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Beta

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Notations

Traditional name

Beta function

Traditional notation

$B(a, b)$

Mathematica StandardForm notation

`Beta[a, b]`

Primary definition

06.18.02.0001.01

$$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Specific values

Specialized values

For fixed *a*

06.18.03.0001.01

$$B(a, 0) = \tilde{\infty}$$

06.18.03.0007.01

$$B(a, a+1) = \frac{1}{a\,(a+1)\,C_a}$$

For fixed *b*

06.18.03.0002.01

$$B(0, b) = \tilde{\infty}$$

For rational variables ||| For rational variables

06.18.03.0003.01

$$B(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

06.18.03.0004.01

$$B\left(m+\frac{p}{q},\, n+\frac{r}{s}\right) == \frac{q^n s^m \left(\prod_{k=1}^{m} (p-q+k\,q) \prod_{k=1}^{n} (r-s+k\,s)\right)}{\prod_{k=1}^{m+n} (q\,r+p\,s-q\,s+k\,q\,s)} B\left(\frac{p}{q},\, \frac{r}{s}\right) /;$$

$m \in \mathbb{Z} \land n \in \mathbb{Z} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land s \in \mathbb{N}^+ \land p < q \land r < s$

06.18.03.0005.01

$$B\left(\frac{p}{q}-m,\, \frac{r}{s}-n\right) == \frac{q^{-n} s^{-m} \prod_{k=1}^{m+n} (-q\,r-p\,s+k\,q\,s)}{\left(\prod_{k=1}^{m} (q\,k-p)\right) \prod_{k=1}^{n} (s\,k-r)} B\left(\frac{p}{q},\, \frac{r}{s}\right) /;$$

$m \in \mathbb{Z} \land n \in \mathbb{Z} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land r \in \mathbb{N}^+ \land s \in \mathbb{N}^+ \land p < q \land r < s$

Values at fixed points

06.18.03.0006.01

$B(1,\, 1) == 1$

General characteristics

Domain and analyticity

$B(a,\, b)$ is an analytical function of a and b which is defined in $\mathbb{C}^2$ with the exception of countably many points $a = -k /; k \in \mathbb{N}$ and $b = -l /; l \in \mathbb{N}$.

06.18.04.0001.01

$(a * b) \longrightarrow B(a,\, b) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

06.18.04.0002.01

$B\left(\overline{a},\, \overline{b}\right) == \overline{B(a,\, b)}$

Permutation symmetry

06.18.04.0003.01

$B(b,\, a) == B(a,\, b)$

Periodicity

No periodicity

Poles and essential singularities

With respect to *a*

For fixed b $(b \neq -a - m /; m \in \mathbb{N})$, the function $B(a,\, b)$ has an infinite set of singular points:

a) $a == -k /; k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k\, \Gamma(b)}{k!\, \Gamma(b-k)}$;

b) $a == \tilde{\infty}$ is the point of convergence of poles, which is similar to considering $\tilde{\infty}$ as an essential singular point.

06.18.04.0004.01

$\mathcal{S}ing_a(B(a,\, b)) == \{\{\{-k,\, 1\} /; k \in \mathbb{N}\},\, \{\tilde{\infty},\, \infty\}\}$

06.18.04.0005.01

$$\text{res}_a(\text{B}(a, b))(-k) = \frac{(-1)^k\, \Gamma(b)}{k!\, \Gamma(b-k)} \text{ /; } k \in \mathbb{N} \bigwedge b - k \neq -m \text{ /; } m \in \mathbb{N}$$

With respect to *b*

For fixed a $(a \neq -b - m \text{ /; } m \in \mathbb{N})$, the function $\text{B}(a, b)$ has an infinite set of singular points:

a) $b = -k \text{ /; } k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k\, \Gamma(a)}{k!\, \Gamma(a-k)}$;

b) $b = \tilde{\infty}$ is the point of convergence of poles, which is similar to considering $\tilde{\infty}$ as an essential singular point.

06.18.04.0006.01

$$\mathcal{S}ing_b(\text{B}(a, b)) = \{\{\{-k, 1\} \text{ /; } k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.18.04.0007.01

$$\text{res}_b(\text{B}(a, b))(-k) = \frac{(-1)^k\, \Gamma(a)}{k!\, \Gamma(a-k)} \text{ /; } k \in \mathbb{N} \bigwedge a - k \neq -m \text{ /; } m \in \mathbb{N}$$

Branch points

With respect to *a*

The function $\text{B}(a, b)$ does not have branch points with respect to a.

