Introductions to AiryAi

Introduction to the Airy functions

General

In 1838, G. B. Airy investigated the simple-looking differential equation:

$$w''(z) - z\, w(z) == 0.$$

This is quite similar to the differential equation $w''(z) - w(z) == 0$ for the hyperbolic sine and hyperbolic cosine functions, which has the general solution $w(z) == c_1 \sinh(z) + c_2 \cosh(z)$. Airy built two partial solutions $w_1(z)$ and $w_2(z)$ for the first equation in the form of a power series $w(z) == \sum_{j=0}^{\infty} a_j\, z^j$. These solutions were named the Airy functions. Much later, H. Jeffreys (1928–1942) investigated these functions more deeply. The current notations Ai and Bi were proposed by J. C. P. Miller (1946).

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are the special solutions of the differential equation:

$$w''(z) - z\, w(z) == 0 \;/;\; w(z) == c_1\, \mathrm{Ai}(z) + c_2\, \mathrm{Bi}(z),$$

satisfying the following initial conditions:

$$w(0) == \mathrm{Ai}(0) == \frac{1}{3^{2/3}\, \Gamma\left(\frac{2}{3}\right)} \bigwedge w'(0) == \mathrm{Ai}'(0) == -\frac{1}{\sqrt[3]{3}\, \Gamma\left(\frac{1}{3}\right)}$$

$$w(0) == \mathrm{Bi}(0) == \frac{1}{\sqrt[6]{3}\, \Gamma\left(\frac{2}{3}\right)} \bigwedge w'(0) == \mathrm{Bi}'(0) == \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}.$$

These functions have different equivalent representations in the form of series or generalized hypergeometric functions. The hypergeometric representation can be conveniently used as a definition of the Airy functions.

Definition of Airy functions

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, and their derivatives $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ can be defined by:

$$\mathrm{Ai}(z) == \frac{1}{3^{2/3}\, \Gamma\left(\frac{2}{3}\right)}\, {}_0F_1\left(; \frac{2}{3}; \frac{z^3}{9}\right) - \frac{z}{\sqrt[3]{3}\, \Gamma\left(\frac{1}{3}\right)}\, {}_0F_1\left(; \frac{4}{3}; \frac{z^3}{9}\right)$$

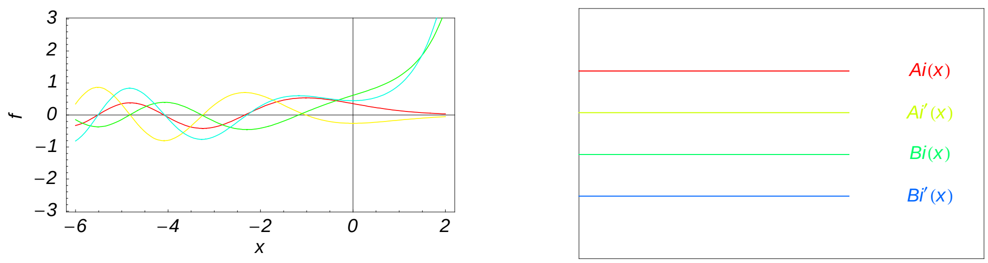
$$\mathrm{Ai}'(z) == \frac{\partial\, \mathrm{Ai}(z)}{\partial z} == \frac{z^2}{2\, 3^{2/3}\, \Gamma\left(\frac{2}{3}\right)}\, {}_0F_1\left(; \frac{5}{3}; \frac{z^3}{9}\right) - \frac{1}{\sqrt[3]{3}\, \Gamma\left(\frac{1}{3}\right)}\, {}_0F_1\left(; \frac{1}{3}; \frac{z^3}{9}\right)$$

$$\mathrm{Bi}(z) == \frac{1}{\sqrt[6]{3}\, \Gamma\left(\frac{2}{3}\right)}\, {}_0F_1\left(; \frac{2}{3}; \frac{z^3}{9}\right) + \frac{\sqrt[6]{3}\, z}{\Gamma\left(\frac{1}{3}\right)}\, {}_0F_1\left(; \frac{4}{3}; \frac{z^3}{9}\right)$$

$$\mathrm{Bi}'(z) == \frac{\partial \mathrm{Bi}(z)}{\partial z} == \frac{z^2}{2\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(;\frac{5}{3};\frac{z^3}{9}\right) + \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(;\frac{1}{3};\frac{z^3}{9}\right).$$

A quick look at the Airy functions

Here is a quick look at the graphics for the Airy functions and their derivatives along the real axis.



