$$
$$\frac{495}{2} \log^8(2\pi) \gamma_2 + 660 \log^7(2\pi) \gamma_3 + 1155 \log^6(2\pi) \gamma_4 + 1386 \log^5(2\pi) \gamma_5 + 1155 \log^4(2\pi) \gamma_6 + 660 \log^3(2\pi) \gamma_7 +$$
$$\frac{495}{2} \log^2(2\pi) \gamma_8 + 55 \log(2\pi) \gamma_9 + \frac{11 \gamma_{10}}{2} - \frac{129\,943 \pi^8 \left(-6 \log(2\pi) \gamma_1 - 3 \gamma_2 + 2 \zeta(3) + \log^3(2\pi)\right)}{1536} +$$
$$\frac{165}{8} \gamma^8 \left(4 \left(6 \log(2\pi) \gamma_1 + 3 \gamma_2 + 14 \zeta(3) + 7 \log^3(2\pi)\right) + 7 \pi^2 \log(2\pi)\right) +$$
$$\frac{33}{8} \gamma^7 \left(40 \pi^2 \left(\gamma_1 + 3 \log^2(2\pi)\right) + 80 \left(6 \log^2(2\pi) \gamma_1 + 6 \log(2\pi) \gamma_2 + 2 \gamma_3 + 24 \log(2\pi) \zeta(3) + 3 \log^4(2\pi)\right) + 19 \pi^4\right) -$$
$$\frac{3025}{64} \pi^6 \left(-20 \left(\log^3(2\pi) + 2 \zeta(3)\right) \gamma_1 - 30 \log^2(2\pi) \gamma_2 - 20 \log(2\pi) \gamma_3 - 5 \gamma_4 + 24 \zeta(5) + 20 \log^2(2\pi) \zeta(3) + \log^5(2\pi)\right) +$$
$$\frac{385}{16} \gamma^6 \left(8 \pi^2 \left(6 \log(2\pi) \gamma_1 + 3 \gamma_2 + 10 \zeta(3) + 5 \log^3(2\pi)\right) + 48 \left(4 \left(\log^3(2\pi) + 2 \zeta(3)\right) \gamma_1 + 6 \log^2(2\pi) \gamma_2 + 4 \log(2\pi) \gamma_3 + \gamma_4 + 24 \zeta(5) + 20 \log^2(2\pi) \zeta(3) + \log^5(2\pi)\right) + 19 \pi^4 \log(2\pi)\right) +$$
$$\frac{11}{16} \gamma^5 \left(798 \pi^4 \left(\gamma_1 + 2 \log^2(2\pi)\right) + 1680 \pi^2 \left(3 \log^2(2\pi) \gamma_1 + 3 \log(2\pi) \gamma_2 + \gamma_3 + 8 \log(2\pi) \zeta(3) + \log^4(2\pi)\right) + 672 \left(15 \left(\log^4(2\pi) + 8 \log(2\pi) \zeta(3)\right) \gamma_1 + 30 \log^2(2\pi) \gamma_3 + 15 \log(2\pi) \gamma_4 + 3 \gamma_5 + 288 \log(2\pi) \zeta(5) +\right.\right.$$

$$30\,\gamma_2\left(\log^3(2\pi)+2\,\zeta(3)\right)+80\,\zeta(3)^2+80\log^3(2\pi)\,\zeta(3)+2\log^6(2\pi)\Big)+275\,\pi^6\Big)-$$

$$\frac{209}{16}\,\pi^4\left(-42\left(\log^5(2\pi)+20\log^2(2\pi)\,\zeta(3)+24\,\zeta(5)\right)\gamma_1-140\log^3(2\pi)\,\gamma_3-105\log^2(2\pi)\,\gamma_4-42\log(2\pi)\,\gamma_5-\right.$$

$$7\,\gamma_6+720\,\zeta(7)+504\log^2(2\pi)\,\zeta(5)-105\,\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\,\zeta(3)\right)+280\log(2\pi)\,\zeta(3)^2-$$

$$\left.280\,\gamma_3\,\zeta(3)+70\log^4(2\pi)\,\zeta(3)+\log^7(2\pi)\right)+\frac{165}{64}\,\gamma^4\left(532\,\pi^4\left(2\log(2\pi)\,\gamma_1+\gamma_2+2\,\zeta(3)+\log^3(2\pi)\right)+\right.$$

$$112\,\pi^2\left(20\left(\log^3(2\pi)+2\,\zeta(3)\right)\gamma_1+30\log^2(2\pi)\,\gamma_2+20\log(2\pi)\,\gamma_3+5\,\gamma_4+72\,\zeta(5)+60\log^2(2\pi)\,\zeta(3)+3\log^5(2\pi)\right)+$$

$$64\left(42\left(\log^5(2\pi)+20\log^2(2\pi)\,\zeta(3)+24\,\zeta(5)\right)\gamma_1+140\log^3(2\pi)\,\gamma_3+105\log^2(2\pi)\,\gamma_4+\right.$$

$$42\log(2\pi)\,\gamma_5+7\,\gamma_6+2160\,\zeta(7)+1512\log^2(2\pi)\,\zeta(5)+105\,\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\,\zeta(3)\right)+$$

$$\left.\left.840\log(2\pi)\,\zeta(3)^2+280\,\gamma_3\,\zeta(3)+210\log^4(2\pi)\,\zeta(3)+3\log^7(2\pi)\right)+275\,\pi^6\log(2\pi)\right)+$$

$$\frac{11}{768}\,\gamma^3\left(66\,000\,\pi^6\left(\gamma_1+\log^2(2\pi)\right)+63\,840\,\pi^4\left(6\log^2(2\pi)\,\gamma_1+6\log(2\pi)\,\gamma_2+2\,\gamma_3+8\log(2\pi)\,\zeta(3)+\log^4(2\pi)\right)+\right.$$

$$26\,880\,\pi^2\left(15\left(\log^4(2\pi)+8\log(2\pi)\,\zeta(3)\right)\gamma_1+30\log^2(2\pi)\,\gamma_3+15\log(2\pi)\,\gamma_4+3\,\gamma_5+\right.$$

$$\left.144\log(2\pi)\,\zeta(5)+30\,\gamma_2\left(\log^3(2\pi)+2\,\zeta(3)\right)+40\,\zeta(3)^2+40\log^3(2\pi)\,\zeta(3)+\log^6(2\pi)\right)+$$

$$11\,520\left(28\left(\log^6(2\pi)+40\log^3(2\pi)\,\zeta(3)+40\,\zeta(3)^2+144\log(2\pi)\,\zeta(5)\right)\gamma_1+140\log^4(2\pi)\,\gamma_3+\right.$$

$$140\log^3(2\pi)\,\gamma_4+84\log^2(2\pi)\,\gamma_5+28\log(2\pi)\,\gamma_6+4\,\gamma_7+5760\log(2\pi)\,\zeta(7)+$$

$$84\,\gamma_2\left(\log^5(2\pi)+20\log^2(2\pi)\,\zeta(3)+24\,\zeta(5)\right)+2688\,\zeta(3)\,\zeta(5)+1344\log^3(2\pi)\,\zeta(5)+$$

$$\left.\left.1120\log^2(2\pi)\,\zeta(3)^2+1120\log(2\pi)\,\gamma_3\,\zeta(3)+280\,\gamma_4\,\zeta(3)+112\log^5(2\pi)\,\zeta(3)+\log^8(2\pi)\right)+11\,813\,\pi^8\right)+$$

$$\frac{11}{256}\,\gamma\left(11\,813\,\pi^8\,\gamma_1+22\,000\,\pi^6\left(3\log^2(2\pi)\,\gamma_1+3\log(2\pi)\,\gamma_2+\gamma_3\right)+12\,768\,\pi^4\right.$$

$$\left(5\left(\log^4(2\pi)+8\log(2\pi)\,\zeta(3)\right)\gamma_1+10\log^2(2\pi)\,\gamma_3+5\log(2\pi)\,\gamma_4+\gamma_5+10\,\gamma_2\left(\log^3(2\pi)+2\,\zeta(3)\right)\right)+$$