06.18.04.0008.01

$$\mathcal{BP}_a(\text{B}(a, b)) = \{\}$$

With respect to *b*

The function $\text{B}(a, b)$ does not have branch points with respect to b.

06.18.04.0009.01

$$\mathcal{BP}_b(\text{B}(a, b)) = \{\}$$

Branch cuts

With respect to *a*

The function $\text{B}(a, b)$ does not have branch cuts with respect to a.

06.18.04.0010.01

$$\mathcal{BC}_a(\text{B}(a, b)) = \{\}$$

With respect to *b*

The function $\text{B}(a, b)$ does not have branch cuts with respect to b.

06.18.04.0011.01

$$\mathcal{BC}_b(\text{B}(a, b)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $a == a_0$

For the function itself

06.18.06.0008.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a_0,b)\left(1+(\psi(a_0)-\psi(b+a_0))(a-a_0)+\frac{1}{2}\left((\psi(a_0)-\psi(b+a_0))^2+\psi^{(1)}(a_0)-\psi^{(1)}(b+a_0)\right)(a-a_0)^2+\ldots\right)/;(a\to a_0)$$

06.18.06.0009.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a_0,b)\left(1+(\psi(a_0)-\psi(b+a_0))(a-a_0)+\frac{1}{2}\left((\psi(a_0)-\psi(b+a_0))^2+\psi^{(1)}(a_0)-\psi^{(1)}(b+a_0)\right)(a-a_0)^2\right)+O\left((a-a_0)^3\right)$$

06.18.06.0010.01

$$\mathrm{B}(a,b)==\sum_{k=0}^{\infty}(-1)^k\,\Gamma(a_0)^{k+1}\,{}_{k+2}\tilde{F}_{k+1}(1-b,c_1,c_2,\ldots,c_{k+1};c_1+1,c_2+1,\ldots,c_{k+1}+1;1)\,(a-a_0)^k\,/;$$

$$c_1==c_2==\ldots==c_{k+1}==a_0\bigwedge k\in\mathbb{N}$$

06.18.06.0011.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a_0,b)\,(1+O(a-a_0))$$

Expansions at $a == 0$

06.18.06.0001.01

$$\mathrm{B}(a,b)==\frac{1}{a}-(b-1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\,(2-b)_k\,a^j}{k!\,(k+1)^{j+2}}\,/;\,|a|<1\bigwedge \mathrm{Re}(b)>0$$

06.18.06.0002.01

$$\mathrm{B}(a,b)==\frac{1}{a}-(b-1)\sum_{j=0}^{\infty}(-1)^j\,{}_{j+3}F_{j+2}\left(2-b,a_1,a_2,\ldots,a_{j+2};a_1+1,a_2+1,\ldots,a_{j+2}+1;1\right)a^j\,/;$$

$$a_1==a_2==\ldots==a_{j+2}==1\bigwedge|a|<1\bigwedge \mathrm{Re}(b)>0$$

Expansions at generic point $b == b_0$

For the function itself

06.18.06.0012.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a,b_0)\left(1+(\psi(b_0)-\psi(a+b_0))(b-b_0)+\frac{1}{2}\left((\psi(b_0)-\psi(a+b_0))^2+\psi^{(1)}(b_0)-\psi^{(1)}(a+b_0)\right)(b-b_0)^2+\ldots\right)/;(b\to b_0)$$

06.18.06.0013.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a,b_0)\left(1+(\psi(b_0)-\psi(a+b_0))(b-b_0)+\frac{1}{2}\left((\psi(b_0)-\psi(a+b_0))^2+\psi^{(1)}(b_0)-\psi^{(1)}(a+b_0)\right)(b-b_0)^2\right)+O\left((b-b_0)^3\right)$$

06.18.06.0014.01

$$\mathrm{B}(a,b)==\sum_{k=0}^{\infty}(-1)^k\,\Gamma(b_0)^{k+1}\,{}_{k+2}\tilde{F}_{k+1}(1-a,c_1,c_2,\ldots,c_{k+1};c_1+1,c_2+1,\ldots,c_{k+1}+1;1)\,(b-b_0)^k\,/;$$

$$c_1==c_2==\ldots==c_{k+1}==a_0\bigwedge k\in\mathbb{N}$$

06.18.06.0015.01

$$\mathrm{B}(a,b)\propto \mathrm{B}(a,b_0)\,(1+O(b-b_0))$$

Expansions at $b == 0$

06.18.06.0003.01

$$B(a, b) = \frac{1}{b} - (a-1)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left((-1)^j (2-a)_k\right) b^j}{k!\,(k+1)^{j+2}} \text{ /; } |b| < 1 \bigwedge \operatorname{Re}(a) > 0$$

