BernoulliB

Notations

Traditional name

Bernoulli number

Traditional notation

B_n

Mathematica StandardForm notation

`BernoulliB[n]`

Primary definition

04.13.02.0001.01

$$B_n = n! \left([t^n] \, \frac{t}{e^t - 1} \right) /; n \in \mathbb{N}$$

Specific values

Specialized values

04.13.03.0001.01

$$B_{2n+1} = 0 /; n \in \mathbb{N}^+$$

Values at fixed points

04.13.03.0002.01

$$B_0 = 1$$

04.13.03.0003.01

$$B_1 = -\frac{1}{2}$$

04.13.03.0004.01

$$B_2 = \frac{1}{6}$$

04.13.03.0005.01

$$B_3 = 0$$

04.13.03.0006.01

$$B_4 == -\frac{1}{30}$$

04.13.03.0007.01

$$B_5 == 0$$

04.13.03.0008.01

$$B_6 == \frac{1}{42}$$

04.13.03.0009.01

$$B_7 == 0$$

04.13.03.0010.01

$$B_8 == -\frac{1}{30}$$

04.13.03.0011.01

$$B_9 == 0$$

04.13.03.0012.01

$$B_{10} == \frac{5}{66}$$

General characteristics

Domain and analyticity

B_n is a nonanalytical function which is defined only for nonnegative integer n.

04.13.04.0001.01

$$n \longrightarrow B_n :: \mathbb{Z} \longrightarrow \mathbb{Q}$$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Generalized power series

04.13.06.0006.01

$$B_n == \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n \text{ /; } n \in \mathbb{N}^+$$

04.13.06.0008.01

$$B_n = \sum_{m=0}^{n} \frac{(-1)^m}{m+1} \sum_{i=0}^{m-1} (-1)^i (m-i)^n \binom{m}{i} /; n \in \mathbb{N}$$

04.13.06.0007.01

$$B_n = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j j^n \frac{\binom{n+1}{k-j}}{\binom{n}{k}} /; n \in \mathbb{N}^+$$

04.13.06.0001.01

$$B_n = -\frac{2\, n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{1}{k^n} \cos\left(\frac{\pi n}{2}\right) /; n-1 \in \mathbb{N}^+$$

04.13.06.0002.01

$$B_{2n} = \frac{(-1)^{n-1}\, 2\, (2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} /; n \in \mathbb{N}^+$$

04.13.06.0009.01

$$B_{2n} = A_n - \sum_{(p_k-1)|(2n)} \frac{1}{p_k} /; A_n \in \mathbb{Z} \bigwedge p_k = \text{prime}(k)$$

von Staudt–Clausen theorem

04.13.06.0010.01

$$B_n = \frac{2^{-n}\, n}{-1+2^n} \sum_{k=0}^{n-1} (-1)^{k+1}\, 2^{-k+n-1}\, k!\, \mathcal{S}_{n-1}^{(k)} /; n \in \mathbb{N}^+$$

Victor Adamchik

Asymptotic series expansions

04.13.06.0003.01

$$B_{2n} \propto 4\sqrt{\pi n}\, (-1)^{n-1} \left(\frac{n}{\pi e}\right)^{2n} \left(1 + O\left(\frac{1}{n}\right)\right) /; (n \to \infty)$$

Residue representations

04.13.06.0005.01

$$B_n = n!\, \text{res}_z\left(\frac{z^{-n}}{e^z - 1}\right)(0)$$

Integral representations

On the real axis

Of the direct function

04.13.07.0009.01

$$B_n = -\frac{2n}{(2\pi)^n} \cos\left(\frac{\pi n}{2}\right) \int_0^1 \frac{1}{1-t} \log^{n-1}\left(\frac{1}{t}\right) dt /; n-1 \in \mathbb{N}^+$$

04.13.07.0014.01

$$B_n = \frac{n\,(n-1)}{4\,(2^n-1)} \int_0^1 E_{n-2}(x)\,dx \,/;\, n-1 \in \mathbb{N}^+$$