$$3840\,\pi^2\left(7\left(\log^6(2\pi)+40\log^3(2\pi)\,\zeta(3)+40\,\zeta(3)^2+144\log(2\pi)\,\zeta(5)\right)\gamma_1+\right.$$

$$35\log^4(2\pi)\,\gamma_3+35\log^3(2\pi)\,\gamma_4+21\log^2(2\pi)\,\gamma_5+7\log(2\pi)\,\gamma_6+\gamma_7+$$

$$\left.21\,\gamma_2\left(\log^5(2\pi)+20\log^2(2\pi)\,\zeta(3)+24\,\zeta(5)\right)+280\log(2\pi)\,\gamma_3\,\zeta(3)+70\,\gamma_4\,\zeta(3)\right)+1280$$

$$\left(9\left(\log^8(2\pi)+112\log^5(2\pi)\,\zeta(3)+1120\log^2(2\pi)\,\zeta(3)^2+1344\log^3(2\pi)\,\zeta(5)+2688\,\zeta(3)\,\zeta(5)+5760\log(2\pi)\,\zeta(7)\right)\right.$$

$$\gamma_1+84\log^6(2\pi)\,\gamma_3+126\log^5(2\pi)\,\gamma_4+126\log^4(2\pi)\,\gamma_5+84\log^3(2\pi)\,\gamma_6+36\log^2(2\pi)\,\gamma_7+9\log(2\pi)\,\gamma_8+$$

$$\gamma_9+36\,\gamma_2\left(\log^7(2\pi)+70\log^4(2\pi)\,\zeta(3)+280\log(2\pi)\,\zeta(3)^2+504\log^2(2\pi)\,\zeta(5)+720\,\zeta(7)\right)+$$

$$12\,096\log(2\pi)\,\gamma_3\,\zeta(5)+3024\,\gamma_4\,\zeta(5)+3360\,\gamma_3\,\zeta(3)^2+3360\log^3(2\pi)\,\gamma_3\,\zeta(3)+$$

$$\left.\left.2520\log^2(2\pi)\,\gamma_4\,\zeta(3)+1008\log(2\pi)\,\gamma_5\,\zeta(3)+168\,\gamma_6\,\zeta(3)\right)\right)-$$

$$\frac{55}{24}\,\pi^2\left(-72\left(\log^7(2\pi)+70\log^4(2\pi)\,\zeta(3)+280\log(2\pi)\,\zeta(3)^2+504\log^2(2\pi)\,\zeta(5)+720\,\zeta(7)\right)\gamma_1-\right.$$

$$504\log^5(2\pi)\,\gamma_3-630\log^4(2\pi)\,\gamma_4-504\log^3(2\pi)\,\gamma_5-252\log^2(2\pi)\,\gamma_6-72\log(2\pi)\,\gamma_7-9\,\gamma_8+$$

$$40\,320\,\zeta(9)+25\,920\log^2(2\pi)\,\zeta(7)-252\,\gamma_2\left(\log^6(2\pi)+40\log^3(2\pi)\,\zeta(3)+40\,\zeta(3)^2+144\log(2\pi)\,\zeta(5)\right)-$$

$$12\,096\,\gamma_3\,\zeta(5)+24\,192\log(2\pi)\,\zeta(3)\,\zeta(5)+3024\log^4(2\pi)\,\zeta(5)+2240\,\zeta(3)^3+3360\log^3(2\pi)\,\zeta(3)^2-$$

$$\left.10\,080\log^2(2\pi)\,\gamma_3\,\zeta(3)-5040\log(2\pi)\,\gamma_4\,\zeta(3)-1008\,\gamma_5\,\zeta(3)+168\log^6(2\pi)\,\zeta(3)+\log^9(2\pi)\right)+$$

$$\frac{11}{512}\,\gamma^2\left(22\,000\,\pi^6\left(6\log(2\pi)\,\gamma_1+3\,\gamma_2+2\,\zeta(3)+\log^3(2\pi)\right)+\right.$$

$$12\,768\,\pi^4\left(20\left(\log^3(2\pi)+2\,\zeta(3)\right)\gamma_1+30\log^2(2\pi)\,\gamma_2+20\log(2\pi)\,\gamma_3+5\,\gamma_4+24\,\zeta(5)+20\log^2(2\pi)\,\zeta(3)+\log^5(2\pi)\right)+$$

$$3840\,\pi^2\left(42\left(\log^5(2\pi)+20\log^2(2\pi)\,\zeta(3)+24\,\zeta(5)\right)\gamma_1+140\log^3(2\pi)\,\gamma_3+105\log^2(2\pi)\,\gamma_4+\right.$$

$$42\log(2\pi)\,\gamma_5+7\,\gamma_6+720\,\zeta(7)+504\log^2(2\pi)\,\zeta(5)+105\,\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\,\zeta(3)\right)+$$

$$\left.280\log(2\pi)\,\zeta(3)^2+280\,\gamma_3\,\zeta(3)+70\log^4(2\pi)\,\zeta(3)+\log^7(2\pi)\right)+$$

$$1280\left(72\left(\log^7(2\pi)+70\log^4(2\pi)\zeta(3)+280\log(2\pi)\zeta(3)^2+504\log^2(2\pi)\zeta(5)+720\zeta(7)\right)\gamma_1+504\log^5(2\pi)\gamma_3+630\log^4(2\pi)\gamma_4+504\log^3(2\pi)\gamma_5+252\log^2(2\pi)\gamma_6+72\log(2\pi)\gamma_7+9\gamma_8+40320\zeta(9)+25920\log^2(2\pi)\zeta(7)+252\gamma_2\left(\log^6(2\pi)+40\log^3(2\pi)\zeta(3)+40\zeta(3)^2+144\log(2\pi)\zeta(5)\right)+12096\gamma_3\zeta(5)+24192\log(2\pi)\zeta(3)\zeta(5)+3024\log^4(2\pi)\zeta(5)+2240\zeta(3)^3+3360\log^3(2\pi)\zeta(3)^2+10080\log^2(2\pi)\gamma_3\zeta(3)+5040\log(2\pi)\gamma_4\zeta(3)+1008\gamma_5\zeta(3)+168\log^6(2\pi)\zeta(3)+\log^9(2\pi)\right)+11813\pi^8\log(2\pi)\Big)-$$

$$1814400\zeta(11)-1108800\log^2(2\pi)\zeta(9)+1425600\log(2\pi)\gamma_2\zeta(7)+475200\gamma_3\zeta(7)-950400\log(2\pi)\zeta(3)\zeta(7)-118800\log^4(2\pi)\zeta(7)-399168\log(2\pi)\zeta(5)^2+332640\log^3(2\pi)\gamma_2\zeta(5)+332640\log^2(2\pi)\gamma_3\zeta(5)+166320\log(2\pi)\gamma_4\zeta(5)+33264\gamma_5\zeta(5)-221760\zeta(3)^2\zeta(5)+665280\gamma_2\zeta(3)\zeta(5)-221760\log^3(2\pi)\zeta(3)\zeta(5)-5544\log^6(2\pi)\zeta(5)-61600\log^2(2\pi)\zeta(3)^3+277200\log^2(2\pi)\gamma_2\zeta(3)^2+184800\log(2\pi)\gamma_3\zeta(3)^2+46200\gamma_4\zeta(3)^2-9240\log^5(2\pi)\zeta(3)^2+27720\log^5(2\pi)\gamma_2\zeta(3)+46200\log^4(2\pi)\gamma_3\zeta(3)+46200\log^3(2\pi)\gamma_4\zeta(3)+27720\log^2(2\pi)\gamma_5\zeta(3)+9240\log(2\pi)\gamma_6\zeta(3)+1320\gamma_7\zeta(3)-165\log^8(2\pi)\zeta(3)-\frac{1}{2}\log^{11}(2\pi)-\frac{95265\pi^{10}\log(2\pi)}{2048}+\frac{99}{2}\gamma^{10}\log(2\pi)+5\gamma^{11}$$