Connections within the group of Airy functions and with other function groups

Representations through more general functions

The Airy functions Ai (z) and Bi (z), and their derivatives Ai$'(z)$ and Bi$'(z)$ are particular cases of the more general Bessel, hypergeometric, and Meijer G functions.

The Airy functions can be represented as combinations of modified Bessel functions $I_\nu(x)$ with argument $x == \frac{2}{3} z^{3/2}$ and parameter $\nu == \pm \frac{1}{3}, \pm \frac{2}{3}$ through the formulas:

$$\mathrm{Ai}(z) == \frac{1}{3}\left(\sqrt[3]{z^{3/2}}\; I_{-\frac{1}{3}}\left(\frac{2 z^{3/2}}{3}\right) - z\left(z^{3/2}\right)^{-\frac{1}{3}} I_{\frac{1}{3}}\left(\frac{2 z^{3/2}}{3}\right)\right)$$

$$\mathrm{Ai}'(z) == \frac{1}{3}\left(z^2\left(z^{3/2}\right)^{-\frac{2}{3}} I_{\frac{2}{3}}\left(\frac{2 z^{3/2}}{3}\right) - \left(z^{3/2}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{2 z^{3/2}}{3}\right)\right)$$

$$\mathrm{Bi}(z) == \frac{1}{\sqrt{3}}\left(\sqrt[3]{z^{3/2}}\; I_{-\frac{1}{3}}\left(\frac{2 z^{3/2}}{3}\right) + z\left(z^{3/2}\right)^{-\frac{1}{3}} I_{\frac{1}{3}}\left(\frac{2 z^{3/2}}{3}\right)\right)$$

$$\mathrm{Bi}'(z) == \frac{1}{\sqrt{3}}\left(\left(z^{3/2}\right)^{2/3} I_{-\frac{2}{3}}\left(\frac{2 z^{3/2}}{3}\right) + z^2\left(z^{3/2}\right)^{-\frac{2}{3}} I_{\frac{2}{3}}\left(\frac{2 z^{3/2}}{3}\right)\right).$$

Airy functions and their derivatives can be expressed as particular cases of single generalized Meijer G functions:

$$\mathrm{Ai}\,(z) == \frac{1}{2\pi\sqrt[6]{3}}\, G_{0,2}^{2,0}\left(3^{-2/3} z, \frac{1}{3} \,\middle|\, 0, \frac{1}{3}\right)$$

$$\mathrm{Ai}'(z) == -\frac{\sqrt[6]{3}}{2\pi}\, G_{0,2}^{2,0}\left(3^{-2/3} z, \frac{1}{3} \,\middle|\, 0, \frac{2}{3}\right)$$

$$\text{Bi}(z) = \frac{2\pi}{\sqrt[6]{3}} G_{2,4}^{2,0}\left(3^{-2/3} z, \frac{1}{3} \middle| \begin{matrix} \frac{1}{6}, \frac{2}{3} \\ 0, \frac{1}{3}, \frac{1}{6}, \frac{2}{3} \end{matrix}\right)$$

$$\text{Bi}'(z) = -2\pi \sqrt[6]{3}\; G_{2,4}^{2,0}\left(3^{-2/3} z, \frac{1}{3} \middle| \begin{matrix} -\frac{1}{6}, \frac{1}{3} \\ 0, \frac{2}{3}, -\frac{1}{6}, \frac{1}{3} \end{matrix}\right).$$

