Introductions to EllipticK

Introduction to the complete elliptic integrals

General

Elliptic integrals were introduced in the investigations of J. Wallis (1655–1659) who studied the integral (in modern notation):

$$E(z) = \int_0^{\pi/2} \sqrt{1 - m\sin^2(t)}\, dt /;\ 0 < m < 1.$$

L. Euler (1733, 1757, 1763, 1766) derived the addition theorem for the following elliptic integrals currently called incomplete elliptic integrals of the first and second kind:

$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m\sin^2(t)}}\, dt /;\ 0 < m < 1$$

$$E(z \mid m) = \int_0^z \sqrt{1 - m\sin^2(t)}\, dt /;\ 0 < m < 1.$$

J.-L. Lagrange (1783) and especially A. M. Legendre (1793, 1811, 1825–1828) devoted considerable attention to study different properties of these integrals. C. F. Gauss (1799, 1818) also used these integrals during his research.

Simultaneously, A. M. Legendre (1811) introduced the incomplete elliptic integral of the third kind:

$$\Pi(n; z \mid m) = \int_0^z \frac{1}{(1 - n\sin^2(t))\sqrt{1 - m\sin^2(t)}}\, dt$$

and the complete versions of the integrals:

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \middle| m\right) = \int_0^{\frac{\pi}{2}} \frac{1}{(1 - n\sin^2(t))\sqrt{1 - m\sin^2(t)}}\, dt$$

$$K(z) = F\left(\frac{\pi}{2} \middle| z\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2(t)}} \, dt$$

$$E(z) = E\left(\frac{\pi}{2} \middle| z\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2(t)} \, dt.$$

C. G. J. Jacobi (1827–1829) introduced inverse functons of the elliptic integrals $F(z \mid m)$ and $E(z \mid m)$, which lead him to build the theory of elliptic functions. In 1829 C. G. J. Jacobi defined the function:

$$\mathrm{Z}(z \mid m) = E(z \mid m) - \frac{E(m)}{K(m)} F(z \mid m),$$

which was later called the Jacobi zeta function. J. Liouville (1840) also studied the elliptic integrals $F(z \mid m)$ and $E(z \mid m)$.

N. H. Abel independently derived some of C. G. J. Jacobi's results and studied the so-called hyperelliptic and Abelian integrals.

Definitions of complete elliptic integrals

The complete elliptic integral of the first kind $K(z)$, the complete elliptic integral of the second kind $E(z)$, and the complete elliptic integral of the third kind $\Pi(n \mid m)$ are defined through the following formulas:

$$K(z) = F\left(\frac{\pi}{2} \middle| z\right)$$

$$E(z) = E\left(\frac{\pi}{2} \middle| z\right)$$

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \middle| m\right),$$

where the incomplete elliptic integrals of the first, second, and third kind are as follows:

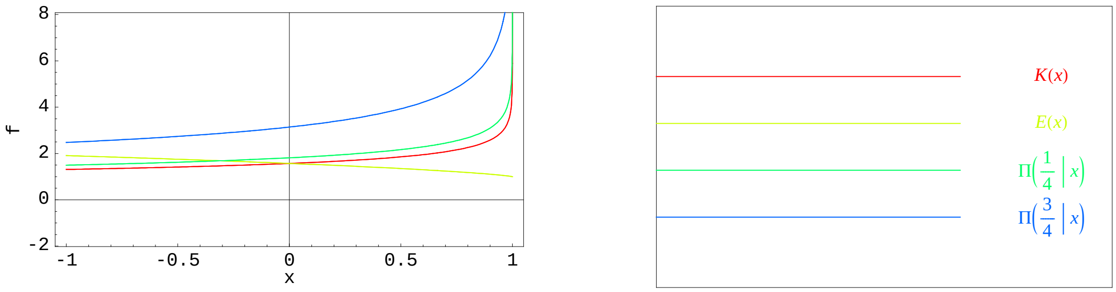
$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m \sin^2(t)}} \, dt$$

$$E(z \mid m) = \int_0^z \sqrt{1 - m \sin^2(t)} \, dt$$

$$\Pi(n; z \mid m) = \int_0^z \frac{1}{\left(1 - n \sin^2(t)\right) \sqrt{1 - m \sin^2(t)}} \, dt.$$

A quick look at the complete elliptic integrals

Here is a quick look at the graphics for the complete elliptic integrals along the real axis.