10.01.20.0017.01

$$\zeta^{(12)}(0)=-\frac{285795\pi^{10}\left(\log^2(2\pi)-2\gamma_1\right)}{1024}+66\log^{10}(2\pi)\gamma_1+\frac{33}{4}\gamma^{10}\left(8\gamma_1+36\log^2(2\pi)+3\pi^2\right)+330\log^9(2\pi)\gamma_2+990\log^8(2\pi)\gamma_3+1980\log^7(2\pi)\gamma_4+2772\log^6(2\pi)\gamma_5+2772\log^5(2\pi)\gamma_6+1980\log^4(2\pi)\gamma_7+990\log^3(2\pi)\gamma_8+330\log^2(2\pi)\gamma_9+66\log(2\pi)\gamma_{10}+6\gamma_{11}+110\gamma^9\left(6\log(2\pi)\gamma_1+3\gamma_2+16\zeta(3)+8\log^3(2\pi)+2\pi^2\log(2\pi)\right)-\frac{129943}{512}\pi^8\left(-12\log^2(2\pi)\gamma_1-12\log(2\pi)\gamma_2-4\gamma_3+8\log(2\pi)\zeta(3)+\log^4(2\pi)\right)+\frac{33}{32}\gamma^8\left(120\pi^2\left(2\gamma_1+7\log^2(2\pi)\right)+240\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+56\log(2\pi)\zeta(3)+7\log^4(2\pi)\right)+133\pi^4\right)-$$

$$\frac{3025}{32}\pi^6\left(-30\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1-60\log^2(2\pi)\gamma_3-30\log(2\pi)\gamma_4-6\gamma_5+\right.$$

$$\left.144\log(2\pi)\zeta(5)-60\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)+40\zeta(3)^2+40\log^3(2\pi)\zeta(3)+\log^6(2\pi)\right)+$$

$$\frac{99}{2}\gamma^7\left(20\pi^2\left(2\log(2\pi)\gamma_1+\gamma_2+4\zeta(3)+2\log^3(2\pi)\right)+8\left(20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+30\log^2(2\pi)\gamma_2+\right.\right.$$

$$\left.\left.20\log(2\pi)\gamma_3+5\gamma_4+144\zeta(5)+120\log^2(2\pi)\zeta(3)+6\log^5(2\pi)\right)+19\pi^4\log(2\pi)\right)+$$

$$\frac{11}{32}\gamma^6\left(1596\pi^4\left(2\gamma_1+5\log^2(2\pi)\right)+1680\pi^2\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+40\log(2\pi)\zeta(3)+5\log^4(2\pi)\right)+\right.$$

$$1344\left(30\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1+60\log^2(2\pi)\gamma_3+30\log(2\pi)\gamma_4+6\gamma_5+720\log(2\pi)\zeta(5)+\right.$$

$$\left.\left.60\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)+200\zeta(3)^2+200\log^3(2\pi)\zeta(3)+5\log^6(2\pi)\right)+1375\pi^6\right)-$$

$$\frac{627}{32}\pi^4\left(-56\left(\log^6(2\pi)+40\log^3(2\pi)\zeta(3)+40\zeta(3)^2+144\log(2\pi)\zeta(5)\right)\gamma_1-280\log^4(2\pi)\gamma_3-280\log^3(2\pi)\gamma_4-\right.$$

$$168\log^2(2\pi)\gamma_5-56\log(2\pi)\gamma_6-8\gamma_7+5760\log(2\pi)\zeta(7)-168\gamma_2\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)+$$

$$2688\zeta(3)\zeta(5)+1344\log^3(2\pi)\zeta(5)+1120\log^2(2\pi)\zeta(3)^2-2240\log(2\pi)\gamma_3\zeta(3)-560\gamma_4\zeta(3)+$$

$$\left.112\log^5(2\pi)\zeta(3)+\log^8(2\pi)\right)+\frac{33}{4}\gamma^5\left(133\pi^4\left(6\log(2\pi)\gamma_1+3\gamma_2+8\zeta(3)+4\log^3(2\pi)\right)+\right.$$

$$84\pi^2\left(20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+30\log^2(2\pi)\gamma_2+20\log(2\pi)\gamma_3+5\gamma_4+96\zeta(5)+80\log^2(2\pi)\zeta(3)+4\log^5(2\pi)\right)+$$

$$48\left(42\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)\gamma_1+140\log^3(2\pi)\gamma_3+105\log^2(2\pi)\gamma_4+\right.$$

$$42\log(2\pi)\gamma_5+7\gamma_6+2880\zeta(7)+2016\log^2(2\pi)\zeta(5)+105\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)+$$

$$\left.\left.1120\log(2\pi)\zeta(3)^2+280\gamma_3\zeta(3)+280\log^4(2\pi)\zeta(3)+4\log^7(2\pi)\right)+275\pi^6\log(2\pi)\right)+$$

$$\frac{33}{512}\gamma^4\left(22\,000\pi^6\left(2\gamma_1+3\log^2(2\pi)\right)+21\,280\pi^4\left(12\log^2(2\pi)\gamma_1+12\log(2\pi)\gamma_2+4\gamma_3+24\log(2\pi)\zeta(3)+3\log^4(2\pi)\right)+\right.$$

$$26\,880\pi^2\left(10\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)\gamma_1+20\log^2(2\pi)\gamma_3+10\log(2\pi)\gamma_4+2\gamma_5+\right.$$

$$\left.144\log(2\pi)\zeta(5)+20\gamma_2\left(\log^3(2\pi)+2\zeta(3)\right)+40\zeta(3)^2+40\log^3(2\pi)\zeta(3)+\log^6(2\pi)\right)+$$

$$3840\left(56\left(\log^6(2\pi)+40\log^3(2\pi)\zeta(3)+40\zeta(3)^2+144\log(2\pi)\zeta(5)\right)\gamma_1+280\log^4(2\pi)\gamma_3+\right.$$

$$280\log^3(2\pi)\gamma_4+168\log^2(2\pi)\gamma_5+56\log(2\pi)\gamma_6+8\gamma_7+17\,280\log(2\pi)\zeta(7)+$$

$$168\gamma_2\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)+8064\zeta(3)\zeta(5)+4032\log^3(2\pi)\zeta(5)+3360\log^2(2\pi)\zeta(3)^2+$$

$$\left.\left.2240\log(2\pi)\gamma_3\zeta(3)+560\gamma_4\zeta(3)+336\log^5(2\pi)\zeta(3)+3\log^8(2\pi)\right)+11\,813\pi^8\right)-\frac{11}{4}\pi^2$$

$$\left(-90\left(\log^8(2\pi)+112\log^5(2\pi)\zeta(3)+1120\log^2(2\pi)\zeta(3)^2+1344\log^3(2\pi)\zeta(5)+2688\zeta(3)\zeta(5)+5760\log(2\pi)\zeta(7)\right)\gamma_1-\right.$$

$$840\log^6(2\pi)\gamma_3-1260\log^5(2\pi)\gamma_4-1260\log^4(2\pi)\gamma_5-840\log^3(2\pi)\gamma_6-360\log^2(2\pi)\gamma_7-90\log(2\pi)\gamma_8-10\gamma_9+$$

$$403\,200\log(2\pi)\zeta(9)-360\gamma_2\left(\log^7(2\pi)+70\log^4(2\pi)\zeta(3)+280\log(2\pi)\zeta(3)^2+504\log^2(2\pi)\zeta(5)+720\zeta(7)\right)+$$

$$172\,800\zeta(3)\zeta(7)+86\,400\log^3(2\pi)\zeta(7)+72\,576\zeta(5)^2-120\,960\log(2\pi)\gamma_3\zeta(5)-$$

$$30\,240\gamma_4\zeta(5)+120\,960\log^2(2\pi)\zeta(3)\zeta(5)+6048\log^5(2\pi)\zeta(5)+22\,400\log(2\pi)\zeta(3)^3-$$

$$33\,600\gamma_3\zeta(3)^2+8400\log^4(2\pi)\zeta(3)^2-33\,600\log^3(2\pi)\gamma_3\zeta(3)-25\,200\log^2(2\pi)\gamma_4\zeta(3)-$$

