FresnelS

Notations

Traditional name

Fresnel integral S

Traditional notation

$S(z)$

Mathematica StandardForm notation

`FresnelS[z]`

Primary definition

06.32.02.0001.01

$$S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

Specific values

Values at fixed points

06.32.03.0001.01

$$S(0) = 0$$

Values at infinities

06.32.03.0002.01

$$S(\infty) = \frac{1}{2}$$

06.32.03.0003.01

$$S(-\infty) = -\frac{1}{2}$$

06.32.03.0004.01

$$S(i\infty) = -\frac{i}{2}$$

06.32.03.0005.01

$$S(-i\infty) = \frac{i}{2}$$

06.32.03.0006.01

$S(\tilde{\infty}) == ¿$

General characteristics

Domain and analyticity

$S(z)$ is an entire analytical function of z which is defined in the whole complex z-plane.

06.32.04.0001.01

$z \longrightarrow S(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$S(z)$ is an odd function.

06.32.04.0002.01

$S(-z) == -S(z)$

Mirror symmetry

06.32.04.0003.01

$S(\bar{z}) == \overline{S(z)}$

Periodicity

No periodicity

Poles and essential singularities

The function $S(z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

06.32.04.0004.01

$\mathcal{S}ing_z(S(z)) == \{\{\tilde{\infty}, \infty\}\}$

Branch points

The function $S(z)$ does not have branch points.

06.32.04.0005.01

$\mathcal{BP}_z(S(z)) == \{\}$

Branch cuts

The function $S(z)$ does not have branch cuts.

06.32.04.0006.01

$\mathcal{BC}_z(S(z)) == \{\}$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

06.32.06.0011.01

$$S(z) \propto S(z_0) + \sin\left(\frac{\pi z_0^2}{2}\right)(z - z_0) + \frac{1}{2}\pi\cos\left(\frac{\pi z_0^2}{2}\right)z_0\,(z - z_0)^2 + \ldots\ /;\ (z \to z_0)$$

06.32.06.0012.01

$$S(z) \propto S(z_0) + \sin\left(\frac{\pi z_0^2}{2}\right)(z - z_0) + \frac{1}{2}\pi\cos\left(\frac{\pi z_0^2}{2}\right)z_0\,(z - z_0)^2 + O\left((z - z_0)^3\right)$$

06.32.06.0013.01

$$S(z) == S(z_0) + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{2^{j-k+1}\,\pi^j\,z_0^{2j-k+1}}{k\,(2j-k+1)!\,(k-j-1)!}\sin\left(\frac{1}{2}\pi\left(z_0^2 + j\right)\right)(z - z_0)^k$$

06.32.06.0014.01

$$S(z) == \pi^{5/2}\sum_{k=0}^{\infty}\frac{2^{2k-\frac{11}{2}}\,z_0^{3-k}}{k!}\,{}_3\tilde{F}_4\left(\frac{3}{4}, 1, \frac{5}{4}; 1 - \frac{k}{4}, \frac{5-k}{4}, \frac{6-k}{4}, \frac{7-k}{4}; -\frac{1}{16}\left(\pi^2 z_0^4\right)\right)(z - z_0)^k$$

06.32.06.0015.01

$$S(z) \propto S(z_0)\,(1 + O(z - z_0))$$

Expansions at $z == 0$

For the function itself

06.32.06.0001.02

$$S(z) \propto \frac{\pi}{6}z^3\left(1 - \frac{\pi^2 z^4}{56} + \frac{\pi^4 z^8}{7040} - \ldots\right)/;\ (z \to 0)$$

06.32.06.0016.01

$$S(z) \propto \frac{\pi}{6}z^3\left(1 - \frac{\pi^2 z^4}{56} + \frac{\pi^4 z^8}{7040} - O\left(z^{12}\right)\right)$$

06.32.06.0002.01

$$S(z) == z^3\sum_{k=0}^{\infty}\frac{2^{-2k-1}\,\pi^{2k+1}\left(-z^4\right)^k}{(4k+3)\,(2k+1)!}$$

06.32.06.0003.01

$$S(z) == \frac{\pi z^3}{6}\,{}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}; -\frac{\pi^2 z^4}{16}\right)$$

06.32.06.0004.02

$$S(z) \propto \frac{\pi}{6}z^3\left(1 + O\left(z^4\right)\right)$$

06.32.06.0017.01

$$S(z) = F_\infty(z) /; \left(\left(F_n(z) = z^3 \sum_{k=0}^{n} \frac{2^{-2k-1} \pi^{2k+1} \left(-z^4\right)^k}{(4k+3)(2k+1)!} = S(z) + \frac{(-1)^n 2^{-2n-3} \pi^{2n+3} z^{4n+7}}{(4n+7)\Gamma(2n+4)} \, {}_2F_3\left(1, n+\frac{7}{4}; n+2, n+\frac{5}{2}, n+\frac{11}{4}; -\frac{1}{16}\pi^2 z^4\right)\right) \land n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Asymptotic series expansions

