EllipticTheta2

Notations

Traditional name

Jacobi theta function ϑ_2

Traditional notation

$\vartheta_2(z, q)$

Mathematica StandardForm notation

`EllipticTheta[2, z, q]`

Primary definition

09.02.02.0001.01

$$\vartheta_2(z, q) = 2 \sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)} \cos((2k+1)z) /; |q| < 1$$

Specific values

Specialized values

For fixed *z*

09.02.03.0001.01

$$\vartheta_2(z, 0) = 0$$

For fixed *q*

09.02.03.0006.01

$$\vartheta_2(0, q) = \frac{2}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{2 i \log(q)}{\pi}\right)^2$$

09.02.03.0005.02

$$\vartheta_2(0, q) = \sqrt{\frac{2}{\pi}} \sqrt[4]{q^{-1}(q)} \sqrt{K\left(q^{-1}(q)\right)}$$

09.02.03.0003.01

$$\vartheta_2\left(0, e^{\pi i \tau}\right) = \frac{2 \eta(2\tau)^2}{\eta(\tau)} /; \operatorname{Im}(\tau) > 0$$

09.02.03.0004.01

$$\vartheta_2\left(0, e^{-\frac{i\pi}{\tau}}\right) == \frac{\sqrt{\tau}}{\sqrt{i}} \vartheta_4(0, q) /; q == e^{i\pi\tau}$$

09.02.03.0007.01

$$\vartheta_2\left(-\frac{\pi}{2}, q\right) == 0$$

09.02.03.0008.01

$$\vartheta_2\left(\frac{\pi}{2}, q\right) == 0$$

09.02.03.0009.01

$$\vartheta_2\left(-\frac{\pi}{6}, e^{\pi i\tau}\right) = \sqrt{3}\ \eta(3\tau) /; \operatorname{Im}(\tau) > 0 \land |\operatorname{Re}(\tau)| < 1$$

09.02.03.0010.01

$$\vartheta_2\left(\frac{\pi}{6}, e^{\pi i\tau}\right) = \sqrt{3}\ \eta(3\tau) /; \operatorname{Im}(\tau) > 0 \land |\operatorname{Re}(\tau)| < 1$$

09.02.03.0011.01

$$\vartheta_2(m\pi, q) == \frac{2(-1)^m}{\eta\left(-\frac{i\log(q)}{\pi}\right)} \eta\left(-\frac{2i\log(q)}{\pi}\right)^2 /; m \in \mathbb{Z}$$

09.02.03.0012.01

$$\vartheta_2\left(\pi\left(m + \frac{1}{2}\right), q\right) == 0 /; m \in \mathbb{Z}$$

09.02.03.0002.01

$$\vartheta_2\left((2m+1)\frac{\pi}{2} + n\pi\tau, q\right) == 0 /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i\pi\tau}$$

General characteristics

Domain and analyticity

$\vartheta_2(z, q)$ is an analytic function of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

09.02.04.0001.01

$$(2 * z * q) \longrightarrow \vartheta_2(z, q) :: (\{2\} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\vartheta_2(z, q)$ is an even function with respect to z.

09.02.04.0002.01

$$\vartheta_2(-z, q) == \vartheta_2(z, q)$$

09.02.04.0003.01

$$\vartheta_2(z, -q) == \exp\left(-\frac{i\pi}{4} \operatorname{sgn}(\operatorname{Im}(q))\right) \vartheta_2(z, q)$$

Mirror symmetry

09.02.04.0004.01

$\vartheta_2(\bar{z}, \bar{q}) = \overline{\vartheta_2(z, q)}$

Periodicity

The function $\vartheta_2(z, q)$ is a periodic function with respect to z with period 2π and a quasi-period $i \log(q)$.