$$\left.10\,080\log(2\pi)\gamma_5\zeta(3)-1680\gamma_6\zeta(3)+240\log^7(2\pi)\zeta(3)+\log^{10}(2\pi)\right)+$$

$$\frac{11}{64}\gamma^3\left(11\,000\pi^6\left(6\log(2\pi)\gamma_1+3\gamma_2+4\zeta(3)+2\log^3(2\pi)\right)+\right.$$

$$6384\pi^4\left(20\left(\log^3(2\pi)+2\zeta(3)\right)\gamma_1+30\log^2(2\pi)\gamma_2+20\log(2\pi)\gamma_3+5\gamma_4+48\zeta(5)+40\log^2(2\pi)\zeta(3)+2\log^5(2\pi)\right)+$$

$$1920\pi^2\left(42\left(\log^5(2\pi)+20\log^2(2\pi)\zeta(3)+24\zeta(5)\right)\gamma_1+140\log^3(2\pi)\gamma_3+105\log^2(2\pi)\gamma_4+\right.$$

$$42\log(2\pi)\gamma_5+7\gamma_6+1440\zeta(7)+1008\log^2(2\pi)\zeta(5)+105\gamma_2\left(\log^4(2\pi)+8\log(2\pi)\zeta(3)\right)+$$

$$560 \log(2\pi)\,\zeta(3)^2 + 280\,\gamma_3\,\zeta(3) + 140 \log^4(2\pi)\,\zeta(3) + 2 \log^7(2\pi)\big) +$$

$$640 \big(72 \big(\log^7(2\pi) + 70 \log^4(2\pi)\,\zeta(3) + 280 \log(2\pi)\,\zeta(3)^2 + 504 \log^2(2\pi)\,\zeta(5) + 720\,\zeta(7)\big)\,\gamma_1 +$$

$$504 \log^5(2\pi)\,\gamma_3 + 630 \log^4(2\pi)\,\gamma_4 + 504 \log^3(2\pi)\,\gamma_5 + 252 \log^2(2\pi)\,\gamma_6 + 72 \log(2\pi)\,\gamma_7 + 9\,\gamma_8 + 80\,640\,\zeta(9) +$$

$$51\,840 \log^2(2\pi)\,\zeta(7) + 252\,\gamma_2 \big(\log^6(2\pi) + 40 \log^3(2\pi)\,\zeta(3) + 40\,\zeta(3)^2 + 144 \log(2\pi)\,\zeta(5)\big) + 12\,096\,\gamma_3\,\zeta(5) +$$

$$48\,384 \log(2\pi)\,\zeta(3)\,\zeta(5) + 6048 \log^4(2\pi)\,\zeta(5) + 4480\,\zeta(3)^3 + 6720 \log^3(2\pi)\,\zeta(3)^2 + 10\,080 \log^2(2\pi)\,\gamma_3\,\zeta(3) +$$

$$5040 \log(2\pi)\,\gamma_4\,\zeta(3) + 1008\,\gamma_5\,\zeta(3) + 336 \log^6(2\pi)\,\zeta(3) + 2 \log^9(2\pi)\big) + 11\,813\,\pi^8 \log(2\pi)\big) +$$

$$\frac{1}{1024} \big(3\,\gamma^2 \big(519\,772\,\pi^8 \big(2\,\gamma_1 + \log^2(2\pi)\big) + 484\,000\,\pi^6 \big(12 \log^2(2\pi)\,\gamma_1 + 12 \log(2\pi)\,\gamma_2 + 4\,\gamma_3 + 8 \log(2\pi)\,\zeta(3) + \log^4(2\pi)\big) +$$

$$187\,264\,\pi^4 \big(30 \big(\log^4(2\pi) + 8 \log(2\pi)\,\zeta(3)\big)\,\gamma_1 + 60 \log^2(2\pi)\,\gamma_3 + 30 \log(2\pi)\,\gamma_4 + 6\,\gamma_5 +$$

$$144 \log(2\pi)\,\zeta(5) + 60\,\gamma_2 \big(\log^3(2\pi) + 2\,\zeta(3)\big) + 40\,\zeta(3)^2 + 40 \log^3(2\pi)\,\zeta(3) + \log^6(2\pi)\big) +$$

$$42\,240\,\pi^2 \big(56 \big(\log^6(2\pi) + 40 \log^3(2\pi)\,\zeta(3) + 40\,\zeta(3)^2 + 144 \log(2\pi)\,\zeta(5)\big)\,\gamma_1 + 280 \log^4(2\pi)\,\gamma_3 +$$

$$280 \log^3(2\pi)\,\gamma_4 + 168 \log^2(2\pi)\,\gamma_5 + 56 \log(2\pi)\,\gamma_6 + 8\,\gamma_7 + 5760 \log(2\pi)\,\zeta(7) +$$

$$168\,\gamma_2 \big(\log^5(2\pi) + 20 \log^2(2\pi)\,\zeta(3) + 24\,\zeta(5)\big) + 2688\,\zeta(3)\,\zeta(5) + 1344 \log^3(2\pi)\,\zeta(5) +$$

$$1120 \log^2(2\pi)\,\zeta(3)^2 + 2240 \log(2\pi)\,\gamma_3\,\zeta(3) + 560\,\gamma_4\,\zeta(3) + 112 \log^5(2\pi)\,\zeta(3) + \log^8(2\pi)\big) +$$

$$11\,264 \big(90 \big(\log^8(2\pi) + 112 \log^5(2\pi)\,\zeta(3) + 1120 \log^2(2\pi)\,\zeta(3)^2 + 1344 \log^3(2\pi)\,\zeta(5) + 2688\,\zeta(3)\,\zeta(5) +$$

$$5760 \log(2\pi)\,\zeta(7)\big)\,\gamma_1 + 840 \log^6(2\pi)\,\gamma_3 + 1260 \log^5(2\pi)\,\gamma_4 + 1260 \log^4(2\pi)\,\gamma_5 +$$

$$840 \log^3(2\pi)\,\gamma_6 + 360 \log^2(2\pi)\,\gamma_7 + 90 \log(2\pi)\,\gamma_8 + 10\,\gamma_9 + 403\,200 \log(2\pi)\,\zeta(9) +$$

$$360\,\gamma_2 \big(\log^7(2\pi) + 70 \log^4(2\pi)\,\zeta(3) + 280 \log(2\pi)\,\zeta(3)^2 + 504 \log^2(2\pi)\,\zeta(5) + 720\,\zeta(7)\big) +$$

$$172\,800\,\zeta(3)\,\zeta(7) + 86\,400 \log^3(2\pi)\,\zeta(7) + 72\,576\,\zeta(5)^2 + 120\,960 \log(2\pi)\,\gamma_3\,\zeta(5) +$$

$$30\,240\,\gamma_4\,\zeta(5) + 120\,960 \log^2(2\pi)\,\zeta(3)\,\zeta(5) + 6048 \log^5(2\pi)\,\zeta(5) + 22\,400 \log(2\pi)\,\zeta(3)^3 +$$

$$33\,600\,\gamma_3\,\zeta(3)^2 + 8400 \log^4(2\pi)\,\zeta(3)^2 + 33\,600 \log^3(2\pi)\,\gamma_3\,\zeta(3) + 25\,200 \log^2(2\pi)\,\gamma_4\,\zeta(3) +$$

$$10\,080 \log(2\pi)\,\gamma_5\,\zeta(3) + 1680\,\gamma_6\,\zeta(3) + 240 \log^7(2\pi)\,\zeta(3) + \log^{10}(2\pi)\big) + 95\,265\,\pi^{10}\big)\big) +$$

$$\frac{33}{128}\,\gamma \big(11\,813\,\pi^8 \,(2 \log(2\pi)\,\gamma_1 + \gamma_2) + 11\,000\,\pi^6 \big(4 \big(\log^3(2\pi) + 2\,\zeta(3)\big)\,\gamma_1 + 6 \log^2(2\pi)\,\gamma_2 + 4 \log(2\pi)\,\gamma_3 + \gamma_4\big) +$$