Connections within the group of complete elliptic integrals and with other function groups

Representations through more general functions

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ can be represented through more general functions.
Through the Gauss
hypergeometric function:

$$K(z) = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$$

$$E(z) = \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right).$$

Through the Meijer G function:

$$K(z) = \frac{1}{2}\, G_{2,2}^{1,2}\left(-z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{matrix}\right)$$

$$E(z) = -\frac{1}{4}\, G_{2,2}^{1,2}\left(-z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right).$$

Through the hypergeometric Appell F_1 function of two variables:

$$K(z) = F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1, z\right)$$

$$E(z) = 2\, F_1\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1, z\right)$$

$$\Pi(n \mid m) = \frac{\pi}{2}\, F_1\left(\frac{1}{2}; \frac{1}{2}, 1; 1; m, n\right).$$

Through the hypergeometric function of two variables:

$$\Pi(n \mid m) = \frac{\pi}{2} F_{1,0,0}^{1,1,1}\left(\begin{matrix} \frac{1}{2};\ 1;\ \frac{1}{2}; \\ 1;;; \end{matrix}\ n,\ m\right).$$

Through the incomplete elliptic integrals:

$$K(z) = \Pi\left(0; \frac{\pi}{2} \middle| z\right)$$

$$K(z) = F\left(\frac{\pi}{2} \middle| z\right)$$

$$K(z) = \frac{1}{\sqrt{z}} F\left(\sin^{-1}\left(\sqrt{z}\right) \middle| \frac{1}{z}\right)$$

$$E(z) = (1 - z)\, \Pi\left(z; \frac{\pi}{2} \middle| z\right)$$

$$E(z) = E\left(\frac{\pi}{2} \middle| z\right)$$

$$E(z) = (1 - z)\, \Pi\left(z; \frac{\pi}{2} \middle| z\right)$$

$$\Pi(n \mid m) = K(m) - \frac{\tan(\phi)}{\sqrt{1 - n}} (E(m)\, F(\phi \mid m) - K(m)\, E(\phi \mid m)) \text{ /; } \phi = \sin^{-1}\left(\sqrt{\frac{n}{m}}\right) \bigwedge 0 < n < 1 \bigwedge 0 < m < 1$$

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \middle| m\right)$$

$$\Pi(n \mid m) = \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m}; \sin^{-1}\left(\sqrt{m}\right) \middle| \frac{1}{m}\right).$$

Through the elliptic theta functions:

$$K(z) = \frac{\pi}{2} \vartheta_3(0, q(z))^2.$$

Through the arithmetic geometric mean:

$$K(z) = \frac{\pi}{2\, \mathrm{agm}\left(1, \sqrt{1 - z}\right)}.$$

Through the Jacobi elliptic functions:

$$K(z) = \mathrm{sn}^{-1}(1 \mid z)$$

$$K(z) = \mathrm{dn}^{-1}\left(\sqrt{1 - z} \mid z\right)$$

$$K(z) = \mathrm{cn}^{-1}(0 \mid z) \text{ /; } z \in \mathbb{R} \land z < 1.$$

Through the Weierstrass elliptic functions and inverse elliptic nome $q^{-1}(t)$:

$$K\left(q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_2}{\omega_1}\right)\right)\right) = \sqrt{e_1 - e_3}\;\omega_1 \;/;$$
$$\{e_1, e_2, e_3\} = \{\wp(\omega_1; g_2, g_3), \wp(\omega_1 + \omega_2; g_2, g_3), \wp(\omega_2; g_2, g_3)\} \bigwedge \{g_2, g_3\} = \{g_2(\omega_1, \omega_2), g_3(\omega_1, \omega_2)\}$$

$$\frac{K(1-z)}{K(z)} = -\frac{i\,\omega_2}{\omega_1} \;/;\; z = q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_2}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_2\} = \{\omega_1(g_2, g_3), \omega_2(g_2, g_3)\}.$$

Through the Legendre P and Q functions:

$$K(z) = \frac{\pi}{2} P_{-\frac{1}{2}}(1 - 2\,z)$$

$$K(z) = Q_{-\frac{1}{2}}(2\,z - 1)$$

$$E(z) = \frac{\pi}{4}\left(P_{\frac{1}{2}}(1 - 2\,z) + P_{-\frac{1}{2}}(1 - 2\,z)\right)$$

$$E(z) = \frac{1}{2}\left(Q_{-\frac{1}{2}}(2\,z - 1) - Q_{\frac{1}{2}}(2\,z - 1)\right).$$

