Factorial2

Notations

Traditional name

Double factorial

Traditional notation

$n!!$

Mathematica StandardForm notation

`Factorial2[n]`

Primary definition

06.02.02.0001.01

$$n!! = \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} 2^{n/2}\,\Gamma\left(\frac{n}{2}+1\right)$$

Specific values

Specialized values

06.02.03.0001.01

$$(2k)!! = \prod_{j=1}^{k} 2j \;/;\; k \in \mathbb{N}$$

06.02.03.0002.01

$$(2k)!! = 2^k\,\Gamma(k+1) \;/;\; k \in \mathbb{N}$$

06.02.03.0003.01

$$(-2k)!! = \tilde{\infty} \;/;\; k \in \mathbb{N}^+$$

06.02.03.0004.01

$$(2k-1)!! = \prod_{j=1}^{k} (2j-1) \;/;\; k \in \mathbb{N}$$

06.02.03.0005.01

$$(2k-1)!! = \frac{2^k}{\sqrt{\pi}}\,\Gamma\left(k+\frac{1}{2}\right) \;/;\; k \in \mathbb{N}$$

Values at fixed points

06.02.03.0006.01

$(-3)!! == -1$

06.02.03.0007.01

$(-2)!! == \tilde{\infty}$

06.02.03.0008.01

$(-1)!! == 1$

06.02.03.0009.01

$0!! == 1$

06.02.03.0010.01

$1!! == 1$

06.02.03.0011.01

$2!! == 2$

06.02.03.0012.01

$3!! == 3$

06.02.03.0013.01

$4!! == 8$

06.02.03.0014.01

$5!! == 15$

06.02.03.0015.01

$6!! == 48$

06.02.03.0016.01

$7!! == 105$

06.02.03.0017.01

$8!! == 384$

06.02.03.0018.01

$9!! == 945$

06.02.03.0019.01

$10!! == 3840$

Values at infinities

06.02.03.0020.01

$\infty!! == \infty$

06.02.03.0021.01

$(-\infty)!! == ¿$

06.02.03.0022.01

$(i\,\infty)!! == \tilde{\infty}$

06.02.03.0023.01

$(-i\,\infty)!! == \tilde{\infty}$

06.02.03.0024.01

$\tilde{\infty}!! == ¿$

General characteristics

Domain and analyticity

$n!!$ is an analytical function of n which is defined in the whole complex n-plane with the exception of countably many points $n = -k /; k - 1 \in \mathbb{N}^+$. $1 / n!!$ is an entire function.

06.02.04.0001.01

$$n \longrightarrow n!! :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.02.04.0002.01

$$\overline{n}!! = \overline{n!!}$$

Periodicity

No periodicity

Poles and essential singularities

The function $n!!$ has an infinite set of singular points:

a) $n = -2k /; k - 1 \in \mathbb{N}^+$ are the simple poles with residues $\frac{(-1)^{k-1}}{(2k-2)!!}$;

b) $n = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.02.04.0003.01

$$\mathcal{S}ing_n(n!!) = \left\{ \left\{ \{-2k, 1\} /; k - 1 \in \mathbb{N}^+ \right\}, \{\tilde{\infty}, \infty\} \right\}$$

06.02.04.0004.01

$$\text{res}_n(n!!)(-2k) = \frac{(-1)^{k-1}}{(2k-2)!!} /; k \in \mathbb{N}^+$$

Branch points

The function $n!!$ does not have branch points.

06.02.04.0005.01

$$\mathcal{BP}_n(n!!) = \{\}$$

Branch cuts

The function $n!!$ does not have branch cuts.

