E

Notations

Traditional name

Base of the natural logarithm

Traditional notation

e

Mathematica StandardForm notation

E

Primary definition

02.05.02.0001.01

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Specific values

02.05.03.0001.01

$$e = 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178\ldots$$

Above approximate numerical value of e shows 90 decimal digits.

General characteristics

The Euler number e is a constant. It is irrational and transcendental over $\mathbb{Q}$ positive real number.

Series representations

Generalized power series

Expansions for e

02.05.06.0001.01

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

02.05.06.0002.01

$$e = {}_0F_0(;;1)$$

02.05.06.0003.01

$$e = \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!}$$

H. J. Brothers

02.05.06.0012.01

$$e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+1)!}$$

H. J. Brothers

02.05.06.0004.01

$$e = 2 \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)!}$$

H. J. Brothers

02.05.06.0005.01

$$e = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k+1}{k!}$$

H. J. Brothers

02.05.06.0006.01

$$e = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z-1+k}{k!}$$

H. J. Brothers

02.05.06.0007.01

$$e = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}}$$

H. J. Brothers

02.05.06.0008.01

$$e = \frac{1}{\sum_{k=0}^{\infty} \frac{1-2k}{(2k)!}}$$

H. J. Brothers

02.05.06.0009.01

$$e = \sum_{k=0}^{\infty} \frac{3-4k^2}{(2k+1)!}$$

H. J. Brothers

02.05.06.0010.01

$$e = \sum_{k=0}^{\infty} \frac{(3k)^2 + 1}{(3k)!}$$

H. J. Brothers

02.05.06.0013.01

$$e = \sum_{k=0}^{\infty} \frac{(3k+1)^2 + 1}{(3k+1)!}$$

H. J. Brothers

02.05.06.0014.01

$$e = \sum_{k=0}^{\infty} \frac{(3k+2)^2 + 1}{(3k+2)!}$$

H. J. Brothers

02.05.06.0011.01

$$e = \frac{2}{3} \sum_{k=0}^{\infty} \frac{(k+3)^{k \bmod 2}}{2^{k \bmod 2}\, k!}$$

H. J. Brothers

02.05.06.0015.01

$$e = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(k+2)^{k \bmod 2}}{k!}$$

H. J. Brothers

02.05.06.0016.01

$$e = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(k+2)^{(k+1) \bmod 2}}{k!}$$

H. J. Brothers

02.05.06.0017.01

$$e = \frac{2}{3} \sum_{k=0}^{\infty} \frac{k+1}{2^{k \bmod 2}\, k!}$$

H. J. Brothers

02.05.06.0018.01

$$e = \frac{2}{3} \sum_{k=0}^{\infty} \frac{(k+3)^{k \bmod 2}}{2^{k \bmod 2}\, k!}$$

H. J. Brothers

02.05.06.0019.01

$$e = \sum_{k=0}^{\infty} \frac{(8k-4)\left(8k^2+1\right)+5}{(4k)!}$$

H. J. Brothers

02.05.06.0020.01

$$e = \sum_{k=0}^{\infty} \frac{3-(2k-1)^2}{(2k)!}$$

H. J. Brothers

02.05.06.0021.01

$$e = 2\sum_{k=0}^{\infty} \frac{-2k^2+2k+1}{(2k)!}$$

H. J. Brothers

02.05.06.0022.01

$$e = 3 - \sum_{k=0}^{\infty} \frac{k+1}{(k+3)!}$$

H. J. Brothers

Expansions for $1/e$

02.05.06.0023.01

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{1-2k}{(2k)!}$$

H. J. Brothers

02.05.06.0024.01

$$\frac{1}{e} = 1 - \sum_{k=0}^{\infty} \frac{2k+1}{(2k+2)!}$$

H. J. Brothers

02.05.06.0025.01

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{2k}{(2k+1)!}$$

H. J. Brothers

Expansions for $\sqrt{e}$

02.05.06.0026.01

$$\sqrt{e} = \sum_{k=0}^{\infty} \frac{4k+3}{(2k+1)!\,2^{2k+1}}$$

H. J. Brothers

Product representations

02.05.08.0001.01

$$e = 2\prod_{k=0}^{\infty}\left(\frac{1}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+2^{k+1}\right)}\,2^{2^{k+1}}\,\Gamma\left(\frac{1}{2}+2^k\right)^2\right)^{2^{-k-1}}$$