They cannot be represented as cases of single classical Meijer G functions. Such representations include combinations of two classical Meijer G functions:

$$\text{Ai}(z) = \frac{\pi}{3^{2/3}} \left(G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, \frac{1}{3}, \frac{1}{2} \end{matrix}\right) - \frac{z}{3^{2/3}} G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{3}, \frac{1}{2} \end{matrix}\right)\right)$$

$$\text{Ai}'(z) = \frac{\pi}{9} \left(\sqrt[3]{3}\, z^2 G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, -\frac{2}{3}, \frac{1}{2} \end{matrix}\right) - 3\, 3^{2/3} G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, \frac{2}{3}, \frac{1}{2} \end{matrix}\right)\right)$$

$$\text{Bi}(z) = \frac{\pi}{3^{5/6}} \left(3^{2/3} G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, \frac{1}{3}, \frac{1}{2} \end{matrix}\right) + z\, G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{3}, \frac{1}{2} \end{matrix}\right)\right)$$

$$\text{Bi}'(z) = \sqrt[6]{3}\, \pi\, G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, \frac{2}{3}, \frac{1}{2} \end{matrix}\right) + \frac{\pi z^2}{3\sqrt[6]{3}} G_{1,3}^{1,0}\left(\frac{z^3}{9} \middle| \begin{matrix} \frac{1}{2} \\ 0, -\frac{2}{3}, \frac{1}{2} \end{matrix}\right).$$

The best-known properties and formulas for Airy functions

Real values for real arguments

For real values of argument z, the values of the Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$, and their derivatives $\text{Ai}'(z)$ and $\text{Bi}'(z)$ are real.

Simple values at zero

The Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$, and their derivatives $\text{Ai}'(z)$ and $\text{Bi}'(z)$ have rather simple values for argument $z = 0$:

$$\text{Ai}(0) = \frac{1}{3^{2/3}\, \Gamma\left(\frac{2}{3}\right)}$$

$$\text{Ai}'(0) = -\frac{1}{\sqrt[3]{3}\; \Gamma\left(\frac{1}{3}\right)}$$

$$\text{Bi}(0) = \frac{1}{\sqrt[6]{3}\; \Gamma\left(\frac{2}{3}\right)}$$

$$\text{Bi}'(0) = \frac{1}{\Gamma\left(\frac{1}{3}\right)} \sqrt[6]{3}\,.$$

Analyticity

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, and their derivatives $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ defined for all complex values of z, are analytic functions of z over the whole complex z-plane, and do not have branch cuts or branch points. These functions are entire functions with an essential singular point at $z = \tilde{\infty}$.

Periodicity

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, and their derivatives $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ are not periodic functions.

Parity and symmetry

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, and their derivatives $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ have mirror symmetry:

$$\mathrm{Ai}(\bar{z}) = \overline{\mathrm{Ai}(z)}$$

$$\mathrm{Ai}'(\bar{z}) = \overline{\mathrm{Ai}'(z)}$$

$$\mathrm{Bi}(\bar{z}) = \overline{\mathrm{Bi}(z)}$$

$$\mathrm{Bi}'(\bar{z}) = \overline{\mathrm{Bi}'(z)}.$$

Series representations

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$, and their derivatives $\mathrm{Ai}'(z)$ and $\mathrm{Bi}'(z)$ have rather simple series representations at the origin:

$$\mathrm{Ai}(z) \propto \frac{1}{3^{2/3}\,\Gamma\left(\frac{2}{3}\right)}\left(1+\frac{z^3}{6}+\frac{z^6}{180}+\ldots\right)-\frac{z}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\left(1+\frac{z^3}{12}+\frac{z^6}{504}+\ldots\right) /;\ (z \to 0)$$

$$\mathrm{Ai}'(z) \propto -\frac{1}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\left(1+\frac{z^3}{3}+\frac{z^6}{72}+\ldots\right)+\frac{z^2}{2\,3^{2/3}\,\Gamma\left(\frac{2}{3}\right)}\left(1+\frac{z^3}{15}+\frac{z^6}{720}+\ldots\right) /;\ (z \to 0)$$

$$\mathrm{Bi}(z) \propto \frac{1}{\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)}\left(1+\frac{z^3}{6}+\frac{z^6}{180}+\ldots\right)+\frac{\sqrt[6]{3}\,z}{\Gamma\left(\frac{1}{3}\right)}\left(1+\frac{z^3}{12}+\frac{z^6}{504}+\ldots\right) /;\ (z \to 0)$$

$$\mathrm{Bi}'(z) \propto \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}\left(1+\frac{z^3}{3}+\frac{z^6}{72}+\ldots\right)+\frac{z^2}{2\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)}\left(1+\frac{z^3}{15}+\frac{z^6}{720}+\ldots\right) /;\ (z \to 0).$$