06.32.06.0005.01

$$S(z) \propto \frac{\sqrt[4]{z^4}}{2z} - \frac{1}{2\pi z}\left(e^{-\frac{i\pi}{2}z^2} \, {}_2F_0\left(1, \frac{1}{2};; \frac{2i}{\pi z^2}\right) + e^{\frac{i\pi}{2}z^2} \, {}_2F_0\left(1, \frac{1}{2};; -\frac{2i}{\pi z^2}\right)\right) /; (|z| \to \infty)$$

06.32.06.0006.01

$$S(z) \propto \frac{\left(z^4\right)^{3/4}}{2z^3} - \frac{1}{2\pi z}\left(e^{\frac{i\pi}{2}z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) + e^{-\frac{i\pi}{2}z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) /; (|z| \to \infty)$$

06.32.06.0007.01

$$S(z) \propto \frac{\sqrt[4]{z^4}}{2z} - \frac{1}{\pi z}\left(\cos\left(\frac{\pi z^2}{2}\right) {}_3F_0\left(\frac{1}{4}, \frac{3}{4}, 1;; -\frac{16}{\pi^2 z^4}\right) + \frac{1}{\pi z^2}\sin\left(\frac{\pi z^2}{2}\right) {}_3F_0\left(\frac{3}{4}, \frac{5}{4}, 1;; -\frac{16}{\pi^2 z^4}\right)\right) /; (|z| \to \infty)$$

06.32.06.0008.01

$$S(z) \propto \frac{\sqrt[4]{z^4}}{2z} - \frac{1}{\pi z}\cos\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) - \frac{1}{\pi^2 z^3}\sin\left(\frac{\pi z^2}{2}\right)\left(1 + O\left(\frac{1}{z^4}\right)\right) /; (|z| \to \infty)$$

Residue representations

06.32.06.0009.01

$$S(z) = \frac{\pi z^{9/4}}{\sqrt{2}\left(z^2\right)^{3/4}(-z)^{3/4}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(-\frac{\pi^2}{16}z^4\right)^{-s}}{\Gamma(s+1)\Gamma\left(\frac{3}{4}-s\right)\Gamma(1-s)} \Gamma\left(s+\frac{3}{4}\right)\right)\left(-j-\frac{3}{4}\right)$$

06.32.06.0010.01

$$S(z) = \frac{\pi e^{-\frac{3\pi i}{4}}}{\sqrt{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(\frac{1}{2}e^{\frac{\pi i}{4}}\sqrt{\pi}\, z\right)^{-4s}}{\Gamma(s+1)\Gamma\left(\frac{3}{4}-s\right)\Gamma(1-s)} \Gamma\left(s+\frac{3}{4}\right)\right)\left(-j-\frac{3}{4}\right)$$

Integral representations

On the real axis

Of the direct function

06.32.07.0001.01

$$S(z) == \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

06.32.07.0002.01

$$S(z) == \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\pi z^2}{2}} \frac{\sin(t)}{\sqrt{t}} dt \;/;\; |\arg(z)| < \frac{\pi}{4}$$

Contour integral representations

06.32.07.0003.01

$$S(z) == \frac{\pi z^{9/4}}{\sqrt{2} (z^2)^{3/4} (-z)^{3/4}} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{3}{4}\right)}{\Gamma(s+1)\Gamma\left(\frac{3}{4}-s\right)\Gamma(1-s)} \left(-\frac{\pi^2}{16} z^4\right)^{-s} ds$$

06.32.07.0004.01

$$S(z) == \frac{\pi e^{-\frac{3\pi i}{4}}}{\sqrt{2}} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{3}{4}\right)}{\Gamma(s+1)\Gamma\left(\frac{3}{4}-s\right)\Gamma(1-s)} \left(\frac{1}{2} e^{\frac{\pi i}{4}} \sqrt{\pi}\, z\right)^{-4s} ds$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.32.13.0001.02

$$z\, w^{(3)}(z) - w''(z) + \pi^2 z^3 w'(z) == 0 \;/;\; w(z) == S(z) \bigwedge w(0) == 0 \bigwedge w'(0) == 0 \bigwedge w^{(3)}(0) == \pi$$

06.32.13.0002.01

$$z\, w^{(3)}(z) - w''(z) + \pi^2 z^3 w'(z) == 0 \;/;\; w(z) == c_1 S(z) + c_2 C(z) + c_3$$