02.05.08.0002.01

$$e = 2\prod_{j=1}^{\infty}\prod_{k=0}^{2^{j-1}-1}\left(\frac{2k+2^j+2}{2k+2^j+1}\right)^{\frac{1}{2^j}}$$

02.05.08.0003.01

$$e = 2\sqrt{\frac{2}{1}}\sqrt[4]{\frac{2\times 4}{3\times 3}}\sqrt[8]{\frac{4\times 6\times 6\times 8}{5\times 5\times 7\times 7}}\sqrt[16]{\frac{8\times 10\times 10\times 12\times 12\times 14\times 14\times 16}{9\times 9\times 11\times 11\times 13\times 13\times 15\times 15}}\;\ldots$$

02.05.08.0007.01

$$e = 2\prod_{k=1}^{\infty}\left(\frac{2^{k-1}\left(\prod_{j=2^{k-2}+1}^{2^{k-1}-1} 2j\right)^2 2^k}{\left(\prod_{j=2^{k-2}}^{2^{k-1}-1}(2j+1)\right)^2}\right)^{\frac{1}{2^k}}$$

02.05.08.0004.01

$$e = \prod_{k=1}^{\infty} k^{-\frac{\mu(k)}{k}}$$

02.05.08.0005.01

$$e - \frac{1}{e} = 2\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2\pi^2}\right)$$

02.05.08.0006.01

$$e + \frac{1}{e} = 2\prod_{k=1}^{\infty}\left(1+\frac{4}{(2k-1)^2\pi^2}\right)$$

Limit representations

02.05.09.0001.01

$$e = \lim_{z\to 0}(z+1)^{1/z}$$

02.05.09.0002.01

$$e = \lim_{z\to\infty}\frac{z}{z!^{1/z}}$$

02.05.09.0003.01

$$e = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{n^2+k}{n^2-k}$$

02.05.09.0010.01

$$e = \lim_{n\to\infty}\left(2\sum_{k=0}^{n}\frac{n^k}{k!}\right)^{1/n}$$

02.05.09.0004.01

$$e = \lim_{z\to\infty}\left(\frac{z^z}{(z-1)^{z-1}} - \frac{(z-1)^{z-1}}{(z-2)^{z-2}}\right)$$

02.05.09.0005.01

$$e = \lim_{z\to\infty}\frac{4z}{{(z)_{z+1}}^{1/z}}$$

02.05.09.0006.01

$$e = \lim_{z\to\infty}\frac{z^z}{H_{z-1}^{(-z)}} + 1$$

02.05.09.0007.01

$$e = \lim_{n\to\infty}\left(\frac{(s+1)\,n!^{-\frac{s}{n}}}{n}H_n^{(-s)}\right)^{1/s} /; s > 0$$

02.05.09.0008.01

$$e = (s+1)^{1/s}\left(\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n} k^s}{n\left(\prod_{k=1}^{n} k^s\right)^{1/n}}\right)^{1/s}\right) /; s > 0$$

02.05.09.0009.01

$$e = \lim_{n\to\infty}\left(\prod_{k=1}^{\pi(n)} p_k\right)^{\frac{1}{p_n}}$$

02.05.09.0011.01

$$e = \lim_{n\to\infty}\left(\frac{1}{\log(p_n)\prod_{k=1}^{n}\left(1-\frac{1}{p_k}\right)}\right)^{1/\gamma}$$

Mertens theorem

02.05.09.0012.01

$$e = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)$$

The above formula is used for the numerical computation of Euler constant in *Mathematica*.