Relations to inverse functions

The complete elliptic integral $K(z)$ is related to Jacobi amplitude by the following formula, which demonstrates that Jacobi amplitude is the some kind of inverse function to the elliptic integral $K(z)$:

$$\operatorname{am}(K(m) \mid m) = \frac{\pi}{2}.$$

Representations through other complete elliptic integrals

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ can be represented through other complete elliptic integrals by the following formulas:

$$K(z) = \Pi(0 \mid z)$$

$$E(z)\,K(1-z) - K(z)\,K(1-z) + E(1-z)\,K(z) = \frac{\pi}{2}$$

$$E(z) = (1 - z)\,\Pi(z \mid z)$$

$$\Pi(n \mid n) = \frac{E(n)}{1 - n}.$$

The best-known properties and formulas for complete elliptic integrals

Real values for real restricted arguments

For real values of arguments z, n, and m (with $z < 1$, $n < 1$, $m < 1$) the values of all complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ are real.

Simple values at zero

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ are equal to $\frac{\pi}{2}$ at the origin:

$$K(0) = \frac{\pi}{2} \quad E(0) = \frac{\pi}{2} \quad \Pi(0 \mid 0) = \frac{\pi}{2}.$$

Specific values

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ can be represented through elementary or other functions when $z = \frac{1}{2}$, 1, or -1, $m = 0$ or 1, or $n = 0$ or 1:

$$K\left(\frac{1}{2}\right) = \frac{8\pi^{3/2}}{\Gamma\left(-\frac{1}{4}\right)^2} \qquad E(1) = 1 \qquad \Pi(n \mid 0) = \frac{\pi}{2\sqrt{1-n}} \qquad \Pi(0 \mid m) = K(m)$$
$$K(1) = \tilde{\infty} \qquad \sqrt{2}\, E\left(\frac{1}{2}\right) - E(-1) = 0 \qquad \Pi(n \mid 1) = -\frac{\infty}{\operatorname{sgn}(n-1)} \qquad \Pi(1 \mid m) = \tilde{\infty}$$
$$K(-1) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{4\sqrt{2\pi}} \qquad \Pi(n \mid n) = \frac{E(n)}{1-n}.$$

At any infinity, the complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ have the following values:

$$K(\infty) = 0 \qquad E(\infty) = i\,\infty \qquad \Pi(\infty \mid m) = 0$$
$$K(-\infty) = 0 \qquad E(-\infty) = \infty \qquad \Pi(-\infty \mid m) = 0$$
$$K(i\,\infty) = 0 \qquad E(i\,\infty) = -(-1)^{3/4}\,\infty \qquad \Pi(n \mid \infty) = 0$$
$$K(-i\,\infty) = 0 \qquad E(-i\,\infty) = \sqrt[4]{-1}\,\infty \qquad \Pi(n \mid -\infty) = 0$$
$$K(\tilde{\infty}) = 0 \qquad E(\tilde{\infty}) = \tilde{\infty}.$$

Analyticity

The complete elliptic integrals $K(z)$ and $E(z)$ are analytical functions of z, which are defined over the whole complex z-plane.

The complete elliptic integral $\Pi(n \mid m)$ is an analytical function of n and m, which is defined over $\mathbb{C}^2$.

Poles and essential singularities

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ do not have poles and essential singularities.

Branch points and branch cuts

The complete elliptic integrals $K(z)$ and $E(z)$ have two branch points: $z = 1$ and $z = \tilde{\infty}$.

They are single-valued functions on the z-plane cut along the interval $(1, \infty)$. They are continuous from below on the interval $(1, \infty)$:

$$\lim_{\epsilon \to +0} K(x - i\,\epsilon) = K(x) \;/;\; x > 1$$

$$\lim_{\epsilon \to +0} K(x + i\,\epsilon) = 2\,i\,K(1 - x) + K(x) \;/;\; x > 1$$

$$\lim_{\epsilon \to +0} E(x - i\,\epsilon) = E(x) \;/;\; x > 1$$

$$\lim_{\epsilon\to+0} E(x+i\,\epsilon) == 2\,i\,(K(1-x)-E(1-x))+E(x)\;/;\;x>1.$$

For fixed n, the function $\Pi(n\mid m)$ has two branch points at $m == 1$ and $m == \tilde{\infty}$. For fixed m, the function $\Pi(n\mid m)$ has two branch points at $n == 1$ and $n == \tilde{\infty}$.