04.13.07.0001.01

$$B_{2n} = (-1)^{n-1}\,4\,n \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0002.01

$$B_{2n} = \frac{(-1)^{n-1}\,4\,n}{1-2^{1-2n}} \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0003.01

$$B_{2n} = (-1)^{n-1}\,2\,n \int_0^\infty t^{2n-1}\,e^{-\pi t}\,\operatorname{csch}(\pi t)\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0004.01

$$B_{2n} = \frac{(-1)^{n-1}\,2\,n}{1-2^{1-2n}} \int_0^\infty t^{2n-1}\,e^{-\pi t}\,\operatorname{sech}(\pi t)\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0005.01

$$B_{2n} = \frac{(-1)^{n-1}\,2\,n}{2^{2n}-1} \int_0^\infty t^{2n-1}\,\operatorname{csch}(\pi t)\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0006.01

$$B_{2n} = (-1)^{n-1}\,\pi \int_0^\infty t^{2n}\,\operatorname{csch}^2(\pi t)\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0007.01

$$B_{2n} = \frac{(-1)^{n-1}\,\pi}{1-2^{1-2n}} \int_0^\infty t^{2n}\,\operatorname{sech}^2(\pi t)\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0008.01

$$B_{2n} = \frac{(-1)^n\,2\,n\,(2\,n-1)}{\pi} \int_0^\infty t^{2n-2}\,\log\left(1-e^{-2\pi t}\right)dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0010.01

$$B_{2n} = \frac{(-1)^n\,4\,n\,(2\,n-1)}{(2\,\pi)^{2n}} \int_0^1 \frac{\log(1-t)\,\log^{2n-2}(t)}{t}\,dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0011.01

$$B_n = -n\,(-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} \int_0^\infty \frac{t^{n-1}}{\cosh(2\,\pi\,t)-1} \left((-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \cos\left(\frac{\pi\,n}{2}\right) - \left(-n+2\left\lfloor \frac{n}{2} \right\rfloor+1\right) e^{-2\pi t}\right) dt \,/;\, n \in \mathbb{N}^+$$

04.13.07.0012.01

$$B_{2n+2} = \frac{2\,(-1)^{n-1}\,(2\,n+1)\,(2\,n+2)}{(2\,\pi)^{2n+2}} \int_0^1 \frac{\log^{2n}(t)\,\log(1-t)}{t}\,dt \,/;\, n \in \mathbb{N}^+$$

Contour integral representations

04.13.07.0013.01

$$B_n = \frac{1}{2\,\pi\,i} \int_{|z|=1} \frac{z^{-n}}{e^z-1}\,dz \,/;\, n \in \mathbb{N}^+$$

Limit representations

04.13.09.0001.01

$$B_n = \lim_{z\to 0} \frac{\partial^n \frac{z}{e^z-1}}{\partial z^n}$$

Continued fraction representations

04.13.10.0001.01

$$2^z = \cfrac{z}{1+\cfrac{z}{\left|\left(-\frac{1}{2}\right)^z\right| + \cfrac{z}{6^{-\mathrm{Re}(z)} + \cfrac{z}{\cfrac{z}{\left|\left(-\frac{1}{30}\right)^z\right| + \ldots}}}}} \;/;\; \mathrm{Re}(z) > 0$$

04.13.10.0002.01

$$2^z = \underset{k=0}{\overset{\infty}{\mathrm{K}}} \frac{z}{\left|B_k^z\right|} \;/;\; \mathrm{Re}\,(z) > 0$$

Generating functions

04.13.11.0001.01

$$B_n = n!\left([t^n]\,\frac{t}{e^t-1}\right) /;\; n \in \mathbb{N}$$

04.13.11.0004.01

$$B_n = \frac{n!}{2^{n-1}}\left([t^n]\,\frac{t}{e^{2t}-1}\right) /;\; n \in \mathbb{N}$$

04.13.11.0005.01

$$B_n = \frac{n!}{2^n}\left([t^n]\,t\,e^{-t}\,\mathrm{csch}(t)\right) /;\; n \in \mathbb{N}$$

04.13.11.0006.01

$$B_n = \frac{n!}{2}\left([t^n]\,t\,e^{-\frac{t}{2}}\,\mathrm{csch}\left(\frac{t}{2}\right)\right) /;\; n \in \mathbb{N}$$

04.13.11.0002.01

$$B_{2n} = (2n)!\left(\left[t^{2n-1}\right]\left(\frac{e^t}{e^t-1} - \frac{1}{2} - \frac{1}{t}\right)\right) /;\; n \in \mathbb{N}$$

04.13.11.0003.01

$$B_{2n} = 2n(2n)!\left(\left[t^{2n}\right]\left(\log\left(\frac{e^t}{e^t-1} - \frac{t}{2}\right)\right)\right) /;\; n \in \mathbb{N}$$

Identities

Functional identities

04.13.17.0001.01

$$B_n = \frac{1}{m(1-m^n)} \sum_{k=0}^{n-1} m^k \binom{n}{k} B_k \sum_{j=1}^{m-1} j^{n-k} \text{ /; } m-1 \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