Continued fraction representations

02.05.10.0001.01

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \ldots}}}}}}$$

02.05.10.0002.01

$$e = 2 + \underset{k=1}{\overset{\infty}{\mathrm{K}}}\left(1, \left(\frac{2(k+1)}{3}\right)^{\frac{1}{2}\left(1-(-1)^{(k+2) \bmod 3}\right)}\right)$$

02.05.10.0003.01

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \cfrac{7}{7 + \ldots}}}}}}$$

02.05.10.0004.01

$$e = 1 + \frac{1}{\mathrm{K}_k(k, k)_1^\infty}$$

02.05.10.0005.01

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cfrac{1}{26 + \ldots}}}}}}}$$

02.05.10.0006.01

$$e = 1 + \frac{2}{1 + \mathrm{K}_k(1, 4k+2)_1^\infty}$$

02.05.10.0007.01

$$e = \cfrac{1}{1 - \cfrac{2}{3 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cfrac{1}{26 + \ldots}}}}}}}}$$

02.05.10.0008.01

$$e = \frac{1}{1 - 2 \big/ \left(3 + \mathrm{K}_k(1, 4k+2)_1^\infty\right)}$$

02.05.10.0009.01

$$e = \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{1}{2 - \cfrac{1}{3 + \cfrac{1}{2 - \cfrac{1}{5 + \cfrac{1}{2 - \ldots}}}}}}}$$

02.05.10.0010.01

$$e = \frac{1}{1 + \mathrm{K}_k\left((-1)^k, -\frac{1}{2}\left((-1)^k - 1\right) k + (-1)^k + 1\right)_1^\infty}$$

02.05.10.0017.01

$$e^2 = 7 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{18 + \cfrac{1}{5 + \cfrac{1}{1 + \ldots}}}}}}}$$

02.05.10.0018.01

$$e^2 = 7 + \mathrm{K}_k\left(1, 6\left(2\left\lceil\frac{k}{5}\right\rceil + 1\right)\delta_{k \bmod 5,0} + \left(3\left\lceil\frac{k}{5}\right\rceil - 1\right)\delta_{k \bmod 5,1} + 1\,\delta_{k \bmod 5,2} + 1\,\delta_{k \bmod 5,3} + 3\left\lceil\frac{k}{5}\right\rceil\delta_{k \bmod 5,4}\right)_1^\infty$$

02.05.10.0011.01

$$\frac{1}{e - 2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \ldots}}}}}}$$

02.05.10.0012.01

$$\frac{1}{e - 2} = 1 + \mathrm{K}_k(k, k + 1)_1^\infty$$

02.05.10.0013.01

$$\frac{e}{e-2} == 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\ldots}}}}}}}}}$$

02.05.10.0014.01

$$\frac{1}{\sqrt{e}-1} == 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{13+\ldots}}}}}}}}}$$

02.05.10.0015.01

$$\frac{e+1}{e-1} == 2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\ldots}}}}}$$

02.05.10.0016.01

$$\frac{e^2+1}{e^2-1} == 1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ldots}}}}}$$

Complex characteristics

Real part

02.05.19.0001.01

$$\operatorname{Re}(e) == e$$

Imaginary part

02.05.19.0002.01

$\operatorname{Im}(e) = 0$

Absolute value

02.05.19.0003.01

$|e| = e$

Argument

02.05.19.0004.01

$\arg(e) = 0$

Conjugate value

02.05.19.0005.01

$\overline{e} = e$

Signum value

02.05.19.0006.01

$\operatorname{sgn}(e) = 1$

Differentiation

Low-order differentiation

02.05.20.0001.01

$$\frac{\partial e}{\partial z} = 0$$

Fractional integro-differentiation

02.05.20.0002.01

$$\frac{\partial^\alpha e}{\partial z^\alpha} = \frac{z^{-\alpha} e}{\Gamma(1-\alpha)}$$

