Erf2

Notations

Traditional name

Generalized error function

Traditional notation

$\mathrm{erf}(z_1, z_2)$

Mathematica StandardForm notation

`Erf[`z_1`, `z_2`]`

Primary definition

06.26.02.0001.01

$\mathrm{erf}(z_1, z_2) = \mathrm{erf}(z_2) - \mathrm{erf}(z_1)$

Specific values

Specialized values

06.26.03.0001.01

$\mathrm{erf}(z_1, 0) = -\mathrm{erf}(z_1)$

06.26.03.0002.01

$\mathrm{erf}(0, z_2) = \mathrm{erf}(z_2)$

Values at fixed points

06.26.03.0003.01

$\mathrm{erf}(0, 0) = 0$

Values at infinities

06.26.03.0004.01

$\mathrm{erf}(z_1, \infty) = 1 - \mathrm{erf}(z_1)$

06.26.03.0005.01

$\mathrm{erf}(z_1, -\infty) = -\mathrm{erf}(z_1) - 1$

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$\mathrm{erf}(\infty, z_2) = \mathrm{erf}(z_2) - 1$

06.26.03.0007.01

$\text{erf}(-\infty, z_2) = \text{erf}(z_2) + 1$

General characteristics

Domain and analyticity

$\text{erf}(z_1, z_2)$ is an analytical function of z_1 and z_2 which is defined in $\mathbb{C}^2$. For fixed z_1 it is an entire function of z_2. For fixed z_2 it is an entire function of z_1.

06.26.04.0001.01

$(z_1 * z_2) \longrightarrow \text{erf}(z_1, z_2) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$\text{erf}(z_1, z_2)$ is an odd function.

06.26.04.0002.01

$\text{erf}(-z_1, -z_2) = -\text{erf}(z_1, z_2)$

Mirror symmetry

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$\text{erf}(\overline{z_1}, \overline{z_2}) = \overline{\text{erf}(z_1, z_2)}$

Permutation symmetry

06.26.04.0004.01

$\text{erf}(z_1, z_2) = -\text{erf}(z_2, z_1)$

Periodicity

No periodicity

Poles and essential singularities

The function $\text{erf}(z_1, z_2)$ has singular points at $z_1 = \tilde{\infty}$ and $z_2 = \tilde{\infty}$. They are essential singular points.

06.26.04.0005.01

$\mathcal{Sing}_{z_k}(\text{erf}(z_1, z_2)) = \{\{\tilde{\infty}, \infty\}\} /; k \in \{1, 2\}$

Branch points

The function $\text{erf}(z_1, z_2)$ does not have branch points.

06.26.04.0006.01

$\mathcal{BP}_{z_k}(\text{erf}(z_1, z_2)) = \{\} /; k \in \{1, 2\}$

Branch cuts

The function $\text{erf}(z_1, z_2)$ does not have branch cuts.

06.26.04.0007.01

$$\mathcal{BC}_{z_k}(\text{erf}(z_1, z_2)) == \{\} /; k \in \{1, 2\}$$

Series representations

Generalized power series

Expansions at $\{z_1, z_2\} == \{0, 0\}$

06.26.06.0001.02

$$\text{erf}(z_1, z_2) \propto \frac{2}{\sqrt{\pi}}\left(z_2 - \frac{z_2^3}{3} + \frac{z_2^5}{10} - \ldots\right) - \frac{2}{\sqrt{\pi}}\left(z_1 - \frac{z_1^3}{3} + \frac{z_1^5}{10} - \ldots\right) /; (z_1 \to 0) \bigwedge (z_2 \to 0)$$

06.26.06.0005.01

$$\text{erf}(z_1, z_2) \propto \frac{2}{\sqrt{\pi}}\left(z_2 - \frac{z_2^3}{3} + \frac{z_2^5}{10} - O\left(z_2^7\right)\right) - \frac{2}{\sqrt{\pi}}\left(z_1 - \frac{z_1^3}{3} + \frac{z_1^5}{10} - O\left(z_1^7\right)\right)$$