04.13.17.0006.01

$$B_n = -2^{1-n} \sum_{k=0}^{n} \left(2^{k-1}-1\right) \binom{n}{k} B_k \text{ /; } n \in \mathbb{Z} \land n > 1$$

G.Huvent (2006)

04.13.17.0002.01

$$B_{2n} = -\frac{1}{(2n+1)(n+1)} \sum_{k=0}^{n-1} (k+n+1) \binom{n+1}{k} B_{k+n} \text{ /; } n \in \mathbb{N}^+$$

04.13.17.0003.01

$$\sum_{k=0}^{m} (k+n+1) \binom{m+1}{k} B_{k+n} = (-1)^{m+n-1} \sum_{k=0}^{n} (k+m+1) \binom{n+1}{k} B_{k+m} \text{ /; } m \in \mathbb{N} \land n \in \mathbb{N} \land m+n > 0$$

04.13.17.0005.01

$$\sum_{k=2}^{n-2} \frac{B_k B_{n-k}}{k(n-k)} - \sum_{k=2}^{n-2} \binom{n}{k} \frac{B_k B_{n-k}}{k(n-k)} = 2 H_n \frac{B_n}{n} \text{ /; } n \in \mathbb{N} \land n \geq 3$$

Miki's identity

Miki_1978

H. Miki

A relation between Bernoulli numbers

Journal of Number Theory

10

297-302

1978

Identities involving determinants

04.13.17.0004.01

$$\left| \left(B_{k+l+2}\right)_{\substack{0 \leq k \leq n \\ 0 \leq l \leq n}} \right| = \frac{1}{6} (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{k!\,(k+1)!^4\,(k+2)!}{(2k+2)!\,(2k+3)!}$$

Complex characteristics

Real part

04.13.19.0001.01

$$\operatorname{Re}(B_n) = B_n$$

Imaginary part

04.13.19.0002.01

$$\text{Im}(B_n) = 0$$

Absolute value

04.13.19.0003.01

$$|B_n| = \sqrt{B_n^2}$$

Argument

04.13.19.0004.01

$$\arg(B_n) = \tan^{-1}(B_n, 0)$$

Conjugate value

04.13.19.0005.01

$$\overline{B_n} = B_n$$

Signum value

04.13.19.0006.01

$$\text{sgn}(B_n) = (-1)^{\frac{n}{2}-1}\left(1 - n + 2\left\lfloor\frac{n}{2}\right\rfloor\right) - \delta_{n-1} + 2\,\delta_n$$

Summation

Finite summation

04.13.23.0001.01

$$\sum_{k=0}^{n-1}\binom{n}{k} B_k = 0 \;/;\; n-1 \in \mathbb{N}^+$$

04.13.23.0002.01

$$\sum_{k=0}^{n+1} 2^k\left(2^k - 1\right)\binom{n+1}{k} B_k = -(n+1)\,E_n \;/;\; n \in \mathbb{N}$$

04.13.23.0003.01

$$\sum_{k=0}^{n} \frac{\left(2 - 2^{n-k+2}\right) z^k}{k!\,(n-k+1)!} B_{n-k+1} = \frac{E_n(z)}{n!}$$

04.13.23.0004.01

$$\sum_{k=0}^{n}\binom{n}{k} B_k\, z^k = z^n\, B_n\!\left(\frac{1}{z}\right)$$

04.13.23.0005.01

$$\sum_{k=0}^{n-1}(k+n+1)\binom{n+1}{k} B_{k+n} = -(2\,n+1)\,(n+1)\,B_{2\,n}$$

04.13.23.0006.01

$$\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k} B_{2k} = \frac{n}{2} /; n-1 \in \mathbb{N}^+$$

04.13.23.0020.01

$$\sum_{k=0}^{n} (4^k - 2) \binom{2n+1}{2k} B_{2k} = 0 /; n \in \mathbb{N}^+$$

G.Huvent (2006)

04.13.23.0007.01

$$\sum_{k=0}^{n-1} H_{m-1}^{(k-1)} \binom{n}{k} B_k m^k = m (1 - m^n) B_n /; m-1 \in \mathbb{N}^+ \land n \in \mathbb{N}^+$$

04.13.23.0008.01

$$\sum_{k=0}^{n} \frac{(1 - 2^{1-k})(1 - 2^{k-n+1}) B_{n-k} B_k}{(n-k)! \, k!} = \frac{(1-n) B_n}{n!}$$

04.13.23.0009.01

$$\sum_{k=0}^{n} \binom{n+2}{k} (2^{n-k+2} - 1)(1 - 2^{1-k}) B_{n-k+2} B_k = -2^{-n-2} (n+1)(n+2) E_n$$