06.18.06.0004.01

$$B(a, b) = \frac{1}{b} - (a-1)\sum_{j=0}^{\infty}(-1)^j\, {}_{j+3}F_{j+2}\left(2-a, a_1, a_2, \ldots, a_{j+2}; a_1+1, a_2+1, \ldots, a_{j+2}+1; 1\right) b^j \text{ /; }$$

$$a_1 = a_2 = \ldots = a_{j+2} = 1 \bigwedge |b| < 1 \bigwedge \operatorname{Re}(a) > 0$$

Asymptotic series expansions

06.18.06.0005.01

$$B(a, b) \propto \Gamma(b)\, a^{-b} \sum_{k=0}^{\infty}\frac{(-1)^k (b)_k\, B(k, 1-b, 0)\, a^{-k}}{k!} \text{ /; } B(n, \alpha, z) = n!\left([t^n]\,\frac{t^\alpha e^{t z}}{(e^t - 1)^\alpha}\right) \bigwedge |\arg(a)| < \pi \bigwedge (|a| \to \infty)$$

06.18.06.0006.01

$$B(a, b) \propto \Gamma(b)\, a^{-b}\left(1 - \frac{b\,(b-1)}{2\,a}\left(1 + O\left(\frac{1}{a}\right)\right)\right) \text{ /; } |\arg(a)| < \pi \bigwedge (|a| \to \infty)$$

Other series representations

06.18.06.0007.01

$$B(a, b) = \sum_{k=0}^{\infty}\frac{(1-b)_k}{(a+k)\,k!} \text{ /; } \operatorname{Re}(a) > 0 \bigwedge \operatorname{Re}(b) > 0$$

Integral representations

On the real axis

Of the direct function

06.18.07.0001.01

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt \text{ /; } \operatorname{Re}(a) > 0 \bigwedge \operatorname{Re}(b) > 0$$

06.18.07.0002.01

$$B(a, b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}(t)\cos^{2b-1}(t)\, dt \text{ /; } \operatorname{Re}(a) > 0 \bigwedge \operatorname{Re}(b) > 0$$

Product representations

06.18.08.0001.01

$$B(a, b) = \frac{1}{a+b-1}\prod_{k=1}^{\infty}\frac{k\,(a+b+k-2)}{(a+k-1)\,(b+k-1)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.18.16.0001.01

$$B(a+1,\, b) == \frac{a\, B(a,\, b)}{a+b}$$

06.18.16.0002.01

$$B(a-1,\, b) == \frac{a+b-1}{a-1}\, B(a,\, b)$$

06.18.16.0003.01

$$B(a+n,\, b) == \frac{(a)_n}{(a+b)_n}\, B(a,\, b) \,/;\, n \in \mathbb{N}$$

06.18.16.0004.01

$$B(a-n,\, b) == \frac{(a+b-n)_n}{(a-n)_n}\, B(a,\, b) \,/;\, n \in \mathbb{N}$$

Multiple arguments

06.18.16.0005.01

$$B(2\, a,\, 2\, b) == \frac{2^{-2\, b}\, B(a,\, b)\, B\left(a+\frac{1}{2},\, b\right)}{B(b,\, b)}$$

06.18.16.0006.01

$$B(n\, a,\, n\, b) == \frac{n^{-n\, b} \prod_{k=0}^{n-1} B\left(a+\frac{k}{n},\, b\right)}{\prod_{j=1}^{n-1} B(j\, b,\, b)} \,/;\, n \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

