EllipticThetaPrime3

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Notations

Traditional name

Derivative of the Jacobi theta function ϑ_3

Traditional notation

$\vartheta_3'(z, q)$

Mathematica StandardForm notation

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EllipticThetaPrime[3, z, q]
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Primary definition

09.07.02.0001.01

$$\vartheta_3'(z, q) = -4 \sum_{k=1}^{\infty} q^{k^2} k \sin(2 k z) /; |q| < 1$$

Specific values

Specialized values

For fixed *z*

09.07.03.0001.01

$$\vartheta_3'(z, 0) = 0$$

For fixed *q*

09.07.03.0002.01

$$\vartheta_3'(0, q) = 0$$

09.07.03.0004.01

$$\vartheta_3'\left(-\frac{\pi}{4}, q\right) = 4\, \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3$$

09.07.03.0005.01

$$\vartheta_3'\left(\frac{\pi}{4}, q\right) = -4\, \eta\left(-\frac{4 i \log(q)}{\pi}\right)^3$$

09.07.03.0003.01

$$\vartheta_3'\left(\frac{\pi m}{2}, q\right) == 0 \;/;\; m \in \mathbb{Z}$$

09.07.03.0006.01

$$\vartheta_3'\left(\frac{\pi m}{2} + \frac{\pi}{4}, q\right) == 4\,(-1)^{m-1}\,\eta\left(-\frac{4\,i\log(q)}{\pi}\right)^3 \;/;\; m \in \mathbb{Z}$$

09.07.03.0007.01

$$\vartheta_3'(-i\log(q), q) == -\frac{2\,i}{q}\sqrt{\frac{2}{\pi}}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.07.03.0008.01

$$\vartheta_3'\left(-\frac{i\log(q)}{2}, q\right) == -i\,\frac{1}{\sqrt[4]{q}}\sqrt{\frac{2}{\pi}}\,\sqrt[4]{q^{-1}(q)}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.07.03.0009.01

$$\vartheta_3'\left(\frac{1}{2}(\pi - i\log(q)), q\right) == 2\,i\,\frac{1}{\sqrt[4]{q}}\,\eta\left(-\frac{i\log(q)}{\pi}\right)^3$$

09.07.03.0010.01

$$\vartheta_3'(m\pi + n\,i\log(q), q) == 2\,n\,i\,q^{-n^2}\sqrt{\frac{2}{\pi}}\,\sqrt{K\left(q^{-1}(q)\right)} \;/;\; \{m, n\} \in \mathbb{Z}$$

General characteristics

Domain and analyticity

$\vartheta_3'(z, q)$ is an analytic function of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

09.07.04.0001.01

$$(3 * z * q) \longrightarrow \vartheta_3'(z, q) :: (\{3\} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\vartheta_3'(z, q)$ is an odd function with respect to z.

09.07.04.0002.01

$$\vartheta_3'(-z, q) == -\vartheta_3'(z, q)$$

09.07.04.0003.01

$$\vartheta_3'(z, -q) == \vartheta_4'(z, q)$$

Mirror symmetry

09.07.04.0004.01

$$\vartheta_3'(\bar{z}, \bar{q}) == \overline{\vartheta_3'(z, q)}$$

Periodicity

$\vartheta_3'(z, q)$, considered as a function of z, has a period of π.

09.07.04.0005.01

$$\vartheta_3'(z+\pi, q) = \vartheta_3'(z, q)$$

09.07.04.0006.01

$$\vartheta_3'(z+m\,\pi, q) = \vartheta_3'(z, q) \;/;\; m \in \mathbb{Z}$$

Poles and essential singularities

With respect to *q*

The function $\vartheta_3'(z, q)$ does not have poles and essential singularities inside of the unit circle $|q| < 1$.

09.07.04.0007.01

$$\mathcal{S}ing_q\left(\vartheta_3'(z, q)\right) = \{\}$$

With respect to *z*

09.07.04.0008.01

$$\mathcal{S}ing_z\left(\vartheta_3'(z, q)\right) = \{\}$$

Branch points

With respect to *q*

The function $\vartheta_3'(z, q)$ does not have branch points with respect to q.

09.07.04.0009.01

$$\mathcal{BP}_q\left(\vartheta_3'(z, q)\right) = \{\}$$

With respect to *z*

The function $\vartheta_3'(z, q)$ does not have branch points with respect to z.

09.07.04.0010.01

$$\mathcal{BP}_z\left(\vartheta_3'(z, q)\right) = \{\}$$

Branch cuts

With respect to *q*

The function $\vartheta_3'(z, q)$ does not have branch cuts with respect to q.