$$4256\,\pi^4 \big(6 \big(\log^5(2\pi) + 20 \log^2(2\pi)\,\zeta(3) + 24\,\zeta(5)\big)\,\gamma_1 + 20 \log^3(2\pi)\,\gamma_3 +$$

$$15 \log^2(2\pi)\,\gamma_4 + 6 \log(2\pi)\,\gamma_5 + \gamma_6 + 15\,\gamma_2 \big(\log^4(2\pi) + 8 \log(2\pi)\,\zeta(3)\big) + 40\,\gamma_3\,\zeta(3)\big) +$$

$$960\,\pi^2 \big(8 \big(\log^7(2\pi) + 70 \log^4(2\pi)\,\zeta(3) + 280 \log(2\pi)\,\zeta(3)^2 + 504 \log^2(2\pi)\,\zeta(5) + 720\,\zeta(7)\big)\,\gamma_1 +$$

$$56 \log^5(2\pi)\,\gamma_3 + 70 \log^4(2\pi)\,\gamma_4 + 56 \log^3(2\pi)\,\gamma_5 + 28 \log^2(2\pi)\,\gamma_6 + 8 \log(2\pi)\,\gamma_7 +$$

$$\gamma_8 + 28\,\gamma_2 \big(\log^6(2\pi) + 40 \log^3(2\pi)\,\zeta(3) + 40\,\zeta(3)^2 + 144 \log(2\pi)\,\zeta(5)\big) +$$

$$1344\,\gamma_3\,\zeta(5) + 1120 \log^2(2\pi)\,\gamma_3\,\zeta(3) + 560 \log(2\pi)\,\gamma_4\,\zeta(3) + 112\,\gamma_5\,\zeta(3)\big) +$$

$$256 \big(10 \big(\log^9(2\pi) + 168 \log^6(2\pi)\,\zeta(3) + 3360 \log^3(2\pi)\,\zeta(3)^2 + 3024 \log^4(2\pi)\,\zeta(5) + 24\,192 \log(2\pi)\,\zeta(3)\,\zeta(5) +$$

$$25\,920 \log^2(2\pi)\,\zeta(7) + 2240 \big(\zeta(3)^3 + 18\,\zeta(9)\big)\big)\,\gamma_1 + 120 \log^7(2\pi)\,\gamma_3 + 210 \log^6(2\pi)\,\gamma_4 + 252 \log^5(2\pi)\,\gamma_5 +$$

$$210 \log^4(2\pi)\,\gamma_6 + 120 \log^3(2\pi)\,\gamma_7 + 45 \log^2(2\pi)\,\gamma_8 + 10 \log(2\pi)\,\gamma_9 + \gamma_{10} + 45\,\gamma_2 \big(\log^8(2\pi) + 112 \log^5(2\pi)$$

$$\zeta(3) + 1120 \log^2(2\pi)\,\zeta(3)^2 + 1344 \log^3(2\pi)\,\zeta(5) + 2688\,\zeta(3)\,\zeta(5) + 5760 \log(2\pi)\,\zeta(7)\big) + 86\,400\,\gamma_3\,\zeta(7) +$$

$$60\,480 \log^2(2\pi)\,\gamma_3\,\zeta(5) + 30\,240 \log(2\pi)\,\gamma_4\,\zeta(5) + 6048\,\gamma_5\,\zeta(5) + 33\,600 \log(2\pi)\,\gamma_3\,\zeta(3)^2 + 8400\,\gamma_4\,\zeta(3)^2 +$$

$$8400 \log^4(2\pi)\,\gamma_3\,\zeta(3) + 8400 \log^3(2\pi)\,\gamma_4\,\zeta(3) + 5040 \log^2(2\pi)\,\gamma_5\,\zeta(3) + 1680 \log(2\pi)\,\gamma_6\,\zeta(3) + 240\,\gamma_7\,\zeta(3)\big)\big) -$$

$$21\,772\,800 \log(2\pi)\,\zeta(11) + 26\,611\,200 \log(2\pi)\,\gamma_1\,\zeta(9) + 13\,305\,600\,\gamma_2\,\zeta(9) -$$

$$8\,870\,400$$

$$\zeta(3)$$

$$\zeta(9) - 4\,435\,200$$

$$\log^3(2\pi)$$

$\zeta(9) + 5\,702\,400$
$\log^3(2\pi)$
γ_1
$\zeta(7) + 8\,553\,600$
$\log^2(2\pi)\,\gamma_2$
$\zeta(7) + 5\,702\,400$
$\log(2\pi)\,\gamma_3$
$\zeta(7) + 1\,425\,600\,\gamma_4$
$\zeta(7) - 6\,842\,880\,\zeta(5)$
$\zeta(7) + 11\,404\,800\,\gamma_1$
$\zeta(3)\,\zeta(7) - 5\,702\,400$
$\log^2(2\pi)\,\zeta(3)$
$\zeta(7) - 285\,120\,\log^5(2\pi)$
$\zeta(7) + 4\,790\,016\,\gamma_1\,\zeta(5)^2 -$
$2\,395\,008\,\log^2(2\pi)\,\zeta(5)^2 + 399\,168\,\log^5(2\pi)\,\gamma_1\,\zeta(5) +$
$997\,920\,\log^4(2\pi)\,\gamma_2\,\zeta(5) +$
$1\,330\,560\,\log^3(2\pi)\,\gamma_3\,\zeta(5) +$
$997\,920\,\log^2(2\pi)\,\gamma_4\,\zeta(5) +$
$399\,168\,\log(2\pi)\,\gamma_5\,\zeta(5) +$
$66\,528\,\gamma_6\,\zeta(5) -$
$2\,661\,120\,\log(2\pi)\,\zeta(3)^2\,\zeta(5) +$
$7\,983\,360\,\log^2(2\pi)\,\gamma_1\,\zeta(3)\,\zeta(5) +$
$7\,983\,360\,\log(2\pi)\,\gamma_2\,\zeta(3)\,\zeta(5) +$
$2\,661\,120\,\gamma_3\,\zeta(3)\,\zeta(5) -$
$665\,280\,\log^4(2\pi)\,\zeta(3)\,\zeta(5) -$
$9504\,\log^7(2\pi)\,\zeta(5) -$
$123\,200\,\zeta(3)^4 +$
$1\,478\,400\,\log(2\pi)\,\gamma_1\,\zeta(3)^3 +$
$739\,200\,\gamma_2\,\zeta(3)^3 -$
$246\,400\,\log^3(2\pi)\,\zeta(3)^3 +$
$554\,400\,\log^4(2\pi)\,\gamma_1\,\zeta(3)^2 +$
$1\,108\,800\,\log^3(2\pi)\,\gamma_2\,\zeta(3)^2 +$
$1\,108\,800\,\log^2(2\pi)\,\gamma_3\,\zeta(3)^2 +$
$554\,400\,\log(2\pi)\,\gamma_4\,\zeta(3)^2 +$
$110\,880\,\gamma_5\,\zeta(3)^2 -$
$18\,480\,\log^6(2\pi)\,\zeta(3)^2 +$
$15\,840\,\log^7(2\pi)\,\gamma_1\,\zeta(3) +$
$55\,440\,\log^6(2\pi)\,\gamma_2\,\zeta(3) +$
$110\,880\,\log^5(2\pi)\,\gamma_3\,\zeta(3) +$
$138\,600\,\log^4(2\pi)\,\gamma_4\,\zeta(3) +$
$110\,880\,\log^3(2\pi)\,\gamma_5\,\zeta(3) +$
$55\,440\,\log^2(2\pi)\,\gamma_6\,\zeta(3) +$
$15\,840\,\log(2\pi)\,\gamma_7\,\zeta(3) +$
$1980\,\gamma_8\,\zeta(3) -$
$220\,\log^9(2\pi)\,\zeta(3) -$

$$\frac{1}{2}\log^{12}(2\pi) +$$
$$60\,\gamma^{11}\log(2\pi) -$$
$$\frac{193\,814\,931\,\pi^{12}}{3\,727\,360} +$$
$$\frac{11\,\gamma^{12}}{2}$$