Periodicity

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n\mid m)$ are not periodical functions.

Parity and symmetry

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n\mid m)$ have mirror symmetry:

$$K(\bar{z}) == \overline{K(z)}\;/;\;z\notin(1,\infty)$$

$$E(\bar{z}) == \overline{E(z)}\;/;\;z\notin(1,\infty)$$

$$\Pi(\bar{n}\mid\bar{m}) == \overline{\Pi(n\mid m)}.$$

Series representations

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n\mid m)$ have the following series expansions at the point $z == 0$:

$$K(z)\propto\frac{\pi}{2}\left(1+\frac{z}{4}+\frac{9\,z^2}{64}+\ldots\right)/;\;(z\to0)$$

$$K(z) == \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k z^k}{k!^2}\;/;\;|z|<1$$

$$E(z)\propto\frac{\pi}{2}\left(1-\frac{z}{4}-\frac{3\,z^2}{64}-\ldots\right)/;\;(z\to0)$$

$$E(z) == \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k z^k}{k!^2}\;/;\;|z|<1$$

$$\Pi(n\mid m)\propto\frac{\pi}{2}\left(1+\frac{1}{4}(m+2\,n)+\frac{3}{64}\left(3\,m^2+4\,m\,n+8\,n^2\right)+\frac{5}{256}\left(5\,m^3+6\,m^2\,n+8\,m\,n^2+16\,n^3\right)+\ldots\right)/;\;(m\to0)\land(n\to0)$$

$$\Pi(n\mid m) == \frac{\pi}{2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(2\,k)!\,(2\,j)!\,m^j\,n^{k-j}}{4^k\,4^j\,k!^2\,j!^2}\;/;\;|m|<1\land|n|<1.$$

The complete elliptic integrals $K(z)$ and $E(z)$ have the following series expansions at the point $z == 1$:

$$K(z)\propto-\frac{1}{2}\log(1-z)\left(1-\frac{z-1}{4}+\frac{9}{64}(z-1)^2+\ldots\right)+\log(4)+\frac{1}{4}(1-\log(4))(z-1)+\frac{3}{128}(6\log(4)-7)(z-1)^2+\ldots\;/;\;(z\to1)$$

$$K(z) == -\frac{1}{2}\log(1-z)\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2}{k!^2}(z-1)^k+\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k^2}{k!^2}\left(\psi(k+1)-\psi\left(k+\frac{1}{2}\right)\right)(z-1)^k\;/;\;|z-1|<1$$

$$E(z) \propto 1 + \frac{z-1}{4}\left(-2\log(4) + 1 + \frac{24\log(2) - 13}{16}(z-1) - \frac{3(5\log(2) - 3)}{16}(z-1)^2 + \ldots\right) + \log(1-z)\,\frac{z-1}{4}\left(1 + \frac{3(1-z)}{8} + \frac{15}{64}(1-z)^2 + \ldots\right) /; (z \to 1)$$

$$E(z) = 1 + \frac{1}{4}(z-1)\log(1-z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}(1-z)^k + \frac{z-1}{4}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k(1-z)^k}{k!\,(k+1)!}\left(-2\,\psi(k+1) + 2\,\psi\left(k+\frac{1}{2}\right) + \frac{1}{2k^2+3k+1}\right) /; |z-1| < 1.$$

The complete elliptic integrals $K(z)$ and $E(z)$ have the following series expansions at the point $|z| = \infty$:

$$K(z) \propto \frac{\log(-z)}{2\sqrt{-z}}\left(1 + \frac{1}{4z} + \frac{9}{64z^2} + \ldots\right) + \frac{1}{\sqrt{-z}}\left(\log(4) + \frac{\log(4)-1}{4z} + \frac{3(6\log(4)-7)}{128z^2} + \ldots\right) /; (|z| \to \infty)$$

$$K(z) = \frac{\log(-z)}{2\sqrt{-z}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2 z^{-k}}{k!^2} + \frac{1}{\sqrt{-z}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{k!^2}\left(\psi(k+1) - \psi\left(\frac{1}{2}-k\right)\right)z^{-k} /; |z| > 1$$