06.26.06.0002.01

$$\text{erf}(z_1, z_2) == \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(z_2^{2k+1} - z_1^{2k+1}\right)}{k!\,(2k+1)}$$

06.26.06.0003.01

$$\text{erf}(z_1, z_2) == \frac{2 z_2}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_2^2\right) - \frac{2 z_1}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_1^2\right)$$

06.26.06.0004.02

$$\text{erf}(z_1, z_2) \propto \frac{2 z_2}{\sqrt{\pi}}\left(1 + O\left(z_2^3\right)\right) - \frac{2 z_1}{\sqrt{\pi}}\left(1 + O\left(z_1^3\right)\right)$$

Integral representations

On the real axis

Of the direct function

06.26.07.0001.01

$$\text{erf}(z_1, z_2) == \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2}\, dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to z_1

06.26.13.0001.01

$$w''(z_1) + 2 z_1 w'(z_1) == 0 /; w(z_1) == c_1 \text{erf}(z_1, z_2) + c_2$$

06.26.13.0003.01

$$W_{z_1}(1, \operatorname{erf}(z_1, z_2)) = -\frac{2 e^{-z_1^2}}{\sqrt{\pi}}$$

With respect to z_2

06.26.13.0002.01

$$w''(z_2) + 2 z_2 w'(z_2) = 0 \;/;\; w(z_2) = c_1 \operatorname{erf}(z_1, z_2) + c_2$$

06.26.13.0004.01

$$W_{z_2}(1, \operatorname{erf}(z_1, z_2)) = \frac{2 e^{-z_2^2}}{\sqrt{\pi}}$$

Differentiation

Low-order differentiation

With respect to z_1

06.26.20.0001.01

$$\frac{\partial \operatorname{erf}(z_1, z_2)}{\partial z_1} = -\frac{2 e^{-z_1^2}}{\sqrt{\pi}}$$

06.26.20.0002.01

$$\frac{\partial^2 \operatorname{erf}(z_1, z_2)}{\partial z_1^2} = \frac{4 e^{-z_1^2} z_1}{\sqrt{\pi}}$$

With respect to z_2

06.26.20.0003.01

$$\frac{\partial \operatorname{erf}(z_1, z_2)}{\partial z_2} = \frac{2 e^{-z_2^2}}{\sqrt{\pi}}$$

06.26.20.0004.01

$$\frac{\partial^2 \operatorname{erf}(z_1, z_2)}{\partial z_2^2} = -\frac{4 e^{-z_2^2} z_2}{\sqrt{\pi}}$$

Symbolic differentiation

With respect to z_1

06.26.20.0011.01

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_1^n} = \operatorname{erf}(z_1, z_2)\, \delta_n - \frac{2}{\sqrt{\pi}} e^{-z_1^2} \sum_{k=0}^{n-1} \frac{(-1)^k (2k - n + 2)_{2(n-k-1)}}{(n-k-1)!\, (2 z_1)^{n-2k-1}} \;/;\; n \in \mathbb{N}$$

06.26.20.0005.01

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_1^n} = \operatorname{erf}(z_1, z_2)\, \delta_n - \operatorname{Boole}\left(n \neq 0, \frac{2^{-n} (n-1)!}{\sqrt{\pi}} e^{-z_1^2} \sum_{k=1}^{n} \frac{(-1)^{k-1} 2^{2k} z_1^{2k-n-1}}{(2k-n-1)!\, (n-k)!}\right) \;/;\; n \in \mathbb{N}$$

06.26.20.0006.02

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_1^n} = -2^n z_1^{1-n} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1-\frac{n}{2}, \frac{3-n}{2}; -z_1^3\right) /; n \in \mathbb{N}$$