09.07.04.0011.01

$$\mathcal{BC}_q\left(\vartheta_3'(z, q)\right) = \{\}$$

With respect to *z*

The function $\vartheta_3'(z, q)$ does not have branch cuts with respect to z.

09.07.04.0012.01

$$\mathcal{BC}_z\left(\vartheta_3'(z, q)\right) = \{\}$$

Natural boundary of analyticity

The unit circle $|q| == 1$ is the natural boundary of the region of analyticity.

09.07.04.0013.01

$$\mathcal{AB}_z\left(\vartheta_3'(z, q)\right) == \left\{e^{i\,(-\pi,\pi)}\right\}$$

Series representations

q-series

Expansions at generic point $q == q_0$

09.07.06.0005.01

$$\vartheta_3'(z, q) \propto \vartheta_3'(z, q_0) + \vartheta_3^{(1,1)}(z, q_0)\,(q - q_0) + \frac{\vartheta_3^{(1,2)}(z, q_0)}{2}\,(q - q_0)^2 + \frac{\vartheta_3^{(1,3)}(z, q_0)}{6}\,(q - q_0)^3 + O\left((q - q_0)^4\right)$$

09.07.06.0006.01

$$\vartheta_3'(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_3^{(1,k)}(z, q_0)}{k!}\,(q - q_0)^k$$

09.07.06.0007.01

$$\vartheta_3'(z, q) \propto \vartheta_3'(z, q_0)\,\left(1 + O(q - q_0)\right)$$

Expansions at $q == 0$

09.07.06.0008.01

$$\vartheta_3'(z, q) \propto -4\sin(2\,z)\,q - 8\sin(4\,z)\,q^4 - 12\sin(6\,z)\,q^9 - 16\sin(8\,z)\,q^{16} + \ldots\ /;\ (q \to 0)$$

09.07.06.0001.01

$$\vartheta_3'(z, q) == -4\sum_{k=1}^{\infty} q^{k^2}\,k\sin(2\,k\,z)\ /;\ |q| < 1$$

09.07.06.0002.01

$$\vartheta_3'(z, q) == 2\,i \sum_{k=-\infty}^{\infty} q^{k^2}\,k\,e^{2\,k\,i\,z}\ /;\ |q| < 1$$

09.07.06.0009.01

$$\vartheta_3'(z, q) \propto -4\sin(2\,z)\,q\left(1 + O\left(q^3\right)\right)\ /;\ (q \to 0)$$

Expansions at $q == 1$

09.07.06.0010.01

$$\vartheta_3'(z, q) \propto \frac{2\sqrt{\pi}\,i}{(q-1)^{3/2}}\,e^{-i\,\pi\left\lfloor -\frac{\arg(q-1)}{2\pi}\right\rfloor}\left(1 + \frac{3\,(q-1)}{4} - \frac{1}{32}\,(q-1)^2 + \frac{3}{128}\,(q-1)^3 + \ldots\right) e^{\frac{z^2}{\log(q)}}$$
$$\left(z + 2\,e^{\frac{\pi^2}{\log(q)}}\left(z\cosh\left(\frac{2\,\pi\,z}{\log(q)}\right) + \pi\sinh\left(\frac{2\,\pi\,z}{\log(q)}\right)\right) + 2\,e^{\frac{4\,\pi^2}{\log(q)}}\left(z\cosh\left(\frac{4\,\pi\,z}{\log(q)}\right) + 2\,\pi\sinh\left(\frac{4\,\pi\,z}{\log(q)}\right)\right) + \ldots\right)\ /;\ (q \to 1) \bigwedge |q| < 1$$

09.07.06.0011.01

$$\vartheta_3'(z,q) = \frac{6\sqrt{\pi}\,i}{(q-1)^{3/2}}\, e^{-3 i \pi \left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \binom{k+\frac{3}{2}}{k} \sum_{j=0}^{k} \frac{(-1)^j}{2j+3} \binom{k}{j} p_{j,k} (q-1)^k e^{\frac{z^2}{\log(q)}} \left(z + 2\sum_{m=1}^{\infty} e^{\frac{m^2\pi^2}{\log(q)}} \left(z \cosh\left(\frac{2 m \pi z}{\log(q)}\right) + m\pi \sinh\left(\frac{2 m \pi z}{\log(q)}\right)\right)\right) /;$$

$$(|q|<1 \wedge |q-1|<1) \bigwedge c_k = \frac{(-1)^{k-1}}{k+1} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = -\frac{1}{k}\sum_{m=1}^{k} (j\,m - k + m)\, c_m\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