These series converge at the whole z-plane and their symbolic forms are the following:

$$\mathrm{Ai}(z) = \frac{1}{3^{2/3}\,\Gamma\left(\frac{2}{3}\right)}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{2}{3}\right)_k k!}\left(\frac{z^3}{9}\right)^k-\frac{z}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{4}{3}\right)_k k!}\left(\frac{z^3}{9}\right)^k$$

$$\mathrm{Ai}'(z) = \frac{z^2}{2\,3^{2/3}\,\Gamma\left(\frac{2}{3}\right)}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{5}{3}\right)_k k!}\left(\frac{z^3}{9}\right)^k-\frac{1}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)}\sum_{k=0}^{\infty}\frac{1}{\left(\frac{1}{3}\right)_k k!}\left(\frac{z^3}{9}\right)^k$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)} \sum_{k=0}^{\infty} \frac{1}{\left(\frac{2}{3}\right)_k k!} \left(\frac{z^3}{9}\right)^k + \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)} z \sum_{k=0}^{\infty} \frac{1}{\left(\frac{4}{3}\right)_k k!} \left(\frac{z^3}{9}\right)^k$$

$$\mathrm{Bi}'(z) = \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)} \sum_{k=0}^{\infty} \frac{1}{\left(\frac{1}{3}\right)_k k!} \left(\frac{z^3}{9}\right)^k + \frac{z^2}{2\sqrt[6]{3}\,\Gamma\left(\frac{2}{3}\right)} \sum_{k=0}^{\infty} \frac{1}{\left(\frac{5}{3}\right)_k k!} \left(\frac{z^3}{9}\right)^k.$$

Two sums in the previous formulas can be combined into one formula, and the resulting formulas can be rewritten as follows:

$$\mathrm{Ai}(z) = \frac{1}{3^{2/3}\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k+1}{3}\right) \sin\left(\frac{2\pi(k+1)}{3}\right) \left(\sqrt[3]{3}\,z\right)^k$$

$$\mathrm{Ai}'(z) = \frac{1}{\sqrt[3]{3}\,\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k+2}{3}\right) \sin\left(\frac{2\pi(k+2)}{3}\right) \left(\sqrt[3]{3}\,z\right)^k$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt[6]{3}\,\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k+1}{3}\right) \left|\sin\left(\frac{2\pi(k+1)}{3}\right)\right| \left(\sqrt[3]{3}\,z\right)^k$$

$$\mathrm{Bi}'(z) = \frac{3^{1/6}}{\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{k+2}{3}\right) \left|\sin\left(\frac{2\pi(k+2)}{3}\right)\right| \left(\sqrt[3]{3}\,z\right)^k.$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed in terms of the generalized hypergeometric function ${}_1F_2$, for example:

$$\mathrm{Ai}(z) = F_\infty(z) \,/;$$

$$\left(\left(F_n(z) = \frac{1}{3^{2/3}\,\Gamma\left(\frac{2}{3}\right)} \sum_{k=0}^{n} \frac{\left(\frac{z^3}{9}\right)^k}{\left(\frac{2}{3}\right)_k k!} - \frac{z}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)} \sum_{k=0}^{n} \frac{\left(\frac{z^3}{9}\right)^k}{\left(\frac{4}{3}\right)_k k!} = \mathrm{Ai}(z) - \frac{1}{3^{2/3}\,\Gamma\left(\frac{2}{3}\right)(n+1)!\left(\frac{2}{3}\right)_{n+1}} \left(\frac{z^3}{9}\right)^{n+1} {}_1F_2\left(1; n+2, n+\frac{5}{3}; \frac{z^3}{9}\right) + \frac{z\left(\frac{z^3}{9}\right)^{n+1}}{\sqrt[3]{3}\,\Gamma\left(\frac{1}{3}\right)(n+1)!\left(\frac{4}{3}\right)_{n+1}} {}_1F_2\left(1; n+2, n+\frac{7}{3}; \frac{z^3}{9}\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the Airy functions Ai(z) and Bi(z) can be described through formulas that depend on the sector of Arg(z) (Stokes phenomenon). The following formulas are examples of asymptotic expansions that are valid for $|\mathrm{Arg}(z)| < \pi$ (for Ai(z) and Ai′(z)) and for the sector $|\mathrm{Arg}(z)| < \frac{\pi}{3}$ (for Bi(z) and Bi′(z)):

