# Cyclotomic



## Notations

### Traditional name

Cyclotomic polynomial

### Traditional notation

$C_n(z)$

### *Mathematica* StandardForm notation

`Cyclotomic[n, z]`

## Primary definition

05.11.02.0001.01

$$C_n(z) = \prod_{k=1}^{n} \left(z - e^{\frac{2\pi i k}{n}}\right)^{\delta_{\gcd(k,n),1}} /; n \in \mathbb{N}$$

## Specific values

### Specialized values

#### For fixed *n*

05.11.03.0016.01

$$C_n(0) = 1 /; n \neq 1$$

#### For fixed *z*

05.11.03.0001.01

$$C_p(z) = \sum_{k=0}^{p-1} z^k /; p \in \mathbb{P}$$

05.11.03.0002.01

$$C_0(z) = 1$$

05.11.03.0003.01

$$C_1(z) = z - 1$$

05.11.03.0004.01

$$C_2(z) = z + 1$$

05.11.03.0005.01

$C_3(z) = z^2 + z + 1$

05.11.03.0006.01

$C_4(z) = z^2 + 1$

05.11.03.0007.01

$C_5(z) = z^4 + z^3 + z^2 + z + 1$

05.11.03.0008.01

$C_6(z) = z^2 - z + 1$

05.11.03.0009.01

$C_7(z) = z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$

05.11.03.0010.01

$C_8(z) = z^4 + 1$

05.11.03.0011.01

$C_9(z) = z^6 + z^3 + 1$

05.11.03.0012.01

$C_{10}(z) = z^4 - z^3 + z^2 - z + 1$

### Values at infinities

05.11.03.0013.01

$C_n(\infty) = \infty$

05.11.03.0014.01

$C_n(-\infty) = \infty$

05.11.03.0015.01

$C_n(\tilde{\infty}) = \tilde{\infty}$

## General characteristics

### Domain and analyticity

$C_n(z)$ is a polynomial of $z$ and as such an analytical function of $z$. $C_n(z)$ is defined in the whole complex $z$-plane and for integer $n$.

05.11.04.0001.01

$(n * z) \longrightarrow C_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

### Symmetries and periodicities

#### Mirror symmetry

05.11.04.0002.01

$C_n(\bar{z}) = \overline{C_n(z)}$

#### Periodicity

No periodicity

### Poles and essential singularities

#### With respect to $z$

The function $C_n(z)$ has a pole of order $\varphi(n)$ at $z == \tilde{\infty}$.

05.11.04.0003.01

$$\mathcal{Sing}_z(C_n(z)) == \{\{\tilde{\infty}, \varphi(n)\}\}$$

### Branch points

#### With respect to $z$

The function $C_n(z)$ does not have branch points.

05.11.04.0004.01

$$\mathcal{BP}_z(C_n(z)) == \{\}$$

### Branch cuts

#### With respect to $z$

The function $C_n(z)$ does not have branch cuts.

05.11.04.0005.01

$$\mathcal{BC}_z(C_n(z)) == \{\}$$

## Series representations

### Generalized power series

#### Expansions at $z == 0$

05.11.06.0001.01

$$C_n(z) == \sum_{j=0}^{\phi(n)} a(n, j)\, z^{\phi(n)-j} \;/;\; a(n, j) == -\frac{\mu(n)}{j} \sum_{m=0}^{j-1} a(n, m)\, \mu(\gcd(n, j-m))\, \phi(\gcd(n, j-m)) \bigwedge a(n, 0) == 1 \bigwedge \sqrt{n} \notin \mathbb{N}$$

## Product representations

05.11.08.0001.01

$$C_n(z) == \prod_{d|n} \left(1 - z^d\right)^{\mu\left(\frac{n}{d}\right)} \;/;\; n > 1$$

## Generating functions

05.11.11.0001.01

$$\left(\left[z^{\varphi(n)}\right] C_n(z)\right) == 1 \;/;\; n > 1$$

05.11.11.0002.01

$$\left(\left[z^0\right] C_n(z)\right) == 1 \;/;\; n > 1$$

# Transformations

## Multiple arguments

05.11.16.0001.01

$$C_{n\,p}(z) == C_n(z^p)\;/;\;\frac{n}{p}\in\mathbb{Z}\bigwedge\frac{n}{p}\geq 0$$

05.11.16.0002.01

$$C_{n\,p}(z)\,C_n(z) == C_n(z^p)\;/;\;\frac{n}{p}\notin\mathbb{Z}\bigwedge n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+$$

05.11.16.0003.01

$$C_{n\,p}\left(z^{p^{k-1}}\right) == C_{n\,p^k}(z)\;/;\;n\in\mathbb{N}\land p\in\mathbb{N}\land k\in\mathbb{N}^+$$

## Power of arguments

05.11.16.0004.01

$$C_n(z^p) == C_{n\,p}(z)\;/;\;n>1$$

05.11.16.0005.01

$$C_n(z^p) == C_{n\,p}(z)\,C_n(z)\;/;\;\frac{n}{p}\notin\mathbb{Z}\bigwedge n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+$$