Derivatives at other points

10.01.20.0003.01

$$\zeta'(-1) = \frac{1}{12} - \log(\text{Glaisher})$$

10.01.20.0018.01

$$\zeta'(-3) = -\frac{\log(A)}{2} - \frac{1}{8}\log(2\pi) - 6\,\psi^{(-5)}(1) + 3\,\psi^{(-4)}(1) - \frac{11}{720}$$

10.01.20.0019.01

$$\zeta'(-5) = \frac{\log(A)}{6} + \frac{1}{24}\log(2\pi) - 120\,\psi^{(-7)}(1) + 60\,\psi^{(-6)}(1) - 10\,\psi^{(-5)}(1) + \frac{137}{15\,120}$$

10.01.20.0020.01

$$\zeta'(-7) = -\frac{\log(A)}{6} - \frac{1}{24}\log(2\pi) - 5040\,\psi^{(-9)}(1) + 2520\,\psi^{(-8)}(1) - 420\,\psi^{(-7)}(1) + 7\,\psi^{(-5)}(1) - \frac{121}{11\,200}$$

10.01.20.0021.01

$$\zeta'(-9) = \frac{3\log(A)}{10} + \frac{3}{40}\log(2\pi) - 362\,880\,\psi^{(-11)}(1) + 181\,440\,\psi^{(-10)}(1) - 30\,240\,\psi^{(-9)}(1) + 504\,\psi^{(-7)}(1) - 12\,\psi^{(-5)}(1) + \frac{7129}{332\,640}$$

10.01.20.0022.01

$$\zeta'(-2n) = \frac{(-1)^n\,(2n)!}{2^{2n+1}\,\pi^{2n}}\,\zeta(2n+1)\;/;\; n \in \mathbb{N}^+$$

10.01.20.0023.01

$$\zeta'\left(\frac{1}{2}\right) = \frac{1}{4}\left(6\log(2) + 2\log(\pi) + \pi + 2\,\gamma\right)\zeta\left(\frac{1}{2}\right)$$

10.01.20.0024.01

$$\zeta'(1) = \tilde{\infty}$$

10.01.20.0025.01

$$\zeta'(2) = \frac{1}{6}\pi^2\left(-12\log(A) + \log(2\pi) + \gamma\right)$$

10.01.20.0026.01

$$\zeta''(-2n) = 2^{-2n}\,\pi^{-2n}\,(-1)^n\,(2n)!\left((\log(2\pi) - \psi(2n+1))\,\zeta(2n+1) - \zeta'(1+2n)\right)\;/;\; n \in \mathbb{N}^+$$

Symbolic differentiation

General case

10.01.20.0005.02

$$\frac{\partial^n \zeta(s)}{\partial s^n} = (-1)^n \sum_{k=2}^{\infty} \frac{\log^n(k)}{k^s}\;/;\; \text{Re}(s) > 1 \land n \in \mathbb{N}$$

Derivatives at special points

10.01.20.0027.01

$$\zeta^{(n)}(0) = (-1)^n \, n! \left(\frac{\operatorname{Im}\left(z^{n+1}\right)}{\pi \, (n+1)!} + \frac{1}{\pi} \sum_{k=1}^{n-1} \frac{a_k \operatorname{Im}\left(z^{n-k}\right)}{(n-k)!} \right) /; n \in \mathbb{N} \land z = -\log(2\pi) - \frac{i\pi}{2} \bigwedge a_k = \left(\left[(s-1)^k \right] \left(\Gamma(s)\,\zeta(s) - \frac{1}{s-1} \right) \right)$$

10.01.20.0028.01

$$\zeta^{(n)}(1) = \tilde{\infty} /; n \in \mathbb{N}$$

Fractional integro-differentiation

10.01.20.0006.01

$$\frac{\partial^\alpha \zeta(s)}{\partial s^\alpha} = \frac{s^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=2}^{\infty} \frac{(-s)^{-\alpha}\,(-s\log(k))^\alpha \, Q(-\alpha,\, 0,\, -s\log(k))}{k^s} /; \operatorname{Re}(s) > 1$$

Integration

Indefinite integration

Involving only one direct function

10.01.21.0001.01

$$\int \zeta(s)\, ds = s - \sum_{k=2}^{\infty} \frac{k^{-s}}{\log(k)} /; \operatorname{Re}(s) > 1$$

Involving one direct function and elementary functions

Involving power function

10.01.21.0002.01

$$\int s^{\alpha-1}\,\zeta(s)\, ds = \frac{s^\alpha}{\alpha} - s^\alpha \sum_{k=2}^{\infty} \frac{\Gamma(\alpha,\, s\log(k))}{(s\log(k))^\alpha}$$

Definite integration

10.01.21.0003.01

$$\int_{-\infty}^{\infty} \left| \frac{\left(1 - 2^{1-(i t+\sigma)}\right) \zeta(i t + \sigma)}{i t + \sigma} \right| dt = \frac{\pi \left(1 - 2^{1-2\sigma}\right) \zeta(2\sigma)}{\sigma} /; \sigma \in \mathbb{R}^+ \bigwedge t \in \mathbb{R}$$

A. Ivi▮: Some Identities for the Riemann Zeta Function math.NT/0305219 (2003) http://arXiv.org/abs/math.NT/0305219

10.01.21.0004.01

$$\int_{-\infty}^{\infty} \frac{\left(3 - \sqrt{8}\, \cos(\log(2)\, t)\right) \left| \zeta\left(i t + \frac{1}{2}\right) \right|}{t^2 + \frac{1}{4}} \, dt = \pi \log(2)$$

A. Ivi▮: Some Identities for the Riemann Zeta Function math.NT/0305219 (2003) http://arXiv.org/abs/math.NT/0305219

Summation

Infinite summation

10.01.23.0004.01

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(k+s) z^k = \zeta(s, 1-z) /; |z| < 1$$

10.01.23.0005.01

$$\sum_{k=0}^{\infty} \zeta(k+2) z^k = \frac{\psi(1-z)+\gamma}{z} /; |z| < 1$$

10.01.23.0006.01

$$\sum_{k=1}^{\infty} k\, \zeta(k+2) z^k = \frac{\psi(1-z)+\gamma}{z} + \psi^{(1)}(1-z) /; |z| < 1$$

10.01.23.0007.01

$$\sum_{k=0}^{\infty} \frac{\zeta(2k)}{2k+1} \left(\frac{1}{2^{2k+1}} - \frac{1}{2^{4k}} \right) = \frac{C}{\pi}$$

G.Huvent (2006)

10.01.23.0001.01

$$\sum_{k=1}^{\infty} \frac{(-1)^k (s)_k \zeta(k+s)}{k!} == -1$$

10.01.23.0002.01

$$\sum_{k=1}^{\infty} \frac{(s)_k \zeta(k+s)}{k!\, 2^k} == (2^s - 2) \zeta(s)$$

10.01.23.0003.01

$$\sum_{k=1}^{\infty} \frac{(s)_k \zeta(k+s)}{k!\, 2^k} == \zeta\left(s, a - \frac{1}{2}\right) - \zeta(s, a)$$

10.01.23.0008.01

$$\sum_{k=0}^{\infty} \frac{(k+1)!}{(a)_k} \zeta(k+2) z^k = \Gamma(a) \psi^{(2-a)}(-z) (-z)^{1-a} + \frac{1-a}{z^2} (\gamma (a+z-2) + (2-a) \log(-z) - (2-a) \psi(a-1) - 1) /; |z| < 1$$

10.01.23.0009.01

$$\sum_{k=0}^{\infty} \frac{z^k}{k+n} \zeta(k+2) = (n-2)!\, z^{-n} \sum_{j=0}^{n-2} \frac{z^j}{j!\,(n-j-2)!} \left(\psi(n-j-1) \zeta(j-n+2, 1-z) + \zeta^{(1,0)}(j-n+2, 1-z) \right) -$$
$$(\psi(n-1) \zeta(2-n, 1) + \zeta'(2-n)) z^{-n} /; n \in \mathbb{Z} \land n > 1$$