06.02.04.0006.01

$$\mathcal{BC}_n(n!!) = \{\}$$

Series representations

Generalized power series

Expansions at $n == n_0$ /; $n_0 \neq -2m$

06.02.06.0001.02

$$n!! \propto n_0!! \left(1 + \frac{1}{4}\left(\log(4) + 2\psi\left(\frac{n_0}{2}+1\right) + \pi \log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right)(n-n_0) + \frac{1}{32}\left(4\log^2(2) + 4\psi\left(\frac{n_0}{2}+1\right)^2 + 4\psi^{(1)}\left(\frac{n_0}{2}+1\right) + 4\psi\left(\frac{n_0}{2}+1\right)\left(2\log(2) + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right) + \pi\log\left(\frac{2}{\pi}\right)\left(4\pi\cos(n_0\pi) + \sin(n_0\pi)\left(4\log(2) + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right)\right)\right)(n-n_0)^2 + \frac{1}{384}\left(8\log^3(2) + 8\psi\left(\frac{n_0}{2}+1\right)^3 + 8\psi^{(2)}\left(\frac{n_0}{2}+1\right) + 6\psi\left(\frac{n_0}{2}+1\right)\left(4\log^2(2) + 4\psi^{(1)}\left(\frac{n_0}{2}+1\right) + \pi\log\left(\frac{2}{\pi}\right)\left(4\pi\cos(n_0\pi) + \sin(n_0\pi)\left(\log(16) + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right)\right)\right) + \pi\log\left(\frac{2}{\pi}\right)\left(24\pi\cos(n_0\pi)\log(2) + \sin(n_0\pi)\left(12\log^2(2) - 16\pi^2 + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\left(\log(64) + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right)\right) + 6\pi^2\log\left(\frac{2}{\pi}\right)\sin(2n_0\pi)\right) + 12\left(\psi\left(\frac{n_0}{2}+1\right)^2 + \psi^{(1)}\left(\frac{n_0}{2}+1\right)\right)\left(\log(4) + \pi\log\left(\frac{2}{\pi}\right)\sin(\pi n_0)\right)\right)(n-n_0)^3 + \ldots\right) /; (n \to n_0) \bigwedge -\frac{n_0}{2} \notin \mathbb{N}^+$$

06.02.06.0003.02

$$n!! \propto n_0!! \left(1 + \frac{1}{4}\left(\log(4) + 2\psi\left(\frac{n_0}{2}+1\right) + \pi\log\left(\frac{2}{\pi}\right)\sin(n_0\pi)\right)(n-n_0) + O\left((n-n_0)^2\right)\right) /; (n \to n_0) \bigwedge -\frac{n_0}{2} \notin \mathbb{N}^+$$

Expansions at $n == -2m$

06.02.06.0004.02

$$n!! \propto \frac{(-1)^{m-1}\, 2^{1-m}}{(m-1)!\,(2m+n)}(1 + O(n+2m)) /; (n \to -2m) \bigwedge m \in \mathbb{N}^+$$

06.02.06.0005.02

$$n!! \propto \frac{(-1)^{m-1}\, 2^{1-m}}{(m-1)!\,(n+2m)}\left(1 + \frac{1}{2}(\log(2) + \psi(m))(n+2m)\right) + O(n+2m) /; (n \to -2m) \bigwedge m \in \mathbb{N}^+$$

06.02.06.0006.02

$$n!! \propto \frac{(-1)^{m-1} 2^{1-m}}{(m-1)!\,(2m+n)} \Bigg(1 + \frac{1}{2}(\log(2)+\psi(m))(n+2m) + \frac{1}{24}\left(3\log^2(2)+\pi^2+3\psi(m)^2+\pi^2\log(8)-3\pi^2\log(\pi)+\log(64)\psi(m)-3\psi^{(1)}(m)\right)(2n+2m)^2 + \frac{1}{48}\Big(\log^3(2)+3\pi^2\log^2(2)+\pi^2\log(2)+\psi(m)^3+\log(8)\psi(m)^2-\pi^2\log(8)\log(\pi) - \log(8)\psi^{(1)}(m)+\psi(m)\left(3\log^2(2)+\pi^2(\log(8)-3\log(\pi)+1)-3\psi^{(1)}(m)\right)+\psi^{(2)}(m)\Big)(n+2m)^3 + \frac{1}{5760}\Big(15\log^4(2)+90\pi^2\log^3(2)+45\pi^4\log^2(2)+30\pi^2\log^2(2)-90\pi^2\log(\pi)\log^2(2)-30\pi^4\log(2)+60\psi(m)^3\log(2)-90\pi^4\log(\pi)\log(2)+60\psi^{(2)}(m)\log(2)+7\pi^4+15\psi(m)^4+45\pi^4\log^2(\pi)+45\psi^{(1)}(m)^2+30\pi^4\log(\pi) - 30\left(3\log^2(2)+\pi^2(\log(8)-3\log(\pi)+1)\right)\psi^{(1)}(m)+30\psi(m)^2\left(3\log^2(2)+\pi^2(\log(8)-3\log(\pi)+1)-3\psi^{(1)}(m)\right) + 60\psi(m)\left(\log^3(2)+3\pi^2\log^2(2)+\pi^2\log(2)-\pi^2\log(8)\log(\pi)-\log(8)\psi^{(1)}(m)+\psi^{(2)}(m)\right)-15\psi^{(3)}(m)\Big)(n+2m)^4 + O\left((n+2m)^5\right)\Bigg) /; (n \to -2m) \land m \in \mathbb{N}^+$$