06.18.17.0001.01

$$B(a,\, b) == \frac{a+b}{a}\, B(a+1,\, b)$$

06.18.17.0002.01

$$B(a,\, b) == \frac{a-1}{a+b-1}\, B(a-1,\, b)$$

06.18.17.0003.01

$$B(a,\, b) == B(a+1,\, b) + B(a,\, b+1)$$

Distant neighbors

06.18.17.0004.01

$$B(a,\, b) == \frac{(a+b)_n}{(a)_n}\, B(a+n,\, b) \,/;\, n \in \mathbb{N}$$

06.18.17.0005.01

$$B(a, b) == \frac{(1-a)_n}{(1-a-b)_n} B(a-n, b) \,/;\, n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.18.17.0006.01

$$B(a, b)\, B(a+b, c) == B(b, c)\, B(b+c, a)$$

Differentiation

Low-order differentiation

With respect to *a*

06.18.20.0001.01

$$\frac{\partial B(a, b)}{\partial a} == B(a, b)\, (\psi(a) - \psi(a+b))$$

06.18.20.0002.01

$$\frac{\partial^2 B(a, b)}{\partial a^2} == B(a, b) \left((\psi(a) - \psi(a+b))^2 + \psi^{(1)}(a) - \psi^{(1)}(a+b) \right)$$

With respect to *b*

06.18.20.0003.01

$$\frac{\partial B(a, b)}{\partial b} == B(a, b)\, (\psi(b) - \psi(a+b))$$

06.18.20.0004.01

$$\frac{\partial^2 B(a, b)}{\partial b^2} == B(a, b) \left((\psi(b) - \psi(a+b))^2 + \psi^{(1)}(b) - \psi^{(1)}(a+b) \right)$$

Symbolic differentiation

With respect to *a*

06.18.20.0005.02

$$\frac{\partial^n B(a, b)}{\partial a^n} == (-1)^n\, n!\, \Gamma(a)^{n+1}\, {}_{n+2}\tilde{F}_{n+1}(1-b, a_1, a_2, \ldots, a_{n+1};\, a_1+1, a_2+1, \ldots, a_{n+1}+1;\, 1) \,/;$$
$$a_1 == a_2 == \ldots == a_{n+1} == a \bigwedge n \in \mathbb{N}$$

With respect to *b*

06.18.20.0006.02

$$\frac{\partial^n B(a, b)}{\partial b^n} == (-1)^n\, n!\, \Gamma(b)^{n+1}\, {}_{n+2}\tilde{F}_{n+1}(1-a, a_1, a_2, \ldots, a_{n+1};\, a_1+1, a_2+1, \ldots, a_{n+1}+1;\, 1) \,/;$$
$$a_1 == a_2 == \ldots == a_{n+1} == b \bigwedge n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *a*

06.18.20.0007.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial a^\alpha} = a^{-\alpha} \int_0^1 t^{a-1} (1-t)^{b-1} (a \log(t))^\alpha\, Q(-\alpha, 0, a\log(t))\, dt \;/;\; \mathrm{Re}(a) > 0 \bigwedge \mathrm{Re}(b) > 0$$

06.18.20.0008.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial a^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(a,-1)\, a^{-\alpha-1} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^j (1-b)_{k+1}\, j!\, a^{j-\alpha}}{(k+1)^{j+2}\, k!\, \Gamma(j-\alpha+1)} \;/;\; |a| < 1 \bigwedge \mathrm{Re}(b) > 0$$

06.18.20.0009.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial a^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(a,-1)\, a^{-\alpha-1} + a^{-\alpha} \sum_{k=0}^{\infty} \frac{(1-b)_{k+1}}{(k+1)^2\, k!}\, {}_2\tilde{F}_1\!\left(1, 1; 1-\alpha; -\frac{a}{k+1}\right) \;/;\; \mathrm{Re}(b) > 0$$

With respect to *b*

06.18.20.0010.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial b^\alpha} = b^{-\alpha} \int_0^1 t^{b-1} (1-t)^{a-1} (b \log(t))^\alpha\, Q(-\alpha, 0, b\log(t))\, dt \;/;\; \mathrm{Re}(a) > 0 \bigwedge \mathrm{Re}(b) > 0$$

06.18.20.0011.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial b^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(b,-1)\, b^{-\alpha-1} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^j (1-a)_{k+1}\, j!\, b^{j-\alpha}}{(k+1)^{j+2}\, k!\, \Gamma(j-\alpha+1)} \;/;\; |b| < 1 \bigwedge \mathrm{Re}(a) > 0$$

06.18.20.0012.01

$$\frac{\partial^\alpha \mathrm{B}(a,b)}{\partial b^\alpha} = \mathcal{FC}_{\exp}^{(\alpha)}(b,-1)\, b^{-\alpha-1} + b^{-\alpha} \sum_{k=0}^{\infty} \frac{(1-a)_{k+1}}{(k+1)^2\, k!}\, {}_2\tilde{F}_1\!\left(1, 1; 1-\alpha; -\frac{b}{k+1}\right) \;/;\; \mathrm{Re}(a) > 0$$