04.13.23.0010.01

$$\sum_{k=0}^{m} \binom{m}{k} B_{k+n} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{k+m}$$

04.13.23.0011.01

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (k+n+1) B_{k+n} = 0$$

04.13.23.0012.01

$$\sum_{k=0}^{n+1} j^{-k+n+1} (k+n+1) \binom{n+1}{k} B_{k+n} = (n+1) \sum_{k=1}^{j-1} ((2n+1) k - (n+1) j) k^n (k-j)^{n-1} /; j \in \mathbb{N}$$

04.13.23.0013.01

$$\sum_{k=0}^{n} \binom{k+n}{2k} \frac{B_k}{k+n} = \frac{3}{4} /; k \bmod 12 = 1$$

04.13.23.0014.01

$$\sum_{k=0}^{n} \binom{k+n}{2k} \frac{B_k}{k+n} = -\frac{1}{4} /; k \bmod 12 = 3 \lor k \bmod 12 = 5$$

04.13.23.0015.01

$$\sum_{k=0}^{n} \binom{k+n}{2k} \frac{B_k}{k+n} = -\frac{1}{4} /; k \bmod 12 = 7 \lor k \bmod 12 = 9$$

04.13.23.0016.01

$$\sum_{k=0}^{n} \binom{k+n}{2k} \frac{B_k}{k+n} = \frac{-3}{4} /; k \bmod 12 = 11$$

04.13.23.0019.01

$$\sum_{k=0}^{m-1} B_n\left(\frac{k+z}{m}\right) == m^{1-n} B_n(z) /; m \in \mathbb{N} \land n \in \mathbb{N}$$

Infinite summation

04.13.23.0017.01

$$\sum_{n=0}^{\infty} \frac{B_n z^n}{n!} == \frac{z}{e^z - 1} /; |z| < 2\pi$$

04.13.23.0018.01

$$\sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k} == \frac{z}{2} \coth\left(\frac{z}{2}\right) /; |z| < 2\pi$$

04.13.23.0021.01

$$\sum_{k=1}^{\infty} \frac{(-1)^k B_{2k}}{k\,\Gamma(2k-\alpha)} z^{2k} =$$

$$-2\,\psi^{(\alpha)}\left(\frac{z}{2\pi}\right)\left(\frac{z}{2\pi}\right)^{\alpha+1} - \frac{2}{\Gamma(-\alpha)} (-2\log(2\pi) + \log(-z) + \log(z) - 2\,\psi(-\alpha) - 2\,\gamma) - 2\left(-\frac{z}{2\pi}\right)^{\alpha+1} \psi^{(\alpha)}\left(-\frac{z}{2\pi}\right)$$

Operations

Limit operation

04.13.25.0001.01

$$\lim_{n\to\infty} |B_{2n}| \left(\frac{\pi e}{n}\right)^{2n+\frac{1}{2}} == 4\pi\sqrt{e}$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

04.13.26.0001.01

$$H_n^{(-m)} == \frac{B_{m+1}(n+1) - B_{m+1}}{m+1} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

Involving polylog

04.13.26.0002.01

$$B_{2n} == \frac{(-1)^{n+1}\, 2\,(2n)!}{(2\pi)^{2n}} \operatorname{Li}_{2n}(1) /; n \in \mathbb{N}$$

04.13.26.0003.01

$$B_{2n} == \frac{(-1)^n\, 2\,(2n)!}{(4^n - 2)\,\pi^{2n}} \operatorname{Li}_{2n}(-1) /; n \in \mathbb{N}$$

Involving Stirling numbers

04.13.26.0004.01

$$B_n = \sum_{m=0}^{n} \frac{(-1)^m m!}{m+1} \mathcal{S}_n^{(m)} /; n \in \mathbb{N}$$

Involving zeta functions

04.13.26.0005.01

$$B_n = -n\,\zeta(1-n) /; n-1 \in \mathbb{N}^+$$

04.13.26.0006.01

$$B_n = (-1)^{n-1} n\,\zeta(1-n) /; n \in \mathbb{N}^+$$

04.13.26.0007.01

$$B_n = (-1)^{n-1} n\,\zeta(1-n, 0) /; n \in \mathbb{N}^+$$

04.13.26.0008.01

$$B_{2n} = (-1)^{n+1} (2\pi)^{-2n}\, 2\,(2n)!\,\zeta(2n) /; n \in \mathbb{N}$$

Representations through equivalent functions

With related functions

04.13.27.0001.01

$$B_n = B_n(0) /; n \in \mathbb{N}$$

04.13.27.0002.01

$$B_n = B_n(1) /; n-1 \in \mathbb{N}^+$$

04.13.27.0003.01

$$B_n = (-1)^n B_n(1) /; n \in \mathbb{N}$$

04.13.27.0004.01

$$B_n = -\frac{n}{2\,(2^n-1)} E_{n-1}(0) /; n \in \mathbb{N}^+$$

04.13.27.0005.01

$$B_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, E_k(0) /; n \in \mathbb{N}$$