Operations

Limit operation

10.01.25.0001.01

$$\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right)=\gamma$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_pF_q$

10.01.26.0001.01

$$\zeta(n)={}_{n+1}F_n(1,a_1,a_2,\ldots,a_n;a_1+1,a_2+1,\ldots,a_n+1;1)\;/;\;a_1=a_2=\ldots=a_n=1\bigwedge n-1\in\mathbb{N}^+$$

10.01.26.0002.01

$$\zeta(n)=\frac{1}{1-2^{1-n}}\,{}_{n+1}F_n(1,a_1,a_2,\ldots,a_n;a_1+1,a_2+1,\ldots,a_n+1;-1)\;/;\;a_1=a_2=\ldots=a_n=1\bigwedge n-1\in\mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

10.01.26.0003.01

$$\zeta(n)=-G_{n+1,n+1}^{1,n+1}\left(-1\,\middle|\,\begin{matrix}1,1,\ldots,1\\1,0,\ldots,0\end{matrix}\right)\;/;\;n-1\in\mathbb{N}^+$$

10.01.26.0004.01

$$\zeta(n)=\frac{1}{1-2^{1-n}}\,G_{n+1,n+1}^{1,n+1}\left(1\,\middle|\,\begin{matrix}1,1,\ldots,1\\1,0,\ldots,0\end{matrix}\right)\;/;\;n-1\in\mathbb{N}^+$$

Through other functions

10.01.26.0005.01

$$\zeta(s)=\Phi(1,s,1)$$

10.01.26.0006.01

$$\zeta(s)=\frac{1}{1-2^{1-s}}\,\Phi(-1,s,1)$$

10.01.26.0007.01

$$\zeta(s)=\frac{1}{2^s-1}\,\Phi\left(1,s,\frac{1}{2}\right)$$

10.01.26.0008.01

$$\zeta(s)=S_{s-1}^{1}(1)$$

10.01.26.0009.01

$$\zeta(s)=\mathrm{Li}_s(1)\;/;\;\mathrm{Re}(s)>1$$

10.01.26.0010.01

$$\zeta(s)=-\frac{1}{1-2^{1-s}}\,\mathrm{Li}_s(-1)$$

10.01.26.0011.01

$$\zeta(s)=\zeta(s,1)$$

10.01.26.0012.01

$$\zeta(s) = \frac{1}{2^s - 1}\,\zeta\left(s,\, \frac{1}{2}\right)$$

10.01.26.0013.01

$$\zeta(s) = \zeta(s,\, n) + \sum_{k=1}^{n-1} \frac{1}{k^s} \;/;\; n \in \mathbb{N}^+$$

10.01.26.0014.01

$$\zeta(s) = q^{-s} \sum_{k=1}^{q} \zeta\left(s,\, \frac{k}{q}\right) \;/;\; q \in \mathbb{N}^+$$

10.01.26.0015.01

$$\zeta(s) = H_n^{(s)} + \zeta(s,\, n+1)$$

Representations through equivalent functions

With related functions

10.01.27.0001.01

$$\zeta(z) = Z\left(\frac{i}{2} - i\,z\right) \exp\left(-i\,\vartheta\left(\frac{i}{2} - i\,z\right)\right)$$

Zeros

Sums over zeros

Modulo the Riemann hypothesis the following the following sums over the nontrivial zeros of the Zeta function hold:

10.01.30.0001.01

$$Z_1 = \frac{1}{2}\,(-\log(4\,\pi) + \gamma + 2) \;/;\; Z_1 = \lim_{T\to\infty} \left(\sum_{|\rho_k|=1} \frac{1}{\rho_k}\right) \bigwedge \zeta(\rho_k) = 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0002.01

$$Z_2 = 2\,\gamma_1 + \gamma^2 - \frac{\pi^2}{8} + 1 \;/;\; Z_2 = \lim_{T\to\infty} \left(\sum_{|\rho_k|=1} \frac{1}{\rho_k^2}\right) \bigwedge \zeta(\rho_k) = 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0003.01

$$Z_3 = 3\,\gamma\,\gamma_1 + \gamma^3 + \frac{3\,\gamma_2}{2} - \frac{7\,\zeta(3)}{8} + 1 \;/;\; Z_3 = \lim_{T\to\infty} \left(\sum_{|\rho_k|=1} \frac{1}{\rho_k^3}\right) \bigwedge \zeta(\rho_k) = 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0004.01

$$Z_4 = 2\,\gamma_1^2 + 4\,\gamma^2\,\gamma_1 + \gamma^4 + 2\,\gamma\,\gamma_2 + \frac{2\,\gamma_3}{3} - \frac{\pi^4}{96} + 1 \;/;\; Z_4 = \lim_{T\to\infty} \left(\sum_{|\rho_k|=1} \frac{1}{\rho_k^4}\right) \bigwedge \zeta(\rho_k) = 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0005.01

$$Z_5 = 5\,\gamma^3\,\gamma_1 + \frac{5\,\gamma_2\,\gamma_1}{2} + \gamma^5 + \frac{5\,\gamma^2\,\gamma_2}{2} + \frac{5}{6}\,\gamma\left(6\,\gamma_1^2 + \gamma_3\right) + \frac{5\,\gamma_4}{24} - \frac{31\,\zeta(5)}{32} + 1 \;/;\; Z_5 = \lim_{T\to\infty} \left(\sum_{|\rho_k|=1} \frac{1}{\rho_k^5}\right) \bigwedge \zeta(\rho_k) = 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0006.01

$$Z_6 == 2\gamma_1^3 + 6\gamma^4\gamma_1 + \gamma_3\gamma_1 + \gamma^6 + \frac{3\gamma_2^2}{4} + 3\gamma^3\gamma_2 + \gamma^2\left(9\gamma_1^2 + \gamma_3\right) + \frac{1}{4}\gamma(24\gamma_1\gamma_2 + \gamma_4) + \frac{\gamma_5}{20} - \frac{\pi^6}{960} + 1\ /;$$

$$Z_6 == \lim_{T\to\infty}\left(\sum_{|\rho_k|=1}\frac{1}{\rho_k^6}\right) \bigwedge \zeta(\rho_k) == 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0007.01

$$Z_7 == 7\gamma^5\gamma_1 + \gamma^7 + \frac{7\gamma^4\gamma_2}{2} + \frac{7}{6}\gamma^3\left(12\gamma_1^2 + \gamma_3\right) + \frac{7}{24}\gamma^2(36\gamma_1\gamma_2 + \gamma_4) + \frac{7}{24}\left(2\gamma_2\left(6\gamma_1^2 + \gamma_3\right) + \gamma_1\gamma_4\right) + \frac{7}{120}\gamma\left(30\gamma_2^2 + 40\gamma_1\left(3\gamma_1^2 + \gamma_3\right) + \gamma_5\right) + \frac{7\gamma_6}{720} - \frac{127\zeta(7)}{128} + 1\ /;\ Z_7 == \lim_{T\to\infty}\left(\sum_{|\rho_k|=1}\frac{1}{\rho_k^7}\right) \bigwedge \zeta(\rho_k) == 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0008.01

$$Z_8 == 2\gamma_1^4 + \frac{4}{3}\gamma_3\gamma_1^2 + 8\gamma^6\gamma_1 + 2\gamma_2^2\gamma_1 + \frac{\gamma_5\gamma_1}{15} + \gamma^8 + \left(16\gamma_1^3 + 4\gamma_3\gamma_1 + 3\gamma_2^2 + \frac{\gamma_5}{15}\right)\gamma^2 + \frac{\gamma_3^2}{9} + 4\gamma^5\gamma_2 + \frac{4}{3}\gamma^4\left(15\gamma_1^2 + \gamma_3\right) + \frac{\gamma_2\gamma_4}{6} + \frac{1}{3}\gamma^3(48\gamma_1\gamma_2 + \gamma_4) + \frac{1}{90}\gamma\left(60\left(2\gamma_2\left(9\gamma_1^2 + \gamma_3\right) + \gamma_1\gamma_4\right) + \gamma_6\right) + \frac{\gamma_7}{630} - \frac{17\pi^8}{161\,280} + 1\ /;$$

$$Z_8 == \lim_{T\to\infty}\left(\sum_{|\rho_k|=1}\frac{1}{\rho_k^8}\right) \bigwedge \zeta(\rho_k) == 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0$$