09.07.06.0012.01

$$\vartheta_3'(z,q) \propto \frac{2\sqrt{\pi}\,i}{(q-1)^{3/2}}\, e^{-i\pi \left\lfloor -\frac{\arg(q-1)}{2\pi} \right\rfloor} (1 + O(q-1))\, e^{\frac{z^2}{\log(q)}} \left(z + 2\, e^{\frac{\pi^2}{\log(q)}} \left(z\cosh\left(\frac{2\pi z}{\log(q)}\right) + \pi \sinh\left(\frac{2\pi z}{\log(q)}\right)\right) + O\left(e^{\frac{4\pi^2}{\log(q)}} \left(z\cosh\left(\frac{4\pi z}{\log(q)}\right) + 2\pi\sinh\left(\frac{4\pi z}{\log(q)}\right)\right)\right)\right) /; |q| < 1$$

Other q-series representations

09.07.06.0003.01

$$\frac{\vartheta_3'(z,q)}{\vartheta_3(z,q)} = 4\sum_{k=1}^{\infty} (-1)^k \frac{q^k}{1-q^{2k}} \sin(2kz)$$

Other series representations

09.07.06.0013.01

$$\vartheta_3'(z,q) = -\frac{2(-1)^{3/4}\sqrt{\pi}}{(-i\log(q))^{3/2}}\, e^{\frac{z^2}{\log(q)}} \left(z + 2\sum_{k=1}^{\infty} e^{\frac{k^2\pi^2}{\log(q)}} \left(z\cosh\left(\frac{2k\pi z}{\log(q)}\right) + k\pi\sinh\left(\frac{2k\pi z}{\log(q)}\right)\right)\right)$$

09.07.06.0004.01

$$\vartheta_3'(z,q) = -\frac{2\, i^{3/2}}{\tau^{3/2}} \sum_{n=-\infty}^{\infty} \left(n + \frac{z}{\pi}\right) \exp\left(-\frac{\pi i}{\tau}\left(\frac{z}{\pi} + n\right)^2\right) /; q = e^{i\pi\tau}$$

Differential equations

Partial differential equations

The elliptic theta functions satisfy the one-dimensional heat equation:

09.07.13.0001.01

$$\frac{\partial \vartheta_3'(z,q)}{\partial \tau} = -\frac{\pi i}{4} \frac{\partial^2 \vartheta_3'(z,q)}{\partial z^2} /; q = e^{i\pi\tau}$$

09.07.13.0002.01

$$4q\frac{\partial \vartheta_3'(z,q)}{\partial q} + \frac{\partial^2 \vartheta_3'(z,q)}{\partial z^2} = 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.07.16.0001.01

$$\vartheta_3'(z, q) = -\frac{(-1)^{3/4} e^{\frac{z^2}{\log(q)}} \sqrt{\pi}}{(-i \log(q))^{3/2}} \left(2 z \vartheta_3\left(\frac{i \pi z}{\log(q)}, e^{\frac{\pi^2}{\log(q)}}\right) + i \pi \vartheta_3'\left(\frac{i \pi z}{\log(q)}, e^{\frac{\pi^2}{\log(q)}}\right)\right)$$

Differentiation

Low-order differentiation

With respect to *z*

09.07.20.0001.01

$$\frac{\partial \vartheta_3'(z, q)}{\partial z} = -8 \sum_{k=1}^{\infty} q^{k^2} k^2 \cos(2 k z) \;/;\; |q| < 1$$

09.07.20.0002.01

$$\frac{\partial^2 \vartheta_3'(z, q)}{\partial z^2} = 16 \sum_{k=1}^{\infty} q^{k^2} k^3 \sin(2 k z) \;/;\; |q| < 1$$

With respect to *q*

09.07.20.0003.01

$$\frac{\partial \vartheta_3'(z, q)}{\partial q} = -4 \sum_{k=1}^{\infty} q^{k^2-1} k^3 \sin(2 k z) \;/;\; |q| < 1$$

09.07.20.0004.01

$$\frac{\partial^2 \vartheta_3'(z, q)}{\partial q^2} = -\frac{4}{q^2} \sum_{k=2}^{\infty} q^{k^2} k^3 \left(k^2 - 1\right) \sin(2 k z) \;/;\; |q| < 1$$