Integration

Indefinite integration

02.05.21.0001.01

$$\int e \, dz = e z$$

02.05.21.0002.01

$$\int z^{\alpha-1} e \, dz = \frac{z^\alpha e}{\alpha}$$

Integral transforms

Fourier exp transforms

02.05.22.0001.01

$\mathcal{F}_t[e](z) = e\sqrt{2\pi}\,\delta(z)$

Inverse Fourier exp transforms

02.05.22.0002.01

$\mathcal{F}_t^{-1}[e](z) = e\sqrt{2\pi}\,\delta(z)$

Fourier cos transforms

02.05.22.0003.01

$\mathcal{F}c_t[e](z) = e\sqrt{\frac{\pi}{2}}\,\delta(z)$

Fourier sin transforms

02.05.22.0004.01

$\mathcal{F}s_t[e](z) = \sqrt{\frac{2}{\pi}}\,\frac{e}{z}$

Laplace transforms

02.05.22.0005.01

$\mathcal{L}_t[e](z) = \frac{e}{z}$

Inverse Laplace transforms

02.05.22.0006.01

$\mathcal{L}_t^{-1}[e](z) = e\,\delta(z)$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

02.05.26.0001.01

$e = {}_0\tilde{F}_0(;;1)$

Involving ${}_pF_q$

02.05.26.0002.01

$e = {}_0F_0(;;1)$

Through Meijer *G*

Classical cases

02.05.26.0003.01

$$e = G_{0,1}^{1,0}(-1 \mid 0)$$

02.05.26.0005.01

$$e = e\, G_{0,1}^{1,0}(z \mid 0) + e\, G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 1 \\ 1, 0 \end{matrix}\right)$$

Through other functions

02.05.26.0004.01

$$e = e^z \;/;\; z = 1$$

Representations through equivalent functions

02.05.27.0001.01

$$e^{\pi i} = -1$$

identity due to L.Euler

02.05.27.0002.01

$$e^{2 \pi i} = 1$$

02.05.27.0003.01

$$e^{\pi i k} = (-1)^k \;/;\; k \in \mathbb{Z}$$

02.05.27.0004.01

$$e^{-\frac{\pi}{2}} = i^i$$

02.05.27.0005.01

$$\log(e) = 1$$

Inequalities

02.05.29.0003.01

$$2 + \frac{7}{10} < e < 2 + \frac{3}{4}$$

02.05.29.0001.01

$$e^\pi \geq \pi^e$$

02.05.29.0002.01

$$\left(1 + \frac{1}{n}\right)^{n + \frac{1}{\log(2)} - 1} \leq e \leq \left(1 + \frac{1}{n}\right)^{n + \frac{1}{2}} \;/;\; n \in \mathbb{N}^+$$

History

– John Napier (1618) in the work on logarithms mentioned existence of special convinient constant for calculation of logarithms on its base but he did not evaluate it;
– William Oughtred (1622) found famouse slide rule, which is used for multiplication, division, evaluation of roots, logarithms and other functions;
– Isaac Newton (1669)published series $2 + 1/2! + 1/3! + \ldots == 2.71828 \ldots$;
– Jacob Bernoulli tried to find the limit of $\left(1 + \frac{1}{n}\right)^n$, when $n -> \infty$;
– Leibniz (1690)was the first to recognize e as a constant, but he used notation b;
– L. Euler (1727, 1728)denoted limit $\lim_{x \to 0} (1 + x)^{1/x}$ by the letter e;
– L. Euler (1731) introduced the notation e in a letter to Goldbach;
– L. Euler (1737) proved that e and e^2 are irrational numbers and represented e through continued fractions;
– L. Euler (1748) represented e as sum of series $1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ and found its 23 digits;
– D. Bernoulli (1760) used e, as base of the natural logarithms;
– J. H. Lambert (1768) proved that $e^{p/q}$is irrational number, if p/q is nonzero rational number;
– A. L. Cauchy (1823) determined $e == \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z$;
– J. Liouville (1844) proved that e does not satisfy any quadratic equation with integral coefficients;
– Ch. Hermite (1873) proved that e is a transcendental number;
– E. Catalan (1873) represented e through infinite product;

The only constant appearing more frequently in mathematics than e is π.

Copyright