09.02.04.0005.01

$\vartheta_2(z + \pi, q) = -\vartheta_2(z, q)$

09.02.04.0022.01

$\vartheta_2(z + 2\pi, q) = \vartheta_2(z, q)$

09.02.04.0006.01

$\vartheta_2(z + m\pi, q) = (-1)^m \vartheta_2(z, q) /; m \in \mathbb{Z}$

09.02.04.0007.01

$$\vartheta_2(z + \pi\tau, q) = \frac{e^{-2iz}}{q} \vartheta_2(z, q) /; q = e^{i\pi\tau} \bigwedge \operatorname{Im}(\tau) > 0$$

09.02.04.0009.01

$$\vartheta_2(z + i\log(q), q) = \frac{e^{2iz}}{q} \vartheta_2(z, q)$$

09.02.04.0008.01

$\vartheta_2(z + m\pi\tau, q) = q^{-m} e^{-i(2mz + m(m-1)\pi\tau)} \vartheta_2(z, q) /; m \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$

09.02.04.0010.01

$\vartheta_2(z + i m \log(q), q) = q^{-m^2} e^{2miz} \vartheta_2(z, q) /; m \in \mathbb{Z}$

09.02.04.0011.01

$\vartheta_2(z + m\pi + n\pi\tau, q) = (-1)^m q^{-n^2} e^{-2nzi} \vartheta_2(z, q) /; \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$

Poles and essential singularities

With respect to *q*

The function $\vartheta_2(z, q)$ does not have poles and essential singularities inside of the unit circle $|q| < 1$

09.02.04.0012.01

$\mathcal{S}ing_q(\vartheta_2(z, q)) = \{\}$

With respect to *z*

09.02.04.0013.01

$\mathcal{S}ing_z(\vartheta_2(z, q)) = \{\}$

Branch points

With respect to *q*

For fixed z, the function $\vartheta_2(z, q)$ has one branch point: $q = 0$. (The point $q = -1$ is the branch cut endpoint.)

09.02.04.0014.01

$\mathcal{BP}_q(\vartheta_2(z, q)) = \{0\}$

09.02.04.0015.01

$\mathcal{R}_q(\vartheta_2(z, q), 0) = 4$

With respect to z

For fixed q, the function $\vartheta_2(z, q)$ does not have branch points.

09.02.04.0016.01

$\mathcal{BP}_z(\vartheta_2(z, q)) = \{\}$

Branch cuts

With respect to q

For fixed z, the function $\vartheta_2(z, q)$ is a single-valued function inside the unit circle of the complex q-plane, cut along the interval $(-1, 0)$, where it is continuous from above.

09.02.04.0017.01

$\mathcal{BC}_q(\vartheta_2(z, q)) = \{\{(-1, 0), -i\}\}$

09.02.04.0018.01

$\lim_{\epsilon \to +0} \vartheta_2(z, q + i\,\epsilon) = \vartheta_2(z, q) \;/;\; -1 < q < 0$

09.02.04.0019.01

$\lim_{\epsilon \to +0} \vartheta_2(z, q - i\,\epsilon) = -i\,\vartheta_2(z, q) \;/;\; -1 < q < 0$

With respect to z

For fixed q, the function $\vartheta_2(z, q)$ does not have branch cuts.

09.02.04.0020.01

$\mathcal{BC}_z(\vartheta_2(z, q)) = \{\}$

Natural boundary of analyticity

The unit circle $|q| = 1$ is the natural boundary of the region of analyticity.

09.02.04.0021.01

$\mathcal{AB}_z(\vartheta_2(q, z)) = \left\{e^{i\,(-\pi, \pi)}\right\}$

Branch cut endpoints

The function $\vartheta_2(z, q)$ has one branch cut endpoint: $q = -1$.