$$\mathrm{Ai}(z) \propto \frac{1}{2\sqrt{\pi}\,\sqrt[4]{z}} e^{-\frac{2}{3} z^{3/2}} \left(1 - \frac{5}{48 z^{3/2}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) /; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$\mathrm{Ai}'(z) \propto -\frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{3/2}} \sqrt[4]{z} \left(1 + \frac{7}{48 z^{3/2}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) /; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$\text{Bi}(z) \propto \frac{e^{\frac{2 z^{3/2}}{3}}}{\sqrt{\pi}\,\sqrt[4]{z}} \left(1 + \frac{5}{48 z^{3/2}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) /; |\text{Arg}(z)| < \frac{\pi}{3} \bigwedge (|z| \to \infty)$$

$$\text{Bi}'(z) \propto \frac{e^{\frac{2 z^{3/2}}{3}}\,\sqrt[4]{z}}{\sqrt{\pi}} \left(1 - \frac{7}{48 z^{3/2}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) /; |\text{Arg}(z)| < \frac{\pi}{3} \bigwedge (|z| \to \infty).$$

By using discontinuous functions such as $\sqrt[3]{-z^3}$, it is possible to write single expansions that are valid for all directions:

$$\text{Ai}(z) \propto \frac{1}{2\sqrt{3\pi}\,(-z^3)^{5/12}} \left(\sqrt[12]{-1}\left(\sqrt[3]{-z^3} - \sqrt[3]{-1}\,z\right) e^{\frac{1}{3}(-2) i \sqrt{-z^3}} \left(1 + \frac{5 i}{48\sqrt{-z^3}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) - (-1)^{11/12}\left(\sqrt[3]{-z^3} + (-1)^{2/3} z\right) e^{\frac{2}{3} i \sqrt{-z^3}} \left(1 - \frac{5 i}{48\sqrt{-z^3}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right)\right) /; (|z| \to \infty)$$

$$\text{Ai}'(z) \propto \frac{1}{2\sqrt{3\pi}\,(-z^3)^{7/12}} \left(-\sqrt[12]{-1}\left((-1)^{1/3} z^2 + \left(-z^3\right)^{2/3}\right) e^{\frac{2}{3} i \sqrt{-z^3}} \left(1 + \frac{7 i}{48\sqrt{-z^3}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) + (-1)^{11/12}\left(-(-1)^{2/3} z^2 + \left(-z^3\right)^{2/3}\right) e^{-\frac{2}{3} i \sqrt{-z^3}} \left(1 - \frac{7 i}{48\sqrt{-z^3}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right)\right) /; (|z| \to \infty)$$

$$\text{Bi}(z) \propto \frac{\left(-z^3\right)^{-5/12}}{2\sqrt{\pi}} \left((-1)^{5/12} e^{\frac{1}{3}(-2) i \sqrt{-z^3}} \left(\frac{1}{\sqrt[3]{-1}}\sqrt[3]{-z^3} + z\right)\left(1 + \frac{5 i}{48\sqrt{-z^3}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) - (-1)^{7/12} e^{\frac{2}{3} i \sqrt{-z^3}} \left(\sqrt[3]{-1}\,\sqrt[3]{-z^3} + z\right)\left(1 - \frac{5 i}{48\sqrt{-z^3}} + \frac{385}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right)\right) /; (|z| \to \infty)$$

$$\text{Bi}'(z) \propto \frac{1}{2\sqrt{\pi}\,\left(-z^3\right)^{7/12}} \left(\sqrt[12]{-1}\, e^{\frac{2}{3} i \sqrt{-z^3}} \left(\left(-z^3\right)^{2/3} + (-1)^{-2/3} z^2\right)\left(1 + \frac{7 i}{48\sqrt{-z^3}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right) - (-1)^{11/12} e^{\frac{1}{3}(-2) i \sqrt{-z^3}} \left(\left(-z^3\right)^{2/3} + (-1)^{2/3} z^2\right)\left(1 - \frac{7 i}{48\sqrt{-z^3}} - \frac{455}{4608 z^3} + O\left(\frac{1}{z^{9/2}}\right)\right)\right) /; (|z| \to \infty)$$