05.11.16.0006.01

$$C_{n\,p^k}(z) == C_{n\,p}\left(z^{p^{k-1}}\right)\;/;\;n\in\mathbb{N}\land p\in\mathbb{N}\land k\in\mathbb{N}^+$$

# Differentiation

## Low-order differentiation

05.11.20.0001.01

$$\frac{\partial C_p(z)}{\partial z} == \sum_{k=1}^{p-1} k\,z^{k-1}\;/;\;p\in\mathbb{P}$$

05.11.20.0002.01

$$\frac{\partial C_n(z)}{\partial z} == \sum_{j=0}^{\phi(n)} a(n,\,j)\,(\phi(n)-j)\,z^{\phi(n)-j-1}\;/;$$

$$a(n,\,j) == -\frac{\mu(n)}{j}\sum_{m=0}^{j-1} a(n,\,m)\,\mu(\gcd(n,\,j-m))\,\phi(\gcd(n,\,j-m))\bigwedge a(n,\,0) == 1\bigwedge\sqrt{n}\notin\mathbb{N}$$

## Fractional integro-differentiation

05.11.20.0003.01

$$\frac{\partial^\alpha C_p(z)}{\partial z^\alpha} == \sum_{k=0}^{p-1}\frac{k!\,z^{k-\alpha}}{\Gamma(k-\alpha+1)}\;/;\;p\in\mathbb{P}$$

05.11.20.0004.01

$$\frac{\partial^\alpha C_n(z)}{\partial z^\alpha} = \sum_{j=0}^{\phi(n)} \frac{(a(n, j)\,(\phi(n)-j)!)\, z^{\phi(n)-j-\alpha}}{\Gamma(\phi(n)-j-\alpha+1)} /;$$

$$a(n, j) = -\frac{\mu(n)}{j} \sum_{m=0}^{j-1} a(n, m)\, \mu(\gcd(n, j-m))\, \phi(\gcd(n, j-m)) \bigwedge a(n, 0) = 1 \bigwedge \sqrt{n} \notin \mathbb{N}$$

# Integration

## Indefinite integration

### Involving only one direct function

05.11.21.0001.01

$$\int C_p(z)\, dz = \sum_{k=0}^{p-1} \frac{z^{k+1}}{k+1} /; p \in \mathbb{P}$$

05.11.21.0002.01

$$\int C_n(z)\, dz = \sum_{j=0}^{\phi(n)} \frac{a(n, j)\, z^{\phi(n)-j+1}}{\phi(n)-j+1} /; a(n, j) = -\frac{\mu(n)}{j} \sum_{m=0}^{j-1} a(n, m)\, \mu(\gcd(n, j-m))\, \phi(\gcd(n, j-m)) \bigwedge a(n, 0) = 1 \bigwedge \sqrt{n} \notin \mathbb{N}$$

# Products

## Finite products

05.11.24.0001.01

$$\prod_{d \mid n} C_d(z) = z^n - 1 /; n \in \mathbb{N}$$

05.11.24.0002.01

$$\frac{\prod_{d \mid 2n} C_d(z)}{\prod_{d \mid n} C_d(z)} = z^n + 1 /; n \in \mathbb{N}$$

# Inequalities

05.11.29.0001.01

$$C_n(m) \geq 2 /; m - 1 \in \mathbb{N}^+ \bigwedge n - 1 \in \mathbb{N}^+$$

# Zeros

05.11.30.0001.01

$$C_n(z) = 0 /; z = e^{\frac{2\pi i k}{n}} \bigwedge \delta_{\gcd(k,n),1} = 1 \bigwedge k \in \mathbb{Z}$$

# Theorems

### Degree of $C_n(x)$

$C_n(x)$ are irreducible polynomials over $\mathbb{Z}$ with degree $\varphi(n)$ .

### Coefficients of $C_n(x)$

$C_{105}(x)$ has coefficients of $-2$ for $x^7$ and $x^{41}$, making it the first cyclotomic polynomial to have a coefficient other than $\pm 1$ and 0 (because 105 is the first number to have three distinct odd prime factors, i.e., $105 == 3 \times 5 \times 7$). The coefficients of $C_{p\,q}(x)$ for distinct primes $p$ and $q$ can be only $0, \pm 1$. The smallest values of $n$ for which $C_n(x)$ has one or more coefficients $\pm 2, \pm 3, \ldots$ are $105, 385, 1365, 1785, 2805, 3135, 6545, \ldots$.

## History

– L. Euler (1771) studied cases $C_n(x) == 0$, $n < 10$
– A.-T. Vandermonde (1771) studed $C_{11}(x) == 0$
– J.-L. Lagrange (1772)
– C. F. Gauss (1796) studed $C_{17}(x) == 0$
– N. H. Abel (c. 1824, published in 1829) showed that $C_n(x) == 0$ can always be solved in radicals
– L. Kronecker (1845)
– F.G.M. Eisenstein (1850)
– A. Migott (1883)
– P. Bachmann (1872)
– L. Carlitz (1936)
– E. Lehmer (1936)

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