Series representations

q-series

Expansions at generic point $z = z_0$

09.02.06.0011.01

$$\vartheta_2(z, q) \propto \vartheta_2(z_0, q) + \vartheta_2^{(1,0)}(z_0, q)\,(z - z_0) + \frac{\vartheta_2^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_2^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\big((z - z_0)^4\big)$$

09.02.06.0012.01

$$\vartheta_2(z, q) \propto \vartheta_2(z_0, q) + \vartheta_2'(z_0, q)\,(z - z_0) + \frac{\vartheta_2^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_2^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\big((z - z_0)^4\big)$$

09.02.06.0013.01

$$\vartheta_2(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_2^{(k,0)}(z_0, q)}{k!}\,(z - z_0)^k$$

09.02.06.0014.01

$$\vartheta_2(z, q) \propto \vartheta_2(z_0, q)\,(1 + O(z - z_0))$$

Expansions at generic point $q == q_0$

09.02.06.0015.01

$$\vartheta_2(z, q) \propto \vartheta_2(z, q_0) + \vartheta_2^{(0,1)}(z, q_0)\,(q - q_0) + \frac{\vartheta_2^{(0,2)}(z, q_0)}{2}\,(q - q_0)^2 + \frac{\vartheta_2^{(0,3)}(z, q_0)}{6}\,(q - q_0)^3 + O\big((q - q_0)^4\big)$$

09.02.06.0016.01

$$\vartheta_2(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_2^{(0,k)}(z, q_0)}{k!}\,(q - q_0)^k$$

09.02.06.0017.01

$$\vartheta_2(z, q) \propto \vartheta_2(z, q_0)\,(1 + O(q - q_0))$$

Expansions on branch cuts

09.02.06.0018.01

$$\vartheta_2(z, q) \propto e^{\frac{\pi i}{2}\left\lfloor \frac{\arg(q-x)}{2\pi} \right\rfloor}\left(\vartheta_2(z, x) + \vartheta_2^{(0,1)}(z, x)\,(q - x) + \frac{\vartheta_2^{(0,2)}(z, x)}{2}\,(q - x)^2 + \frac{\vartheta_2^{(0,3)}(z, x)}{6}\,(q - x)^3 + O\big((q - x)^4\big)\right) /;$$
$$x \in \mathbb{R} \land -1 < x < 0$$

09.02.06.0019.01

$$\vartheta_2(z, q) = e^{\frac{\pi i}{2}\left\lfloor \frac{\arg(q-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{\vartheta_2^{(0,k)}(z, x)}{k!}\,(q - x)^k \;/;\; x \in \mathbb{R} \land -1 < x < 0$$

09.02.06.0020.01

$$\vartheta_2(z, q) \propto e^{\frac{\pi i}{2}\left\lfloor \frac{\arg(q-x)}{2\pi} \right\rfloor} \vartheta_2(z, x)\,(1 + O(q - x)) \;/;\; x \in \mathbb{R} \land -1 < x < 0$$

Expansions at $q == 0$

09.02.06.0021.01

$$\vartheta_2(z, q) \propto 2\sqrt[4]{q}\,\big(\cos(z) + \cos(3z)\,q^2 + \cos(5z)\,q^6 + \cos(7z)\,q^{12} + \ldots\big) \;/;\; (q \to 0)$$

09.02.06.0001.01

$$\vartheta_2(z, q) == 2\sqrt[4]{q}\,\sum_{k=0}^{\infty} q^{k(k+1)} \cos((2k + 1)\,z) \;/;\; |q| < 1$$

09.02.06.0002.01

$$\vartheta_2(z, q) = \sqrt[4]{q} \sum_{k=-\infty}^{\infty} q^{k(k+1)} e^{(2n+1) i z}$$

09.02.06.0003.01

$$\vartheta_2(0, q) = 2\sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)}$$

09.02.06.0022.01

$$\vartheta_2(z, q) \propto 2\sqrt[4]{q}\left(\cos(z) + O\left(q^2\right)\right) /; (q \to 0)$$