With respect to z_2

06.26.20.0012.01

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_2^n} = \operatorname{erf}(z_1, z_2)\, \delta_n + \frac{2}{\sqrt{\pi}} e^{-z_2^2} \sum_{k=0}^{n-1} \frac{(-1)^k (2k-n+2)_{2(-k+n-1)}}{(n-k-1)!\, (2z_2)^{-2k+n-1}} /; n \in \mathbb{N}$$

06.26.20.0007.01

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_2^n} = \operatorname{erf}(z_1, z_2)\, \delta_n + \operatorname{Boole}\left(n \neq 0, \frac{2^{-n} (n-1)!}{\sqrt{\pi}} e^{-z_2^2} \sum_{k=1}^{n} \frac{(-1)^{k-1} 2^{2k} z_2^{2k-n-1}}{(2k-n-1)!\, (n-k)!}\right) /; n \in \mathbb{N}$$

06.26.20.0008.02

$$\frac{\partial^n \operatorname{erf}(z_1, z_2)}{\partial z_2^n} = 2^n z_2^{1-n} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1-\frac{n}{2}, \frac{3-n}{2}; -z_2^3\right) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z_1

06.26.20.0009.01

$$\frac{\partial^\alpha \operatorname{erf}(z_1, z_2)}{\partial z_1^\alpha} = \frac{\operatorname{erf}(z_2)\, z_1^{-\alpha}}{\Gamma(1-\alpha)} - 2^\alpha z_1^{1-\alpha} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -z_1^2\right)$$

With respect to z_2

06.26.20.0010.01

$$\frac{\partial^\alpha \operatorname{erf}(z_1, z_2)}{\partial z_2^\alpha} = 2^\alpha z_2^{1-\alpha} \, {}_2\tilde{F}_2\left(\frac{1}{2}, 1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -z_2^2\right) - \frac{z_2^{-\alpha}}{\Gamma(1-\alpha)} \operatorname{erf}(z_1)$$

Integration

Indefinite integration

Involving only one direct function with respect to z_1

06.26.21.0001.01

$$\int \operatorname{erf}(z_1, z_2)\, dz_1 = z_1 \operatorname{erf}(z_1, z_2) - \frac{e^{-z_1^2}}{\sqrt{\pi}}$$

Involving one direct function and elementary functions with respect to z_1

Involving power function

06.26.21.0002.01

$$\int z_1^{\alpha-1} \operatorname{erf}(z_1, z_2)\, dz_1 = \frac{z_1^\alpha}{\alpha} \operatorname{erf}(z_1, z_2) - \frac{1}{\sqrt{\pi}\,\alpha} z_1^{\alpha+1} \left(z_1^2\right)^{-\frac{\alpha+1}{2}} \Gamma\left(\frac{\alpha+1}{2}, z_1^2\right)$$

Involving only one direct function with respect to z_2

06.26.21.0003.01

$$\int \operatorname{erf}(z_1, z_2)\, dz_2 = \operatorname{erf}(z_1, z_2)\, z_2 + \frac{e^{-z_2^2}}{\sqrt{\pi}}$$

Involving one direct function and elementary functions with respect to z_2

Involving power function

06.26.21.0004.01

$$\int z_2^{\alpha-1} \operatorname{erf}(z_1, z_2)\, dz_2 = \frac{z_2^{\alpha}}{\alpha} \operatorname{erf}(z_1, z_2) + \frac{1}{\sqrt{\pi}\,\alpha} z_2^{\alpha+1} \left(z_2^2\right)^{-\frac{\alpha+1}{2}} \Gamma\left(\frac{\alpha+1}{2}, z_2^2\right)$$

Representations through more general functions

Through hypergeometric functions

Involving $_1F_1$

06.26.26.0001.01

$$\operatorname{erf}(z_1, z_2) = \frac{2 z_2}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_2^2\right) - \frac{2 z_1}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z_1^2\right)$$