Asymptotic series expansions

06.02.06.0007.01

$$n!! \propto \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} \sqrt{\pi}\, n^{\frac{n+1}{2}} e^{-\frac{n}{2}} /; (n \to \infty)$$

06.02.06.0008.01

$$n!! \propto \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} \sqrt{\pi}\, n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \left(1 + \frac{1}{6n} + \frac{1}{72n^2} - \frac{139}{6480n^3} - \frac{571}{155\,520n^4} + \frac{163\,879}{6\,531\,840n^5} + \frac{5\,246\,819}{1\,175\,731\,200n^6} - \frac{534\,703\,531}{7\,054\,387\,200n^7} - \frac{4\,483\,131\,259}{338\,610\,585\,600n^8} + \frac{432\,261\,921\,612\,371}{1\,005\,673\,439\,232\,000n^9} + O\left(\frac{1}{n^{10}}\right)\right) /; |\arg(n)| < \pi \land (|n| \to \infty)$$

06.02.06.0009.01

$$n!! \propto \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} \sqrt{\pi}\, n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \left(1 + \sum_{k=1}^{\infty}\sum_{j=1}^{2k} \frac{(-1)^j P(2(j+k), j)\, n^{-k}}{2^j (j+k)!}\right) /; (|\arg(n)| < \pi \land (|n| \to \infty)) \land P(m, j) = (m-1)((m-2)P(m-3, j-1) + P(m-1, j)) \land P(0, 0) = 1 \land P(m, 1) = (m-1)! \land P(m, j) = 0 /; m \le 3j - 1$$

06.02.06.0010.01

$$n!! \propto \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} \sqrt{\pi}\, n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \left(1 + O\left(\frac{1}{n}\right)\right) /; |\arg(n)| < \pi \land (|n| \to \infty)$$

Product representations

06.02.08.0001.01

$$(2k)!! = \prod_{j=1}^{k} 2j /; k \in \mathbb{N}$$

06.02.08.0002.01

$$(2k-1)!! = \prod_{j=1}^{k}(2j-1) \;/;\; k \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.02.16.0001.01

$$(n-1)!! = \frac{n!}{n!!}$$

06.02.16.0002.01

$$(-n)!! = \frac{n}{n!!}\left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

06.02.16.0003.01

$$(-n)!! = \frac{1}{(n-2)!!}\left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

06.02.16.0004.01

$$(-n-2)!! = -\frac{1}{n!!}\left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

06.02.16.0005.01

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} \;/;\; n \in \mathbb{Z}$$

06.02.16.0006.01

$$(n+2)!! = (n+2)\,n!!$$

06.02.16.0007.01

$$(2m+n)!! = 2^m \left(\frac{n}{2}+1\right)_m n!! \;/;\; m \in \mathbb{Z}$$

06.02.16.0008.01

$$(n-2)!! = \frac{n!!}{n}$$

06.02.16.0009.01

$$(n-2m)!! = \frac{(-1)^m\, 2^{-m}\, n!!}{\left(-\frac{n}{2}\right)_m} \;/;\; m \in \mathbb{Z}$$

Multiple arguments

06.02.16.0010.01

$$(2n)!! = 2^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}\sin^2(n\pi)} (n-1)!!\, n!!$$

06.02.16.0011.01

$$(3n)!! == \frac{1}{\sqrt{2\pi}} 3^{\frac{3n+1}{2}} \left(\frac{2}{\pi}\right)^{-\frac{1}{4}\cos(3n\pi)} n!! \left(n-\frac{2}{3}\right)!! \left(n-\frac{4}{3}\right)!!$$

06.02.16.0012.01

$$(m\,n)!! == m^{\frac{1}{2}(m\,n+1)}\, n\, 2^{\frac{1-m}{2}} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(1-m+\cos(m\,n\,\pi))} \prod_{k=0}^{m-1} \left(\frac{2k}{m}+n-2\right)!! \;/;\; m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Products of the direct function

06.02.16.0013.01

$$(n-1)!!\, n!! == n!$$

06.02.16.0014.01

$$(-2n-1)!!\,(2n-1)!! == (-1)^n \;/;\; n \in \mathbb{Z}$$

06.02.16.0015.01

$$(-n-2)!!\, n!! == -\left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

06.02.16.0016.01

$$(-n)!!\, n!! == n \left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

06.02.16.0017.01

$$n!!\, m!! == \frac{1}{\binom{\frac{m+n}{2}}{\frac{n}{2}}} 2^{\frac{m+n}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{4}(2-\cos(m\pi)-\cos(n\pi))} \frac{m+n}{2}!$$