Expansions at $q == 1$

09.02.06.0023.01

$$\vartheta_2(z, q) \propto \frac{i\sqrt{\pi}}{\sqrt{q-1}} e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor}\left(1 + \frac{q-1}{4} - \frac{1}{96} 7 (q-1)^2 + \ldots\right) e^{\frac{z^2}{\log(q)}}\left(1 - 2 e^{\frac{\pi^2}{\log(q)}} \cosh\left(\frac{2\pi z}{\log(q)}\right) + 2 e^{\frac{4\pi^2}{\log(q)}} \cosh\left(\frac{4\pi z}{\log(q)}\right) + \ldots\right) /; (q \to 1) \land |q| < 1$$

09.02.06.0024.01

$$\vartheta_2(z, q) = \frac{i\sqrt{\pi}}{\sqrt{q-1}} e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \binom{k+\frac{1}{2}}{k} \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{k}{j} p_{j,k} (q-1)^k e^{\frac{z^2}{\log(q)}} \left(1 + 2\sum_{k=1}^{\infty} (-1)^k e^{\frac{k^2\pi^2}{\log(q)}} \cosh\left(\frac{2k\pi z}{\log(q)}\right)\right) /;$$

$$(|q| < 1 \land |q-1| < 1) \bigwedge c_k = \frac{(-1)^k}{k+1} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m - k + m)\, c_m\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

09.02.06.0025.01

$$\vartheta_2(z, q) \propto \frac{i\sqrt{\pi}}{\sqrt{q-1}} e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} (1 + O(q-1))\, e^{\frac{z^2}{\log(q)}} \left(1 + O\left(e^{\frac{\pi^2}{\log(q)}} \cosh\left(\frac{2\pi z}{\log(q)}\right)\right)\right) /; |q| < 1$$

Other q-series representations

09.02.06.0004.01

$$\frac{\vartheta_2'(z, q)}{\vartheta_2(z, q)} = -\tan(z) + 4\sum_{k=1}^{\infty} (-1)^k \frac{q^{2k}}{1 - q^{2k}} \sin(2kz)$$

09.02.06.0005.01

$$\log\left(\frac{\vartheta_2(a+b, q)}{\vartheta_2(a-b, q)}\right) = \log\left(\frac{\cos(a+b)}{\cos(a-b)}\right) + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{q^{2k}}{1-q^{2k}} \sin(2ka) \sin(2kb)$$

09.02.06.0006.01

$$\log(\vartheta_2(z, q)) = \log(\vartheta_2(0, q)) + \log\left(\sin\left(z + \frac{\pi}{2}\right)\right) + 4\sum_{k=1}^{\infty} (-1)^k \frac{q^{2k}}{k\left(1-q^{2k}\right)} \sin^2(kz)$$

09.02.06.0007.01

$$\frac{\vartheta_1'(0, q)\,\vartheta_1(z, q)}{4\,\vartheta_2(0, q)\,\vartheta_2(z, q)} = \frac{\tan(z)}{4} + \sum_{k=1}^{\infty} (-1)^k \frac{q^{2k}}{1+q^{2k}} \sin(2kz) /; |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.02.06.0008.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_1(z,q)}{4\,\vartheta_2(0,q)\,\vartheta_2(z,q)} == \frac{\tan(z)}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k \left(q^{2k}\sin(2z)\right)}{1+2\cos(2z)\,q^{2k}+q^{4k}} \;/;\; |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q == e^{i\pi\tau}$$

Other series representations

09.02.06.0026.01

$$\vartheta_2(z,q) = \frac{\sqrt[4]{-1}\,\sqrt{\pi}}{\sqrt{-i\log(q)}}\, e^{\frac{z^2}{\log(q)}} \left(1 + 2\sum_{k=1}^{\infty}(-1)^k e^{\frac{k^2\pi^2}{\log(q)}} \cosh\left(\frac{2k\pi z}{\log(q)}\right)\right)$$

09.02.06.0009.01

$$\vartheta_2(z,q) == \exp\left(-\frac{i z^2}{\pi\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(i\pi\tau\left(n+\frac{1}{2}+\frac{z}{\pi\tau}\right)^2\right) /;\; q == e^{i\pi\tau}$$