Involving hypergeometric U

06.26.26.0002.01

$$\operatorname{erf}(z_1, z_2) = \frac{z_2}{\sqrt{z_2^2}} \left(1 - \frac{1}{\sqrt{\pi}} e^{-z_2^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z_2^2\right)\right) - \frac{z_1}{\sqrt{z_1^2}} \left(1 - \frac{1}{\sqrt{\pi}} e^{-z_1^2}\, U\left(\frac{1}{2}, \frac{1}{2}, z_1^2\right)\right)$$

Through Meijer G

Classical cases for the direct function itself

06.26.26.0003.01

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(z_2\, G_{1,2}^{1,1}\left(z_2^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right) - z_1\, G_{1,2}^{1,1}\left(z_1^2 \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)\right)$$

06.26.26.0004.01

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{z_2^2}}{z_2}\, G_{1,2}^{1,1}\left(z_2^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) - \frac{\sqrt{z_1^2}}{z_1}\, G_{1,2}^{1,1}\left(z_1^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)\right)$$

06.26.26.0005.01

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(G_{1,2}^{1,1}\left(z_2^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) - G_{1,2}^{1,1}\left(z_1^2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)\right) /; \operatorname{Re}(z) > 0$$

06.26.26.0006.01

$$\operatorname{erf}\left(\sqrt{z_1}, \sqrt{z_2}\right) = \frac{1}{\sqrt{\pi}} \left(G_{1,2}^{1,1}\left(z_2 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right) - G_{1,2}^{1,1}\left(z_1 \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0 \end{matrix}\right)\right)$$

Generalized cases for the direct function itself

06.26.26.0007.01

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(G_{1,2}^{1,1}\left(z_2, \frac{1}{2} \middle| \begin{array}{c} 1 \\ \frac{1}{2}, 0 \end{array}\right) - G_{1,2}^{1,1}\left(z_1, \frac{1}{2} \middle| \begin{array}{c} 1 \\ \frac{1}{2}, 0 \end{array}\right)\right)$$

06.26.26.0008.01

$$\operatorname{erf}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left(G_{1,2}^{2,0}\left(z_1, \frac{1}{2} \middle| \begin{array}{c} 1 \\ 0, \frac{1}{2} \end{array}\right) - G_{1,2}^{2,0}\left(z_2, \frac{1}{2} \middle| \begin{array}{c} 1 \\ 0, \frac{1}{2} \end{array}\right)\right)$$

Through other functions

06.26.26.0009.01

$$\operatorname{erf}(z_1, z_2) = \frac{\sqrt{z_2^2}}{z_2} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, z_2^2\right)\right) - \frac{\sqrt{z_1^2}}{z_1} \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, z_1^2\right)\right)$$

06.26.26.0010.01

$$\operatorname{erf}(z_1, z_2) = \frac{\sqrt{z_2^2}}{z_2} \left(1 - Q\left(\frac{1}{2}, z_2^2\right)\right) - \frac{\sqrt{z_1^2}}{z_1} \left(1 - Q\left(\frac{1}{2}, z_1^2\right)\right)$$

06.26.26.0011.01

$$\operatorname{erf}(z_1, z_2) = \frac{z_1}{\sqrt{\pi}} E_{\frac{1}{2}}\left(z_1^2\right) - \frac{z_2}{\sqrt{\pi}} E_{\frac{1}{2}}\left(z_2^2\right) + \frac{\sqrt{z_2^2}}{z_2} - \frac{\sqrt{z_1^2}}{z_1}$$

Representations through equivalent functions

With inverse function

06.26.27.0001.01

$$\operatorname{erf}\left(z_1, \operatorname{erf}^{-1}(z_1, z_2)\right) = z_2$$

Zeros

06.26.30.0001.01

$$\operatorname{erf}(z_1, z_2) = 0 \; /; \; z_1 = 0 \bigwedge z_2 = 0$$

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