06.02.16.0018.01

$$\frac{n!!}{m!!} == 2^{\frac{n-m}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{4}(\cos(m\pi)-\cos(n\pi))} \left(\frac{m}{2}+1\right)_{\frac{n-m}{2}}$$

06.02.16.0019.01

$$\frac{n!!}{m!!} == 2^{\frac{n-m}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{4}(\cos(m\pi)-\cos(n\pi))} \frac{n-m}{2}! \binom{\frac{n}{2}}{\frac{n-m}{2}}$$

06.02.16.0020.01

$$\frac{m!!\, n!!}{(m+n+2)!!} == \frac{1}{2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(m\pi)+\cos(n\pi)-\cos(\pi(m+n))-1)} \mathrm{B}\left(\frac{m}{2}+1, \frac{n}{2}+1\right)$$

Identities

Recurrence identities

Consecutive neighbors

06.02.17.0001.01

$$n!! = \frac{(n+2)!!}{n+2}$$

06.02.17.0002.01

$$n!! = n\,(n-2)!!$$

Distant neighbors

06.02.17.0003.01

$$n!! = \frac{2^{-m}\,(2\,m+n)!!}{\left(\frac{n}{2}+1\right)_m} \;/;\; m \in \mathbb{Z}$$

06.02.17.0004.02

$$n!! = (-1)^m\,2^m \left(-\frac{n}{2}\right)_m (n-2\,m)!! \;/;\; m \in \mathbb{Z}$$

Functional identities

Relations of special kind

06.02.17.0005.01

$$f(n) = n\,f(n-2) \;/;\; f(n) = n!!\,g(n) \bigwedge g(n) = g(n-2) \bigwedge f(1) = 1$$

Differentiation

Low-order differentiation

06.02.20.0001.01

$$\frac{\partial n!!}{\partial n} = \frac{1}{2}\,n!! \left(\log(2) + \psi\left(\frac{n}{2}+1\right) + \frac{1}{2}\,\pi \log\left(\frac{2}{\pi}\right) \sin(n\,\pi)\right)$$

06.02.20.0002.01

$$\frac{\partial^2 n!!}{\partial n^2} = \frac{1}{4}\,n!! \left(\left(\log(2) + \psi\left(\frac{n}{2}+1\right) + \frac{\pi}{2}\log\left(\frac{2}{\pi}\right)\sin(n\,\pi)\right)^2 + \pi^2 \cos(n\,\pi)\log\left(\frac{2}{\pi}\right) + \psi^{(1)}\left(\frac{n}{2}+1\right)\right)$$

Summation

Finite summation

06.02.23.0001.01

$$\sum_{k=0}^{n} \frac{(-1)^k\,(2\,k+1)!!}{(n-k)!\,k!\,(k+1)!} = \frac{(-1)^n \left(\frac{\sqrt{n+1}\;(n-1)!!}{n!!}\right)^{(-1)^n}}{\sqrt{n!\,(n+1)!}} \;/;\; n \in \mathbb{N}^+$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

06.02.26.0001.02

$$n!! = \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} 2^{n/2} \Gamma\left(\frac{n}{2}+1, 0\right) /; \operatorname{Re}(n) > -2$$

Representations through equivalent functions

With related functions

06.02.27.0001.01

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} \frac{n}{2}!$$

06.02.27.0002.01

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} \Gamma\left(\frac{n}{2}+1\right)$$

06.02.27.0003.02

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} (1)_{\frac{n}{2}}$$

06.02.27.0004.01

$$z!! = 2^{\frac{1}{4}(-2z-\cos(\pi z)-3)} \pi^{\frac{1}{2}\cos^2\left(\frac{\pi z}{2}\right)} \frac{z+3}{2}! \, C_{\frac{z+1}{2}}$$

Inequalities

06.02.29.0001.01

$$\frac{1}{4} < \frac{(2n)!!^2}{\pi (2n-1)!!^2} - n < \frac{1}{2} /; n \in \mathbb{N}^+$$

Zeros

06.02.30.0001.01

$$n!! \neq 0 /; \forall n$$

History

– J. Keiper and O.I. Marichev (1994) extended $n!!$ to arbitrary complex n

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