$$E(z) \propto \sqrt{-z} + \frac{1}{\sqrt{-z}}\left(\frac{1}{4} + \log(2) + \frac{8\log(2)-3}{64z} + \frac{6\log(2)-3}{128z^2} + \ldots\right) + \frac{\log(-z)}{4\sqrt{-z}}\left(1 + \frac{1}{8z} + \frac{3}{64z^2} + \ldots\right) /; (|z| \to \infty)$$

$$E(z) = \sqrt{-z} + \frac{\log(-z)}{\sqrt{-z}}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{k+1}^2 z^{-k}}{k!\,(k+1)!} + \frac{1}{\sqrt{-z}}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{k+1}^2 z^{-k}}{k!\,(k+1)!}\left(2\,\psi(k+1) - 2\,\psi\left(\frac{1}{2}-k\right) + \frac{1}{k+1}\right) /; |z| > 1.$$

The complete elliptic integral $\Pi(n \mid m)$ has the following series expansions at the point $n = 1$:

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}} + \frac{E(m)}{m-1} + K(m) + \frac{(m+1)E(m) + (m-1)K(m)}{3(m-1)^2}(n-1) - \frac{\left(2m^2-7m-3\right)E(m) - \left(m^2+2m-3\right)K(m)}{15(m-1)^3}(n-1)^2 + \ldots /; (n \to 1)$$

$$\Pi(n \mid m) = \frac{\pi(-1)^{\left\lfloor \frac{1}{2} - \frac{\operatorname{Arg}(1-m)}{2\pi} - \frac{\operatorname{Arg}\left(\frac{n-m}{n(1-m)}\right)}{2\pi}\right\rfloor}}{2\sqrt{1-m}\,\sqrt{\frac{n-m}{n(1-m)}}\,\sqrt{1-n}} - \frac{\pi m}{4}\sum_{k=0}^{\infty}(-1)^k\,{}_2F_1\left(\frac{3}{2}, k+\frac{3}{2}; 2; m\right)(n-1)^k.$$

The complete elliptic integral $\Pi(n \mid m)$ has the following series expansions at the point $m = 1$:

$$\Pi(n \mid m) \propto \frac{1}{2}\left(-\frac{\log(1-m)}{1-n}\left(1+\frac{(n+1)(m-1)}{4(n-1)}-\frac{3\left(n^2-6n-3\right)(m-1)^2}{64(n-1)^2}+\ldots\right)+\frac{\sqrt{n}\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)-4\log(2)}{n-1}-\frac{1}{2(n-1)^2}\left(2n\log(2)+2\log(2)+\sqrt{n}\left(\log\left(1-\sqrt{n}\right)-\log\left(\sqrt{n}+1\right)\right)-1\right)(m-1)+\frac{1}{64(n-1)^3}\left(-5n^2+12n+24\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)\sqrt{n}+12((n-6)n-3)\log(2)+21\right)(m-1)^2+\ldots\right)\text{/; } (m \to 1)$$

$$\Pi(n \mid m) = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{(1-m)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\sum_{j=0}^{\infty}\frac{\left(k+\frac{1}{2}\right)_j\left(2\psi(k+1)-\psi\left(k+\frac{1}{2}\right)-\psi\left(k+j+\frac{1}{2}\right)\right)n^j}{\left(\frac{1}{2}\right)_j}-\log(1-m)\sum_{j=0}^{\infty}\frac{(1-m)^j\left(\frac{1}{2}\right)_j^2}{(j!)^2}\,{}_2F_1\left(1,\, j+\frac{1}{2};\, \frac{1}{2};\, n\right)\right)\text{/; } |m-1|<1.$$

The complete elliptic integral $\Pi(n \mid m)$ has the following series expansions at the point $|n| == \infty$:

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{-n}}\left(1+\frac{m+1}{2n}+\frac{3m^2+2m+3}{8n^2}+\ldots\right)+\frac{1}{4n}\left(4(E(m)-K(m))-\frac{4(m+2)K(m)-8(m+1)E(m)}{3\,n}+\frac{4\left(\left(8m^2+7m+8\right)E(m)-\left(4m^2+3m+8\right)K(m)\right)}{15n^2}+\ldots\right)\text{/; } (|n| \to \infty)$$

$$\Pi(n \mid m) = \frac{\pi}{2\sqrt{-n}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\,{}_2F_1\left(\frac{1}{2}, -k;\, \frac{1}{2}-k;\, m\right)n^{-k}-\frac{\pi m}{4n}\sum_{k=0}^{\infty}\frac{\left(\frac{3}{2}\right)_k m^k}{(k+1)!}\,{}_2F_1\left(\frac{1}{2}, k+\frac{3}{2};\, k+2;\, m\right)n^{-k}.$$