09.02.06.0010.01

$$\vartheta_2(z,q) == \frac{\sqrt{i}}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(-\frac{\pi i}{\tau}\left(\frac{z}{\pi}+n\right)^2\right) /;\; q == e^{i\pi\tau}$$

Product representations

09.02.08.0001.01

$$\vartheta_2(0,q) == 2\sqrt[4]{q} \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+q^{2n}\right)^2$$

09.02.08.0002.01

$$\vartheta_2(z,q) == 2\sqrt[4]{q}\, \cos(z) \prod_{k=1}^{\infty} \left(1-q^{2k}\right)\left(1+2q^{2k}\cos(2z)+q^{4k}\right)$$

Differential equations

Partial differential equations

The elliptic theta functions satisfy the (one-dimensional) heat equation:

09.02.13.0001.01

$$\frac{\partial\vartheta_2(z,q)}{\partial\tau} == -\frac{\pi i}{4}\,\frac{\partial^2\vartheta_2(z,q)}{\partial z^2} /;\; q == e^{i\pi\tau}$$

09.02.13.0002.01

$$4q\,\frac{\partial\vartheta_2(z,q)}{\partial q} + \frac{\partial^2\vartheta_2(z,q)}{\partial z^2} == 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.02.16.0007.01

$$\vartheta_2(z, q) = \frac{\sqrt[4]{-1}\ \sqrt{\pi}\ e^{\frac{z^2}{\log(q)}}}{\sqrt{-i \log(q)}}\ \vartheta_4\left(\frac{i \pi z}{\log(q)}, e^{\frac{\pi^2}{\log(q)}}\right)$$

09.02.16.0001.01

$$\vartheta_2\left(\frac{z}{\tau}, e^{-\frac{i \pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i z^2}{\pi \tau}\right) \vartheta_4\left(z, e^{i \pi \tau}\right)$$

n th root of q

09.02.16.0002.01

$$\vartheta_2\left(z, q^{1/n}\right) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{\frac{2r}{n}}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_2\left(z + \frac{i r \log(q)}{n}, q\right) /; \frac{n+1}{2} \in \mathbb{Z}^+$$

Multiple angle formulas

09.02.16.0003.01

$$\vartheta_2(n z, q^n) = (-1)^{\frac{n-1}{2}} \left(\prod_{r=1}^{\infty} \frac{1 - q^{2 n r}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_2\left(z + \frac{\pi r}{n}, q\right) /; \frac{n+1}{2} \in \mathbb{Z}^+$$

09.02.16.0004.01

$$\vartheta_2(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2 n r}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_2\left(z + \frac{\pi r}{n}, q\right) /; n \in \mathbb{Z}^+$$

09.02.16.0005.01

$$\frac{\vartheta_2\left(\pi\left((n-1) z + \frac{1}{2}\right), e^{i \pi \tau}\right)}{\vartheta_2\left(\pi\left(z + \frac{1}{2}\right), e^{i \pi \tau}\right)} = \left(\prod_{k=1}^{n-1} \frac{\vartheta_2\left(\pi\left(n z + k \tau + \frac{1}{2}\right), e^{i n \pi \tau}\right)}{\vartheta_2\left(\pi\left(k \tau + \frac{1}{2}\right), e^{i n \pi \tau}\right)}\right) \sum_{k=1}^{n-1} \frac{\vartheta(k \tau, n \tau)\, e^{\left(n^2 - n - 2k\right) i \pi z}}{\vartheta(n z + k \tau, n \tau)} /; \operatorname{Im}(\tau) > 0 \land n \in \mathbb{N}$$