04.13.27.0006.01

$$B_n = -\frac{n}{2^n\,(1-2^n)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} E_k /; n-1 \in \mathbb{N}^+$$

Inequalities

04.13.29.0001.01

$$\frac{2\,(2n)!}{(2\pi)^{2n}} < (-1)^{n+1} B_{2n} < \frac{2\,(2n)!}{(2\pi)^{2n}\left(1-2^{1-2n}\right)} /; n \in \mathbb{N}^+$$

04.13.29.0002.01

$$|B_{2n}| > \frac{(n\,(2n-1))}{6\,(2n+1)} |B_{2n-2}| /; n \in \mathbb{N}^+$$

04.13.29.0003.01

$$|B_{2n}| > |B_{2n}(x)| /; n \in \mathbb{N}^+ \bigwedge 0 < x < 1$$

04.13.29.0004.01

$$(-1)^{n-1} B_{2n} > 0 /; n \in \mathbb{N}^+ \bigwedge n \geq 1$$

Zeros

04.13.30.0001.01

$$B_{2n+1} = 0 /; n \in \mathbb{N}^+$$

Other identities

Congruence properties

04.13.32.0002.01

$$\operatorname{den}(B_{2n}) = \prod_{k=1}^{2n+1} \operatorname{boole}(k \in \mathbb{P}, 1) \operatorname{boole}\left(\frac{2n}{k-1} \in \mathbb{N}, 1\right) k /; n \in \mathbb{N}^+$$

von Staudt–Clausen theorem

04.13.32.0003.01

$$\frac{(1-p^{n-1}) B_n}{n} \bmod p^e = \frac{r^n \left(r^{\phi(p^e)} - 1\right)}{r^{n-1} p^e} - \sum_{k=1}^{\phi(p^e)} r^{k(n-1)} \left\lfloor \frac{r^k}{p^e} \right\rfloor /; p \in \mathbb{P} \bigwedge p \geq 5 \bigwedge n \in \mathbb{Z} \bigwedge n > 0 \bigwedge \frac{p-1}{n} \notin \mathbb{Z} \bigwedge e \in \mathbb{Z} \bigwedge e \geq 1$$

Here r is a primitive root of p^e.

04.13.32.0004.01

$$\frac{(1-p^{n-1}) B_n}{n} \bmod p^e = \frac{(1-p^{m-1}) B_m}{m} \bmod p^e /; p \in \mathbb{P} \bigwedge p \geq 5 \bigwedge n \in \mathbb{Z} \bigwedge n > 0 \bigwedge m \in \mathbb{Z} \bigwedge m > 0 \bigwedge n \bmod p^e = m \bmod p^e$$

Divisibility properties of denominators

04.13.32.0001.01

$$q = \prod_{d|2n} (d+1)\, \chi_{\mathbb{P}}(d+1) /; B_{2n} = \frac{p}{q} n \in \mathbb{N}^+ \bigwedge n \geq 1$$

Theorems

The Euler-Maclaurin formula

$$\int_a^{a+m h} f(x)\, dx \approx h\left(\frac{f(a)}{2} + \sum_{i=1}^{m-1} f(a + i h) + \frac{f(a+m h)}{2}\right)$$

$$+ \sum_{k=1}^{n} \frac{(-1)^k |B_{2k}| h^{2k}}{(2k)!} \left(f^{(2k-1)}(a + m h) - f^{(2k-1)}(a)\right) /; \qquad k = 1, 2, \ldots, n$$

History

– J. Faulhaber (1631) gave the first 8 Bernoulli numbers
– Jac. Bernoulli (1705, 1713); T. Seki (1712)
– A. de Moivre (1730) found the recursion relation
– L. Euler (1755, 1769, 1781) employed name Bernoullian numbers
– G.S.Klügel (1823) and K.G.C. von Staudt (1840) employed the notation $B^{(n)}$
– J. L. Raabe (1848,1851) used the name "Bernoulli polynomials"
– J. Binet (1839), Martin Ohm (1840), J.L. Raabe (1851), C. Hermite (1876), M.A. Stern (1878), J.C. Adams (1878) and G. Peano (1903) introduced modern notations
– C. Hermite (1875); C.J. Malmstén (1884)

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