Integral representations

The Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$, and their derivatives $\text{Ai}'(z)$ and $\text{Bi}'(z)$ have rather simple integral representations through sine, cosine, and power functions:

$$\text{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + z t\right) dt /; \text{Im}(z) = 0$$

$$\text{Ai}'(z) = -\frac{1}{\pi}\int_0^\infty t\sin\left(\frac{t^3}{3}+z\,t\right)dt\,/;\,\text{Im}(z) = 0$$

$$\text{Bi}(z) = \frac{1}{\pi}\int_0^\infty\left(\sin\left(\frac{t^3}{3}+z\,t\right)+e^{z\,t-\frac{t^3}{3}}\right)dt\,/;\,z<0$$

$$\text{Bi}'(z) = \frac{1}{\pi}\int_0^\infty t\left(\cos\left(\frac{t^3}{3}+z\,t\right)+e^{z\,t-\frac{t^3}{3}}\right)dt\,/;\,z<0.$$

Transformations

The argument of the Airy functions $\text{Ai}(z)$ and Bi (z), and their derivatives $\text{Ai}'(z)$ and $\text{Bi}'(z)$ can be simplified for third roots:

$$\text{Ai}\left(\sqrt[3]{z^3}\right) = \frac{1}{2}\left(\frac{\sqrt[3]{z^3}}{z}+1\right)\text{Ai}(z)-\frac{1}{2\sqrt{3}}\left(\frac{\sqrt[3]{z^3}}{z}-1\right)\text{Bi}(z)$$

$$\text{Ai}'\left(\sqrt[3]{z^3}\right) = \frac{1}{2}\left(\frac{\left(z^3\right)^{2/3}}{z^2}+1\right)\text{Ai}'(z)-\frac{1}{2\sqrt{3}}\left(1-\frac{\left(z^3\right)^{2/3}}{z^2}\right)\text{Bi}'(z)$$

$$\text{Bi}\left(\sqrt[3]{z^3}\right) = \frac{1}{2}\left(\sqrt{3}\left(1-\frac{\sqrt[3]{z^3}}{z}\right)\text{Ai}(z)+\left(\frac{\sqrt[3]{z^3}}{z}+1\right)\text{Bi}(z)\right)$$

$$\text{Bi}'\left(\sqrt[3]{z^3}\right) = \frac{1}{2}\left(\left(\frac{\left(z^3\right)^{2/3}}{z^2}+1\right)\text{Bi}'(z)-\sqrt{3}\left(1-\frac{\left(z^3\right)^{2/3}}{z^2}\right)\text{Ai}'(z)\right)$$

Simple representations of derivatives

The derivatives of the Airy functions $\text{Ai}(z)$ and Bi (z), and their derivatives $\text{Ai}'(z)$ and $\text{Bi}'(z)$ have simple representations that can also be expressed through Airy functions:

$$\frac{\partial\,\text{Ai}(z)}{\partial z} = \text{Ai}'(z)$$

$$\frac{\partial\,\text{Ai}'(z)}{\partial z} = z\,\text{Ai}(z)$$

$$\frac{\partial\,\text{Bi}(z)}{\partial z} = \text{Bi}'(z)$$

$$\frac{\partial\,\text{Bi}'(z)}{\partial z} = z\,\text{Bi}(z).$$

The symbolic n^{th}-order derivatives have more complicated representations in terms of the regularized hypergeometric function ${}_2\tilde{F}_3$:

$$\frac{\partial^n \mathrm{Ai}(z)}{\partial z^n} = 3^{n-\frac{4}{3}} z^{-n} \left(3^{2/3} \Gamma\left(\frac{1}{3}\right) {}_2\tilde{F}_3\left(\frac{1}{3}, 1; \frac{1-n}{3}, \frac{2-n}{3}, 1-\frac{n}{3}; \frac{z^3}{9}\right) - z\, \Gamma\left(\frac{2}{3}\right) {}_2\tilde{F}_3\left(\frac{2}{3}, 1; \frac{2-n}{3}, 1-\frac{n}{3}, \frac{4-n}{3}; \frac{z^3}{9}\right)\right) /; n \in \mathbb{N}.$$