The complete elliptic integral $\Pi(n \mid m)$ has the following series expansions at the point $|m| == \infty$:

$$\Pi(n \mid m) \propto \frac{\log(-m)}{2\sqrt{-m}}\left(1+\frac{2n+1}{4m}+\frac{3\left(8n^2+4n+3\right)}{64m^2}+\ldots\right)+\frac{1}{2\sqrt{-m}}\left(4\log(2)+\frac{2\log(2)+n(4\log(2)-1)-1}{2m}-\frac{28n^2+26n-12\left(8n^2+4n+3\right)\log(2)+21}{64m^2}+\ldots\right)+\frac{\sqrt{n}\,\sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\,\sqrt{-m}}\left(1+\frac{n}{2m}+\frac{3n^2}{8m^2}+\ldots\right)\text{/; } (|m| \to \infty)$$

$$\Pi(n \mid m) = \frac{\sqrt{n}\,\sin^{-1}(\sqrt{n})}{\sqrt{1-n}\,\sqrt{-m}} \sum_{k=0}^{\infty} \frac{n^k \left(\frac{1}{2}\right)_k}{k!} m^{-k} + \frac{\log(-m)}{2\sqrt{-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} {}_2F_1\left(1, -k; \frac{1}{2}-k; n\right) m^{-k} + \frac{1}{2\sqrt{-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{k!} \sum_{j=0}^{k} \frac{(-n)^j \left(-\psi\left(j-k+\frac{1}{2}\right) + \psi(k+1) - \psi\left(k+\frac{1}{2}\right) + \psi(-j+k+1)\right)}{\left(\frac{1}{2}-k\right)_j (k-j)!} m^{-k}.$$

The previous formulas can be rewritten in summed forms of the truncated series expansion near corresponding points $z == 0$, 1, or $\tilde{\infty}$:

$$K(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{\pi}{2} \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k z^k}{k!^2} == K(z) - \frac{z^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)^2}{2\,(n+1)!^2} {}_3F_2\left(1, n+\frac{3}{2}, n+\frac{3}{2}; n+2, n+2; z\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$K(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{1}{2} \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} \left(-\log(1-z) + 2\,\psi(k+1) - 2\,\psi\left(k+\frac{1}{2}\right)\right)(1-z)^k == K(z) - \frac{1}{2\pi} G_{4,4}^{2,4}\left(1-z \,\middle|\, \begin{matrix} n+1, n+1, \frac{1}{2}, \frac{1}{2} \\ n+1, n+1, 0, 0 \end{matrix}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$K(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{1}{2\sqrt{-z}} \sum_{k=0}^{m} \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} \left(\log(-z) + 2\,\psi(k+1) - \psi\left(\frac{1}{2}-k\right) - \psi\left(k+\frac{1}{2}\right)\right) z^{-k} == K(z) - \frac{1}{2} G_{4,4}^{3,2}\left(-z \,\middle|\, \begin{matrix} -m-\frac{1}{2}, -m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, -m-\frac{1}{2}, -m-\frac{1}{2}, 0 \end{matrix}\right)\right) \bigwedge m \in \mathbb{N}\right)$$

$$E(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{\pi}{2} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k z^k}{k!^2} == E(z) + \frac{z^{n+1}\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{4\,(n+1)!^2} {}_3F_2\left(1, n+\frac{1}{2}, n+\frac{3}{2}; n+2, n+2; z\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$E(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{1-z}{4} \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_k}{k!\,(k+1)!} \left(-\log(1-z) + 2\,\psi(k+1) - \psi\left(k+\frac{1}{2}\right) - \psi\left(k+\frac{3}{2}\right) + \frac{1}{k+1}\right)(1-z)^k + 1 == E(z) - \frac{1}{2\pi} G_{4,4}^{2,4}\left(1-z \,\middle|\, \begin{matrix} n+2, n+2, \frac{1}{2}, \frac{3}{2} \\ n+2, n+2, 0, 1 \end{matrix}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$E(z) == F_\infty(z) /; \left(\left(F_n(z) == \frac{1}{\sqrt{-z}} \sum_{k=0}^{m} \frac{1}{k!\,(k+1)!} \left(-\frac{1}{2}\right)_{k+1}^2 \left(\log(-z) + 2\,\psi(k+1) - \psi\left(\frac{1}{2}-k\right) - \psi\left(k+\frac{1}{2}\right) + \frac{1}{k+1}\right) z^{-k} + \sqrt{-z} == E(z) - \frac{1}{4} G_{4,4}^{3,2}\left(-z \,\middle|\, \begin{matrix} -m-\frac{1}{2}, -m-\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \\ 0, -m-\frac{1}{2}, -m-\frac{1}{2}, 0 \end{matrix}\right)\right) \bigwedge m \in \mathbb{N}\right).$$