09.02.16.0006.01

$$\frac{\vartheta_2\left(\pi\left((n-1) z - \frac{1}{2}\right), e^{i \pi \tau}\right)}{\vartheta_2\left(\pi\left(z - \frac{1}{2}\right), e^{i \pi \tau}\right)} = \left(\prod_{k=1}^{n-1} \frac{\vartheta_2\left(\pi\left(n z + k \tau - \frac{1}{2}\right), e^{i n \pi \tau}\right)}{\vartheta_2\left(\pi\left(k \tau - \frac{1}{2}\right), e^{i n \pi \tau}\right)}\right) \sum_{k=1}^{n-1} \frac{\vartheta(k \tau, n \tau)\, e^{\left(n^2 - n - 2k\right) i \pi z}}{\vartheta(n z + k \tau, n \tau)} /; \operatorname{Im}(\tau) > 0 \land n \in \mathbb{N}$$

Identities

Functional identities

09.02.17.0001.01

$$\left(\frac{3 \vartheta_2(0, q^9)}{\vartheta_2(0, q)} - 1\right)^3 = \frac{9 \vartheta_2(0, q^3)^4}{\vartheta_2(0, q)^4} - 1$$

Differentiation

Low-order differentiation

With respect to *z*

09.02.20.0001.01

$$\frac{\partial \vartheta_2(z, q)}{\partial z} = \vartheta_2'(z, q)$$

09.02.20.0002.01

$$\frac{\partial^2 \vartheta_2(z, q)}{\partial z^2} = -2 \sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)} (2k+1)^2 \cos((2k+1)z) /; |q| < 1$$

With respect to *q*

09.02.20.0009.01

$$\frac{\partial \vartheta_2(z, q)}{\partial q} = -\frac{1}{4q} \vartheta_2(z, q) \left(\frac{\vartheta_1'(z, q)}{\vartheta_1(z, q)} - \vartheta_2(0, q)^2 \frac{\vartheta_3(z, q) \vartheta_4(z, q)}{\vartheta_1(z, q) \vartheta_2(z, q)} \right)^2 + \frac{1}{4q} \vartheta_3(0, q)^2 \vartheta_4(0, q)^2 \frac{\vartheta_1(z, q)^2}{\vartheta_2(z, q)} + \frac{1}{q \pi^2} \vartheta_2(z, q) \left(\frac{\pi^2}{12} \left(\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4 \right) + \zeta\left(1; g_2\left(1, \frac{\log(q)}{\pi i}\right), g_3\left(1, \frac{\log(q)}{\pi i}\right)\right) \right)$$

09.02.20.0003.01

$$\frac{\partial \vartheta_2(z, q)}{\partial q} = \frac{\vartheta_2(z, q)}{4q} + 2 \sum_{k=1}^{\infty} k(k+1) q^{k(k+1)-\frac{3}{4}} \cos((2k+1)z) /; |q| < 1$$

09.02.20.0004.01

$$\frac{\partial^2 \vartheta_2(z, q)}{\partial q^2} = 2 q^{-\frac{7}{4}} \sum_{k=0}^{\infty} q^{k(k+1)} \left(k^2 + k - \frac{3}{4}\right) \left(k^2 + k + \frac{1}{4}\right) \cos((2k+1)z) /; |q| < 1$$

Symbolic differentiation

With respect to *z*

09.02.20.0005.01

$$\frac{\partial^n \vartheta_2(z, q)}{\partial z^n} = 2 \sqrt[4]{q} \sum_{k=0}^{\infty} q^{k(k+1)} (2k+1)^n \cos\left(\frac{\pi n}{2} + (2k+1)z\right) /; |q| < 1 \land n \in \mathbb{N}^+$$

With respect to *q*

09.02.20.0006.01

$$\frac{\partial^n \vartheta_2(z, q)}{\partial q^n} = 2 q^{\frac{1}{4}-n} \sum_{k=0}^{\infty} q^{k(k+1)} \left(k(k+1) - n + \frac{5}{4}\right)_n \cos((2k+1)z) /; |q| < 1 \land n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.02.20.0007.01

$$\frac{\partial^\alpha \vartheta_2(z, q)}{\partial z^\alpha} = 2^{\alpha+1} \sqrt{\pi} \sqrt[4]{q} z^{-\alpha} \sum_{k=0}^{\infty} q^{k(k+1)} {}_1\tilde{F}_2\left(1; \frac{1-\alpha}{2}, 1 - \frac{\alpha}{2}; -\frac{1}{4}(2k+1)^2 z^2\right) /; |q| < 1$$