Symbolic differentiation

With respect to *z*

09.07.20.0005.01

$$\frac{\partial^n \vartheta_3'(z, q)}{\partial z^n} = -2^{n+2} \sum_{k=1}^{\infty} q^{k^2} k^{n+1} \sin\left(\frac{\pi n}{2} + 2 k z\right) \;/;\; |q| < 1 \land n \in \mathbb{N}^+$$

With respect to *q*

09.07.20.0006.01

$$\frac{\partial^n \vartheta_3'(z, q)}{\partial q^n} = -4 \sum_{k=1}^{\infty} q^{k^2-n} k \left(k^2 - n + 1\right)_n \sin(2 k z) \;/;\; |q| < 1 \land n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.07.20.0007.01

$$\frac{\partial^\alpha \vartheta_3'(z, q)}{\partial z^\alpha} = -2^{\alpha+2} \sqrt{\pi}\, z^{1-\alpha} \sum_{k=1}^{\infty} q^{k^2} k^2 \, {}_1\tilde{F}_2\left(1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}; -k^2 z^2\right) /; |q| < 1$$

With respect to *q*

09.07.20.0008.01

$$\frac{\partial^\alpha \vartheta_3'(z, q)}{\partial q^\alpha} = -4\, q^{-\alpha} \sum_{k=1}^{\infty} \frac{q^{k^2} k\, \Gamma(k^2+1) \sin(2\,k\,z)}{\Gamma(k^2-\alpha+1)} /; |q| < 1$$

Integration

Indefinite integration

Involving only one direct function

09.07.21.0001.01

$$\int \vartheta_3'(z, q)\, dz = \vartheta_3(z, q)$$

Involving only one direct function with respect to *q*

09.07.21.0002.01

$$\int \vartheta_3'(z, q)\, dq = -4 \sum_{k=1}^{\infty} \frac{k\, q^{k^2+1} \sin(2\,k\,z)}{k^2+1} /; |q| < 1$$

Representations through equivalent functions

With related functions

Involving theta functions

Involving $\vartheta_1(z, q)$

09.07.27.0005.01

$$\vartheta_3'(z, q) = \frac{\vartheta_3(z, q)}{\vartheta_1(z, q)} \vartheta_1'(z, q) - \vartheta_3(0, q)^2 \frac{\vartheta_2(z, q)\, \vartheta_4(z, q)}{\vartheta_1(z, q)}$$

09.07.27.0006.01

$$\vartheta_3'(z, q) = -i\, e^{-i z} \sqrt[4]{q} \left(\vartheta_1\left(z - \frac{1}{2} \pi (\tau+1), q\right) + i\, \vartheta_1'\left(z - \frac{1}{2} \pi (\tau+1), q\right)\right) /; q = e^{i \pi \tau}$$

09.07.27.0007.01

$$\vartheta_3'(z, q) = e^{-i (2 m+1) z} q^{\left(m+\frac{1}{2}\right)^2} \left(\vartheta_1'\left(z - \frac{1}{2} \pi (2\, m\, \tau + \tau + 1), q\right) - i\, (2\, m+1)\, \vartheta_1\left(z - \frac{1}{2} \pi (2\, m\, \tau + \tau + 1), q\right)\right) /; m \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}$$

09.07.27.0008.01

$$\vartheta_3'(z, q) = e^{-i z} \sqrt[4]{q} \left(\vartheta_1'\left(z + \frac{1}{2} (i \log(q) + \pi), q\right) - i\, \vartheta_1\left(z + \frac{1}{2} (i \log(q) + \pi), q\right)\right)$$

09.07.27.0009.01

$$\vartheta_3'(z,q) = i\, e^{-i(2m+1)z}\, q^{\left(m+\frac{1}{2}\right)^2} \left((2m+1)\,\vartheta_1\!\left(z + \frac{1}{2} i (2m+1)\log(q) - \frac{\pi}{2}, q\right) + i\, \vartheta_1'\!\left(z + \frac{1}{2} i (2m+1)\log(q) - \frac{\pi}{2}, q\right)\right) /;\ m \in \mathbb{Z}$$

Involving $\vartheta_2(z, q)$

09.07.27.0010.01

$$\vartheta_3'(z,q) = e^{-iz}\, \sqrt[4]{q} \left(i\, \vartheta_2\!\left(z - \frac{\pi\tau}{2}, q\right) - \vartheta_2'\!\left(z - \frac{\pi\tau}{2}, q\right)\right) /;\ q = e^{i\pi\tau}$$