$$\frac{\partial^n \mathrm{Ai}'(z)}{\partial z^n} = 3^{n-\frac{8}{3}} z^{-n} \left(\Gamma\left(\frac{1}{3}\right) z^2\, {}_2\tilde{F}_3\left(1, \frac{4}{3}; 1-\frac{n}{3}, \frac{4-n}{3}, \frac{5-n}{3}; \frac{z^3}{9}\right) + 3\sqrt[3]{3}\, \Gamma\left(-\frac{1}{3}\right) {}_2\tilde{F}_3\left(\frac{2}{3}, 1; \frac{1-n}{3}, \frac{2-n}{3}, 1-\frac{n}{3}; \frac{z^3}{9}\right)\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{Bi}(z)}{\partial z^n} = 3^{n-\frac{5}{6}} z^{-n} \left(3^{2/3} \Gamma\left(\frac{1}{3}\right) {}_2\tilde{F}_3\left(\frac{1}{3}, 1; \frac{1-n}{3}, \frac{2-n}{3}, 1-\frac{n}{3}; \frac{z^3}{9}\right) + z\, \Gamma\left(\frac{2}{3}\right) {}_2\tilde{F}_3\left(\frac{2}{3}, 1; \frac{2-n}{3}, 1-\frac{n}{3}, \frac{4-n}{3}; \frac{z^3}{9}\right)\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{Bi}'(z)}{\partial z^n} = 3^{n-\frac{13}{6}} z^{-n} \left(\Gamma\left(\frac{1}{3}\right) {}_2\tilde{F}_3\left(1, \frac{4}{3}; 1-\frac{n}{3}, \frac{4-n}{3}, \frac{5-n}{3}; \frac{z^3}{9}\right) z^2 + 9\sqrt[3]{3}\, \Gamma\left(\frac{2}{3}\right) {}_2\tilde{F}_3\left(\frac{2}{3}, 1; \frac{1-n}{3}, \frac{2-n}{3}, 1-\frac{n}{3}; \frac{z^3}{9}\right)\right) /; n \in \mathbb{N}.$$

Differential equations

The Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ appeared as special solutions of the simple-looking linear second-order differential equation:

$$w''(z) - z\, w(z) = 0 /; w(z) = c_1 \mathrm{Ai}(z) + c_2 \mathrm{Bi}(z),$$

where c_1 and c_2 are arbitrary constants.

Additional restrictions on $w(0)$ and $w'(0)$ lead to corresponding Airy functions:

$$w''(z) - z\, w(z) = 0 /; w(z) = \mathrm{Ai}(z) \bigwedge w(0) = \frac{1}{3^{2/3}\, \Gamma\left(\frac{2}{3}\right)} \bigwedge w'(0) = -\frac{1}{\sqrt[3]{3}\, \Gamma\left(\frac{1}{3}\right)}$$

$$w''(z) - z\, w(z) = 0 /; w(z) = \mathrm{Bi}(z) \bigwedge w(0) = \frac{1}{\sqrt[6]{3}\, \Gamma\left(\frac{2}{3}\right)} \bigwedge w'(0) = \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)}.$$

Similar properties are valid for derivatives of Airy functions:

$$z\, w''(z) - w'(z) - z^2\, w(z) = 0 /; w(z) = c_1 \mathrm{Ai}'(z) + c_2 \mathrm{Bi}'(z)$$

$$z\, w''(z) - w'(z) - z^2\, w(z) = 0 /; w(z) = \mathrm{Ai}'(z) \bigwedge w(0) = -\frac{1}{\sqrt[3]{3}\, \Gamma\left(\frac{1}{3}\right)} \bigwedge w'(0) = 0$$

$$w''(z)\, z - w'(z) - z^2\, w(z) = 0 /; w(z) = \mathrm{Bi}'(z) \bigwedge w(0) = \frac{\sqrt[6]{3}}{\Gamma\left(\frac{1}{3}\right)} \bigwedge w'(0) = 0.$$

Applications of Airy functions

Applications of Airy functions include quantum mechanics of linear potential, electrodynamics, electromagnetism, combinatorics, analysis of the algorithmic complexity, optical theory of the rainbow, solid state physics, radiative transfer, and semiconductors in electric fields.