With respect to *q*

09.02.20.0008.01

$$\frac{\partial^{\alpha} \vartheta_2(z, q)}{\partial q^{\alpha}} = 2\, q^{\frac{1}{4}-\alpha} \sum_{k=0}^{\infty} \frac{q^{k(k+1)}\, \Gamma\left(k^2+k+\frac{5}{4}\right) \cos((2k+1)z)}{\Gamma\left(k^2+k-\alpha+\frac{5}{4}\right)} \;/;\; |q|<1$$

Integration

Indefinite integration

Involving only one direct function

09.02.21.0001.01

$$\int \vartheta_2(z, q)\, dz = 2\sqrt[4]{q} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}}{2k+1} \sin((2k+1)z) \;/;\; |q|<1$$

Involving only one direct function with respect to *q*

09.02.21.0002.01

$$\int \vartheta_2(z, q)\, dq = 2 \sum_{k=0}^{\infty} \frac{q^{k(k+1)+\frac{5}{4}} \cos((2k+1)z)}{k(k+1)+\frac{5}{4}} \;/;\; |q|<1$$

Representations through equivalent functions

With related functions

Involving theta functions

Involving $\vartheta_1(z, q)$

09.02.27.0001.02

$$\vartheta_2(z, q) = -\vartheta_1\left(z - \frac{\pi}{2}, q\right)$$

09.02.27.0012.01

$$\vartheta_2(z, q) = \vartheta_1\left(z + \frac{\pi}{2}, q\right)$$

09.02.27.0002.02

$$\vartheta_2(z, q) = (-1)^m \vartheta_1\left(\frac{1}{2}\pi(2m+1)+z, q\right) \;/;\; m \in \mathbb{Z}$$

Involving $\vartheta_3(z, q)$

09.02.27.0003.02

$$\vartheta_2(z, q) = \sqrt[4]{q}\, e^{-iz}\, \vartheta_3\left(z - \frac{\pi\tau}{2}, q\right) \;/;\; q = e^{i\pi\tau}$$

09.02.27.0004.02

$$\vartheta_2(z, q) = q^{m^2+m+\frac{1}{4}}\, e^{-i(2m+1)z}\, \vartheta_3\left(z - \frac{1}{2}(2m+1)\pi\tau, q\right) \;/;\; m \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.02.27.0005.02

$$\vartheta_2(z, q) = \sqrt[4]{q}\, e^{i z}\, \vartheta_3\left(z - \frac{1}{2} i \log(q), q\right)$$

09.02.27.0006.02

$$\vartheta_2(z, q) = q^{m^2+m+\frac{1}{4}}\, e^{-i(2m+1)z}\, \vartheta_3\left(\frac{i \log(q)}{2}\,(2m+1) + z, q\right) /; m \in \mathbb{Z}$$

Involving $\vartheta_4(z, q)$

09.02.27.0007.02

$$\vartheta_2(z, q) = \sqrt[4]{q}\, e^{i z}\, \vartheta_4\left(z + \frac{1}{2}\pi(\tau + 1), q\right) /; q = e^{i \pi \tau}$$

09.02.27.0013.01

$$\vartheta_2(z, q) = q^{m^2+m+\frac{1}{4}}\, e^{-i(2m+1)z}\, \vartheta_4\left(\frac{\pi(2m+1)}{2}(1 - \tau) + z, q\right) /; m \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}$$

09.02.27.0014.01

$$\vartheta_2(z, q) = \sqrt[4]{q}\, e^{i z}\, \vartheta_4\left(z + \frac{1}{2}(\pi - i \log(q)), q\right)$$