Integration

Indefinite integration

Involving only one direct function with respect to *a*

06.18.21.0001.01

$$\int \mathrm{B}(a,b)\, da = \int_0^1 \frac{t^{a-1}(1-t)^{b-1}}{\log(t)}\, dt \;/;\; \mathrm{Re}(a) > 0 \bigwedge \mathrm{Re}(b) > 1$$

06.18.21.0002.01

$$\int \mathrm{B}(a,b)\, da = \log(a) + \sum_{k=0}^{\infty} \frac{(1-b)_{k+1}}{(k+1)!} \log\!\left(1 + \frac{a}{k+1}\right) \;/;\; \mathrm{Re}(b) > 0$$

Involving one direct function and elementary functions with respect to *a*

Involving power function

06.18.21.0003.01

$$\int a^{\alpha-1}\, \mathrm{B}(a,b)\, da = \frac{a^{\alpha-1}}{\alpha-1} - \frac{(b-1)\, a^\alpha}{\alpha} \sum_{k=0}^{\infty} \frac{(2-b)_k}{(k+1)\,(k+1)!}\, {}_2F_1\!\left(\alpha, 1; \alpha+1; -\frac{a}{k+1}\right) \;/;\; \mathrm{Re}(b) > 0$$

Involving only one direct function with respect to *b*

06.18.21.0004.01

$$\int B(a,\,b)\,d b = \int_0^1 \frac{t^{b-1}\,(1-t)^{a-1}}{\log(t)}\,d t\,/;\,\mathrm{Re}(b) > 0 \bigwedge \mathrm{Re}(a) > 1$$

06.18.21.0005.01

$$\int B(a,\,b)\,d b = \log(b) + \sum_{k=0}^{\infty} \frac{(1-a)_{k+1}}{(k+1)!} \log\left(1 + \frac{b}{k+1}\right)/;\,\mathrm{Re}(a) > 0$$

Involving one direct function and elementary functions with respect to *b*

Involving power function

06.18.21.0006.01

$$\int b^{\alpha-1}\,B(a,\,b)\,d b = \frac{b^{\alpha-1}}{\alpha-1} - \frac{(a-1)\,b^{\alpha}}{\alpha} \sum_{k=0}^{\infty} \frac{(2-a)_k}{(k+1)\,(k+1)!}\,{}_2F_1\left(\alpha,\,1;\,\alpha+1;\,-\frac{b}{k+1}\right)/;\,\mathrm{Re}(a) > 0$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

06.18.26.0001.01

$$B(a,\,b) = B_1(a,\,b)\,/;\,\mathrm{Re}(b) > 0$$

Representations through equivalent functions

With related functions

06.18.27.0001.01

$$B(a,\,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

06.18.27.0002.01

$$B(a,\,b) = \frac{(a-1)!\,(b-1)!}{(a+b-1)!}$$

06.18.27.0003.01

$$B(a,\,b) = 2^{\frac{1}{4}(\cos(2\,a\,\pi)+\cos(2\,b\,\pi)-\cos(2\,(a+b)\,\pi)+3)}\,\pi^{\frac{1}{2}\left(\sin^2(a\,\pi)+\sin^2(b\,\pi)-\sin^2((a+b)\,\pi)\right)}\,\frac{(2\,a-2)!!\,(2\,b-2)!!}{(2\,a+2\,b-2)!!}$$

Zeros

06.18.30.0001.01

$$B(a,\,b) = 0\,/;\,a+b = -k \bigwedge k \in \mathbb{N}$$

Theorems

Transcendentality of the beta function

For all positive rational noninteger α and β, the value of $\mathrm{B}(\alpha, \beta)$ is transcendental.

The determinant of one matrix with beta functions

The determinant of the matrix $\boldsymbol{A} = \{a_{i\,j}\}_{1 \le i, j \le n}$ with entries $a_{i\,j} = 1 / \mathrm{B}(i, j)$ is $\Gamma(n)$.

History

– J. Wallis (1655); L. Euler (1730); J. Stirling (1730)
– A. M. Legendre (1826) used the name "Euler function"
– P. M. Binet (1839) used the name "Beta function"
– J. J. Schönholzer (1877) derived a contour integral representation
– U. Bigler (1888) found integral representations for complex arguments

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