09.07.27.0011.01

$$\vartheta_3'(z,q) = e^{-i(2m+1)z} \left(e^{i\pi\tau}\right)^{\left(m+\frac{1}{2}\right)^2} \left(i(2m+1)\,\vartheta_2\!\left(z - \frac{1}{2}(2m+1)\pi\tau, e^{i\pi\tau}\right) - \vartheta_2'\!\left(z - \frac{1}{2}(2m+1)\pi\tau, e^{i\pi\tau}\right)\right) /;\ m \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.07.27.0012.01

$$\vartheta_3'(z,q) = e^{iz}\, \sqrt[4]{q} \left(i\, \vartheta_2\!\left(z - \frac{1}{2} i \log(q), q\right) + \vartheta_2'\!\left(z - \frac{1}{2} i \log(q), q\right)\right)$$

09.07.27.0013.01

$$\vartheta_3'(z,q) = e^{i(2m+1)z}\, q^{m^2+m+\frac{1}{4}} \left(i(2m+1)\,\vartheta_2\!\left(z - \frac{1}{2} i (2m+1)\log(q), q\right) + \vartheta_2'\!\left(z - \frac{1}{2} i (2m+1)\log(q), q\right)\right) /;\ m \in \mathbb{Z}$$

Involving $\vartheta_3(z, q)$

09.07.27.0001.02

$$\vartheta_3'\!\left(z, e^{i\pi\tau}\right) = e^{-i(2z-\pi\tau)}\, \vartheta_3'\!\left(z - \pi\tau, e^{i\pi\tau}\right) - 2\, e^{-i(2z-\pi\tau)}\, i\, \vartheta_3\!\left(z - \pi\tau, e^{i\pi\tau}\right) /;\ \mathrm{Im}(\tau) > 0$$

09.07.27.0014.01

$$\vartheta_3'(z,q) = e^{2iz}\, q \left(2 i\, \vartheta_3(z + \pi\tau, q) + \vartheta_3'(z + \pi\tau, q)\right) /;\ q = e^{i\pi\tau}$$

09.07.27.0015.01

$$\vartheta_3'(z,q) = e^{2inz}\, q^{n^2} \left(2 i n\, \vartheta_3(z + n\pi\tau, q) + \vartheta_3'(z + n\pi\tau, q)\right) /;\ n \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.07.27.0016.01

$$\vartheta_3'(z,q) = e^{2inz}\, q^{n^2} \left(2 i n\, \vartheta_3(z + m\pi + n\pi\tau, q) + \vartheta_3'(z + m\pi + n\pi\tau, q)\right) /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.07.27.0017.01

$$\vartheta_3'(z,q) = e^{2iz}\, q \left(2 i\, \vartheta_3(z - i\log(q), q) + \vartheta_3'(z - i\log(q), q)\right) /;\ q = e^{i\pi\tau}$$

09.07.27.0018.01

$$\vartheta_3'(z,q) = e^{-2iz}\, q \left(-2 i\, \vartheta_3(z + i\,\mathrm{og}(q), q) + \vartheta_3'(z + i\log(q), q)\right) /;\ q = e^{i\pi\tau}$$

09.07.27.0019.01

$$\vartheta_3'(z,q) = e^{-2inz}\, q^{n^2} \left(-2 i n\, \vartheta_3(z + i n \log(q), q) + \vartheta_3'(z + i n \log(q), q)\right) /;\ n \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.07.27.0020.01

$$\vartheta_3'(z,q) = e^{-2inz}\, q^{n^2} \left(-2 i n\, \vartheta_3(z + m\pi + i n \log(q), q) + \vartheta_3'(z + m\pi + i n \log(q), q)\right) /;\ \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

Involving $\vartheta_4'(z, q)$

09.07.27.0021.01

$$\vartheta_3'(z,q) = \vartheta_4'\!\left(z - \frac{\pi}{2}, q\right)$$

09.07.27.0004.01

$$\vartheta_3'(z, q) = \vartheta_4'\left(z + \frac{\pi}{2}, q\right)$$

09.07.27.0022.01

$$\vartheta_3'(z, q) = \vartheta_4'\left(z + \frac{1}{2}\pi(2m+1), q\right) /; m \in \mathbb{Z}$$

Zeros

09.07.30.0003.01

$$\vartheta_3'(z, 0) = 0$$

09.07.30.0002.01

$$\vartheta_3'(0, q) = 0$$

09.07.30.0001.01

$$\vartheta_3'\left(\frac{\pi m}{2}, q\right) = 0 /; m \in \mathbb{Z}$$

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