Other series representations

Some elliptic integrals have special series representations through the elliptic nome $q(z)$ and inverse Jacobi elliptic functions by the formulas:

$$K(z) = \frac{\pi}{2}\left(2\sum_{k=1}^{\infty} q(z)^{k^2} + 1\right)^2$$

$$K(z) = \frac{\pi}{2}\left(1 + 4\sum_{k=1}^{\infty} \frac{q(z)^k}{q(z)^{2k} + 1}\right)$$

$$\Pi(n \mid m) = \sqrt{\frac{n}{(m-n)(n-1)}}\; K(m)\left(\frac{2 i \pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^k}{1 - q(m)^{2k}} \sin\left(\frac{k\pi}{K(m)} \operatorname{sn}^{-1}\left(\sqrt{\frac{n}{m}} \,\middle|\, m\right)\right) + \sqrt{\frac{(m-n)(n-1)}{n}}\right) /;$$
$$-1 \leq n \leq 1 \land -1 \leq m \leq 1.$$

Integral representations

The complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ have the following integral representations:

$$K(z) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - z \sin^2(t)}}\, dt \;/;\; |\operatorname{Arg}(1-z)| < \pi$$

$$K(z) = \int_0^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1 - z t^2}}\, dt \;/;\; |\operatorname{Arg}(1-z)| < \pi$$

$$K(z) = \int_1^{\infty} \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2 - z}}\, dt \;/;\; |\operatorname{Arg}(1-z)| < \pi$$

$$E(z) = \int_0^{\frac{\pi}{2}} \sqrt{1 - z \sin^2(t)}\, dt \;/;\; |\operatorname{Arg}(1-z)| < \pi$$

$$E(z) = \int_0^1 \frac{\sqrt{1 - z t^2}}{\sqrt{1-t^2}}\, dt \;/;\; |\operatorname{Arg}(1-z)| < \pi$$

$$\Pi(n \mid m) = \int_0^{\frac{\pi}{2}} \frac{1}{\left(1 - n \sin^2(t)\right)\sqrt{1 - m \sin^2(t)}}\, dt$$

$$\Pi(n \mid m) = \int_0^1 \frac{1}{\left(1 - n t^2\right)\sqrt{1-t^2}\,\sqrt{1 - m t^2}}\, dt$$

$$\Pi(n \mid m) = \int_0^{K(m)} \frac{1}{1 - n \operatorname{sn}(t \mid m)^2}\, dt.$$

Identities

The complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ satisfy numerous identities, for example:

$$K(z) == \frac{1}{\sqrt{1-z}} K\left(\frac{z}{z-1}\right) /; z \notin \{1, \infty\}$$

$$K\left(\frac{1}{z}\right) == \sqrt{z}\, (K(z) - i\, K(1-z)) /; \operatorname{Im}(z) > 0$$

$$K(z) == \frac{2}{z_1 + 1} K\left(\left(\frac{1-z_1}{1+z_1}\right)^2\right) /; z_1 == \sqrt{1-z}$$

$$E\left(1 - \frac{1}{z}\right) == \frac{E(1-z)}{\sqrt{z}} /; |\operatorname{Arg}(z)| < \pi$$

$$E(z) == \sqrt{1-z}\, E\left(\frac{z}{z-1}\right) /; |\operatorname{Arg}(1-z)| < \pi.$$

Representations of derivatives

The first derivatives of all complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ with respect to their variables can also be represented through complete
elliptic integrals by the following formulas:

$$\frac{\partial K(z)}{\partial z} == \frac{E(z) - (1-z) K(z)}{2 (1-z) z}$$

$$\frac{\partial E(z)}{\partial z} == \frac{E(z) - K(z)}{2 z}$$

$$\frac{\partial \Pi(n \mid m)}{\partial n} == \frac{1}{2 (m-n) (n-1)} \left(E(m) + \frac{m-n}{n} K(m) + \frac{n^2 - m}{n} \Pi(n \mid m)\right)$$

$$\frac{\partial \Pi(n \mid m)}{\partial m} == \frac{1}{2 (n-m)} \left(\frac{E(m)}{m-1} + \Pi(n \mid m)\right).$$