10.01.30.0009.01

$$P_n == -\frac{1}{2}n(\log(4\pi) + \gamma) - \sum_{j=1}^{n}\eta_{j-1}\binom{n}{j} - \sum_{j=2}^{n}(-1)^{j-1}\binom{n}{j}\left(1 - 2^{-j}\right)\zeta(j) + 1\ /;$$

$$P_n == \lim_{T\to\infty}\left(\sum_{|\rho_k|=1} 1 - \left(1 - \frac{1}{\rho_k}\right)^n\right) \bigwedge \zeta(\rho_k) == 0 \bigwedge \mathrm{Re}(\rho_k) \neq 0 \bigwedge n \in \mathbb{N}^+ \bigwedge \log(s\,\zeta(s+1)) == -\sum_{k=0}^{\infty}\frac{\eta_k\,s^{k+1}}{k+1}$$

Theorems

The Riemann hypothesis on the zeros of the zeta-function

All nontrivial zeros of ζ(s) lie on the straight line $\mathrm{Re}(s) == \frac{1}{2}$.

The equivalent version of the Riemann hypothesis

The Riemann hypothesis is equivalent to $\sum_{\rho}\frac{1}{|\rho|^2} == 2 + \gamma - \log(4\pi)\ /;\ \zeta(\rho) == 0 \bigwedge \mathrm{Im}(\rho) \neq 0$.

A generalization of this result due to Li, Bombieri, Lagarias is:

$$\frac{1}{(n-1)!}\frac{\partial^n}{\partial s^n}\left(s^{n-1}\ln\left(\frac{1}{2}s(s-1)\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)\right)\right)|_{s=1} = \sum_{\rho} 1 - \left(1 - \frac{1}{\rho}\right)^n > 0\ \forall\ n \in \mathbb{N}^+$$

A. Weil's "explicit formula"

Let α be any function from $C_0^\infty(\mathbb{R})$, $\Phi(s) = \int_{-\infty}^{\infty} \alpha(t) \exp(s\, t)\, dt$. Then the following identity holds (the ρ -sum runs over all nontrivial zeros of $\zeta(z)$)

$$\Phi(0) + \Phi(1) - \sum_{\rho} \Phi(\rho) \equiv$$

$$\sum_{j}^{\infty} \log(p_j) \sum_{k=1}^{\infty} \alpha(k \log(p_j)) + \sum_{j}^{\infty} \log(p_j) \sum_{k=-\infty}^{-1} p_j^k\, \alpha(k \log(p_j)) + \alpha(0) \log(\pi) + \int_0^{\infty} \left(\frac{\alpha(t) - e^{-t}\, \alpha(-t)}{1 - e^{-2t}} - \alpha(0) \frac{e^{-2t}}{t} \right) dt$$

The distribution of the zeros

The sequence of zeros of $\zeta(z)$ along the critical line $1/2 + i\, t$ is homogeneously distributed mod 1.

Zeta function regularization

If $\sum_{k=1}^{\infty} e^{-a_k z}$ is defined for $\mathrm{Re}(z) > 0$ and can be analytically continued to a domain containing $z \equiv 0$, then

$$\lim_{z \to 0} \sum_{k=1}^{\infty} a_k\, e^{-a_k z} \equiv \lim_{s \to -1} \sum_{k=1}^{\infty} a_k^{-s}$$

One map with zeta function

The map $h : \mathbb{R} \longrightarrow \mathbb{C}^n$ defined by $h(t) = \{\zeta(\sigma + i\, t),\ \zeta'(\sigma + i\, t),\ \zeta''(\sigma + i\, t),\ \ldots,\ \zeta^{(n)}(\sigma + i\, t)\}$ with constant $1/2 < \sigma < 1$ is dense in $\mathbb{C}^n$.

One max-property

If $f(z)$ is any nonvanishing continous analytic function in the disk $|z| < 1/4$, then there exists a real $t(\varepsilon)$ such that $\max_{|z|<1/4} \left| \zeta\left(z + \frac{3}{4} + i\, t(\varepsilon)\right) - f(z) \right| < \varepsilon$.

Montgomery conjecture

The two-point correlation function $R_2(r)$ for the zeros of $\zeta(z)$ on the critical line is $R_2(r) \equiv 1 - \sin(\pi\, r)^2 / (\pi\, r)^2$.

Keating–Snaith conjecture

For $k \in \mathbb{R}$, $k > -1/2$ the following is conjectured:

$$\lim_{T \to \infty} \int_0^T \left| \zeta\left(\frac{1}{2} + i\, t\right) \right|^{2k} dt \propto \frac{G(k+1)^2}{G(2\, k+1)}\, a(k) \left(\log\left(\frac{T}{2\, \pi}\right) \right)^{k^2}$$

Here $G(z)$ is the Barnes G function and $a(k) = \prod_{n=1}^{\infty} (1 - p_n)^{k^2} \sum_{j=0}^{\infty} (\Gamma(j + k) / (j!\, \Gamma(k)))^2\, p_n^{-j}$.

Hughes–Keating–O'Connel conjecture

For $k \in \mathbb{R}$, $k > -3/2$ the following is conjectured:

$$\lim_{T\to\infty} \sum_{0<s_n<T} \left|\zeta'\left(\frac{1}{2}+i\, s_n\right)\right|^{2k} dt \propto \frac{G(k+2)^2}{G(2\,k+3)}\, a(k) \left(\log\left(\frac{T}{2\,\pi}\right)\right)^{k(k+2)}$$

Here the sum extends over all zeros on the critical line and $G(z)$ is the Barnes G function and $a(k) = \prod_{n=1}^{\infty} (1-p_n)^{k^2} \sum_{j=0}^{\infty} (\Gamma(j+k)/(j!\,\Gamma(k)))^2\, p_n^{-j}$.

GUE hypothesis

A fixed set of nontrivial zeros of $\zeta(z)$ behaves asymptotically like the eigenvalues of a Gaussian unitary ensemble.

Asymptotical behaviour of zeros

The number $N(t)$ of zeros $\frac{1}{2}+i\,\gamma_n$ of $\zeta\left(\frac{1}{2}+i\,t\right)$ behaves asymptotically as $N(t) == \frac{t}{2\pi} \log\left(\frac{t}{2\,\pi\, e}\right) + O(\log(t))$.

The lattice-packing density for any convex symmetrical body

The lattice-packing density $\delta_n(\mathcal{K})$ for any convex symmetrical body $\mathcal{K}$ in n dimensions satisfies the inequality $\delta_n(\mathcal{K}) \geq \frac{n\,\zeta(n)}{e(1-e^{-n})\, 2^{n-1}}$.

The probability of a lattice point to be visible

The probability of a lattice point from $\mathbb{Z}^d$ being visible from the origin (i.e., an d–tuple of integers is relatively prime) is $\frac{1}{\zeta(d)}$.

The scattering matrix

The scattering matrix $\mathcal{S}$ of the Laplace–Beltrami operator in the modular domain has the form $\mathcal{S}(\omega) = \frac{\Gamma(1/2)\,\Gamma(i\,\omega)}{\Gamma(1/2+i\,\omega)} \frac{\zeta(2\,i\,\omega)}{\zeta(1+2\,i\,\omega)}$.

History

– L. Euler (1737)
– P. G. L. Dirichlet
– P. L. Chebyshev
– B. Riemann (1859)
– J. Hadamard (1893)
– H. von Mangoldt (1894)
– Ch. J. de la Vallee–Poussin (1896)

Applications include number theory, Bose–Einstein and Fermi–Dirac statistics, analytic approximation and evaluation of integrals and products, regularization techniques in quantum field theory, Scharnhorst effect of quantum electrodynamics, Brownian motion.

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