09.02.27.0015.01

$$\vartheta_2(z, q) = q^{m^2+m+\frac{1}{4}}\, e^{-i(2m+1)z}\, \vartheta_4\left(\frac{1}{2}(2m+1)(i \log(q) + \pi) + z, q\right) /; m \in \mathbb{Z}$$

Involving Jacobi functions

09.02.27.0016.01

$$\frac{\vartheta_2(z, q(m))}{\vartheta_1(z, q(m))} = \frac{1}{\sqrt[4]{1-m}} \operatorname{cs}\left(\frac{2K(m)z}{\pi} \,\middle|\, m\right)$$

09.02.27.0017.01

$$\frac{\vartheta_2(z, q(m))}{\vartheta_3(z, q(m))} = \sqrt[4]{m}\, \operatorname{cd}\left(\frac{2K(m)z}{\pi} \,\middle|\, m\right)$$

09.02.27.0008.02

$$\frac{\vartheta_2(z, q(m))}{\vartheta_4(z, q(m))} = \frac{\sqrt[4]{m}}{\sqrt[4]{1-m}} \operatorname{cn}\left(\frac{2K(m)z}{\pi} \,\middle|\, m\right)$$

Involving Weierstrass functions

09.02.27.0009.01

$$\vartheta_2(z, q) = 2\sqrt[4]{q}\left(\prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\right)\left(\prod_{n=1}^{\infty}\left(1 + q^{2n}\right)\right)^2 \exp\left(-\frac{2\eta_1 \omega_1 z^2}{\pi^2}\right)\sigma_1(u; g_2, g_3) /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.02.27.0010.01

$$\frac{\vartheta_2(z,q)}{\vartheta_2(0,q)} = \exp\left(-\frac{2\eta_1\omega_1 z^2}{\pi^2}\right)\sigma_1\left(\frac{2\omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.02.27.0011.01

$$\frac{\vartheta_2'(z,q)}{\vartheta_2(z,q)} = \frac{2\omega_1}{\pi}\zeta\left(\frac{2\omega_1}{\pi}\left(z+\frac{\pi}{2}\right); g_2, g_3\right) - \frac{2\omega_1\eta_1}{\pi} - \frac{4\omega_1\eta_1 z}{\pi^2} /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

Zeros

09.02.30.0002.01

$$\vartheta_2(z, 0) = 0$$

09.02.30.0003.01

$$\vartheta_2\left(\frac{\pi}{2}, q\right) = 0 /; m \in \mathbb{Z}$$

09.02.30.0004.01

$$\vartheta_2\left((2m+1)\frac{\pi}{2}, q\right) = 0 /; m \in \mathbb{Z}$$

09.02.30.0001.01

$$\vartheta_2\left((2m+1)\frac{\pi}{2} + n\pi\tau, q\right) = 0 /; \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

Theorems

The effective potential for the Lagrangian

The effective potential $V(T, \mu, m; \boldsymbol{A})$ for the Lagrangian $\mathcal{L} = \overline{\psi}\gamma^\nu(\partial_\nu - i e A_\nu)\psi - m\overline{\psi}\psi$ at temperature T and chemical potential μ in the one loop approximation is given by

$$V(T, \mu, m; \boldsymbol{A}) = -\frac{\mathrm{Tr}[\mathbf{1}]}{2k_B T}\int_0^\infty \frac{1}{s}\vartheta_2\left(\frac{2s\mu}{k_B T}, \frac{4 i\pi s}{k_B^2 T^2}\right) s\, I[A] \coth(s\, I[A])\, e^{-s(m^2-\mu^2)}\, ds,$$

where $I[\boldsymbol{A}] = e\sqrt{\boldsymbol{E}^2 - \boldsymbol{B}^2}$ and $\mathrm{Tr}[\mathbf{1}]$ is the corresponding gamma trace.

History

- J. Bernoulli (1713)
- L. Euler; J. Fourier
- C. G. J. Jacobi (1827)
- C. W. Borchardt (1838)
- K. Weierstrass (1862–1863)

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