The symbolic n^{th} derivatives of all complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ with respect to their variables can be represented through Gauss classical
or regularized hypergeometric functions by the following formulas:

$$\frac{\partial^n K(z)}{\partial z^n} == \frac{(-1)^n \pi\, \Gamma\left(n + \frac{1}{2}\right)}{2\, n!\, \Gamma\left(\frac{1}{2} - n\right)} {}_2F_1\left(n + \frac{1}{2}, n + \frac{1}{2}; n+1; z\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n K(z)}{\partial z^n} == \frac{\pi z^{-n}}{2} {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; z\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n E(z)}{\partial z^n} == -\frac{\Gamma\left(n - \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{4\, n!} {}_2F_1\left(n - \frac{1}{2}, n + \frac{1}{2}; n+1; z\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n E(z)}{\partial z^n} == \frac{\pi z^{-n}}{2} {}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; 1-n; z\right) /; n \in \mathbb{N}$$

$$\frac{\partial^p \Pi(n \mid m)}{\partial n^p} = \frac{\sqrt{\pi}\; n^{-p}}{2} \sum_{k=0}^{\infty} \frac{n^k\, \Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k-p+1)}\, {}_2F_1\left(k+\frac{1}{2}, \frac{1}{2}; k+1; m\right) /; p \in \mathbb{N}$$

$$\frac{\partial^p \Pi(n \mid m)}{\partial m^p} = \frac{m^{-p}}{2} \sum_{k=0}^{\infty} \frac{m^k\, \Gamma\left(k+\frac{1}{2}\right)^2}{\Gamma(k-p+1)\, k!}\, {}_2F_1\left(1, k+\frac{1}{2}; k+1; n\right) /; p \in \mathbb{N}.$$

Integration

The indefinite integrals of all complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ with respect to their variables can be expressed through complete elliptic integrals (or through hypergeometric functions of two variables) by the following formulas:

$$\int K(z)\, dz = 2\,(E(z) + (z-1)\, K(z))$$

$$\int E(z)\, dz = \frac{2}{3}\,((z+1)\, E(z) + (z-1)\, K(z))$$

$$\int \Pi(n \mid m)\, dm = 2\,(E(m) - K(m) + (m-n)\, \Pi(n \mid m))$$

$$\int \Pi(n \mid m)\, dn = \frac{\pi\, n}{2}\, F_{1\times 1\times 0}^{1\times 2\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}, 1; 1; \\ 1;; 2; \end{matrix}\; m, n\right).$$

Differential equations

All complete elliptic integrals $K(z)$, $E(z)$, and $\Pi(n \mid m)$ satisfy ordinary linear differential equations:

$$(1-z)\, z\, w''(z) + (1-2z)\, w'(z) - \frac{1}{4}\, w(z) = 0 /; w(z) = c_1\, K(z) + c_2\, K(1-z)$$

$$(1-z)\, z\, w''(z) + (1-z)\, w'(z) + \frac{1}{4}\, w(z) = 0 /; w(z) = c_1\, E(z) + c_2(K(1-z) - E(1-z))$$

$$8\,(m-1)\, m\,(m-n)\, \frac{\partial^3 w(m)}{\partial m^3} + 4\left(11\, m^2 - 6\, n\, m - 7\, m + 2\, n\right) \frac{\partial^2 w(m)}{\partial m^2} + 6\,(7\, m - n - 2)\, \frac{\partial w(m)}{\partial m} + 3\, w(m) = 0 /; w(m) = \Pi(n \mid m)$$

$$2\,(n-1)\,(m-n)\, n\, \frac{\partial^3 w(n)}{\partial n^3} + \left(-13\, n^2 + 8\, m\, n + 8\, n - 3\, m\right) \frac{\partial^2 w(n)}{\partial n^2} + 4\,(m - 4\, n + 1)\, \frac{\partial w(n)}{\partial n} - 2\, w(n) = 0 /; w(n) = \Pi(n \mid m).$$

Applications of complete elliptic integrals

Applications of complete elliptic integrals include geometry, physics, mechanics, electrodynamics, statistical mechanics, astronomy, geodesy, geodesics on conics, and magnetic field calculations.

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