06.32.13.0003.01

$$W_z(1, S(z), C(z)) == -\pi z$$

06.32.13.0004.01

$$w^{(3)}(z) - \left(\frac{g'(z)}{g(z)} + \frac{3 g''(z)}{g'(z)}\right) w''(z) + \left(\pi^2 g(z)^2 g'(z)^2 + \frac{g''(z)}{g(z)} + \frac{3 g''(z)^2 - g'(z) g^{(3)}(z)}{g'(z)^2}\right) w'(z)\, w(z) = 0 \;/;$$

$$w(z) = c_1 S(g(z)) + c_2\, \mathrm{SC}(g(z)) + c_3$$

06.32.13.0005.01

$$W_z(S(g(z)), C(g(z)), 1) == -\pi g(z) g'(z)^3$$

06.32.13.0006.01

$$w^{(3)}(z)-\left(\frac{g'(z)}{g(z)}+\frac{3\,h'(z)}{h(z)}+\frac{3\,g''(z)}{g'(z)}\right)w''(z)+\left(\pi^2\,g(z)^2\,g'(z)^2+\frac{2\,h'(z)\,g'(z)}{h(z)\,g(z)}+\frac{6\,h'(z)^2}{h(z)^2}+\frac{6\,h'(z)\,g''(z)}{g'(z)\,h(z)}+\frac{g''(z)}{g(z)}+\frac{3\,g''(z)^2-g'(z)\,g^{(3)}(z)}{g'(z)^2}-\frac{3\,h''(z)}{h(z)}\right)w'(z)+\left(-\frac{6\,h'(z)^3}{h(z)^3}-\frac{2\,g'(z)\,h'(z)^2}{g(z)\,h(z)^2}-\frac{6\,g''(z)\,h'(z)^2}{h(z)^2\,g'(z)}+\frac{6\,h''(z)\,h'(z)}{h(z)^2}-\frac{3\,g''(z)^2\,h'(z)}{h(z)\,g'(z)^2}+\frac{g'(z)\,h''(z)-h'(z)\,g''(z)}{g(z)\,h(z)}+\frac{3\,g''(z)\,h''(z)+h'(z)\,g^{(3)}(z)}{h(z)\,g'(z)}-\frac{\pi^2\,g(z)^2\,h'(z)\,g'(z)^2+h^{(3)}(z)}{h(z)}\right)w(z)=0\;/;\;w(z)=c_1\,h(z)\,S(g(z))+c_2\,h(z)\,\mathrm{SC}(g(z))+c_3\,h(z)$$

06.32.13.0007.01

$$W_z(h(z)\,S(g(z)),\,h(z)\,C(g(z)),\,h(z))==-\pi\,g(z)\,h(z)^3\,g'(z)^3$$

06.32.13.0008.01

$$z^3\,w^{(3)}(z)+(-4\,r-3\,s+3)\,z^2\,w''(z)+\left(\left(a^4\,\pi^2\,z^{4\,r}+3\right)r^2+(8\,s-4)\,r+3\,(s-1)\,s+1\right)z\,w'(z)-s\left(\left(a^4\,\pi^2\,z^{4\,r}+3\right)r^2+4\,s\,r+s^2\right)w(z)=0\;/;\;w(z)=c_1\,z^s\,S(a\,z^r)+c_2\,z^s\,C(a\,z^r)+c_3\,z^s$$

06.32.13.0009.01

$$W_z(z^s\,S(a\,z^r),\,z^s\,C(a\,z^r),\,z^s)==-a^4\,\pi\,r^3\,z^{4\,r+3\,s-3}$$

06.32.13.0010.01

$$w^{(3)}(z)+(-4\log(r)-3\log(s))\,w''(z)+\left(\left(a^4\,\pi^2\,r^{4\,z}+3\right)\log^2(r)+8\log(s)\log(r)+3\log^2(s)\right)w'(z)-\log(s)\left(\left(a^4\,\pi^2\,r^{4\,z}+3\right)\log^2(r)+4\log(s)\log(r)+\log^2(s)\right)w(z)=0\;/;\;w(z)=c_1\,s^z\,S(a\,r^z)+c_2\,s^z\,\mathrm{SC}(a\,r^z)+c_3\,s^z$$

06.32.13.0011.01

$$W_z(s^z\,S(a\,r^z),\,s^z\,C(a\,r^z),\,s^z)==-a^4\,\pi\,r^{4\,z}\,s^{3\,z}\log^3(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.32.16.0001.01

$$S(-z)==-S(z)$$

06.32.16.0002.01

$$S(i\,z)==-i\,S(z)$$

06.32.16.0003.01

$$S(-i\,z)==i\,S(z)$$

06.32.16.0004.01

$$S\left(\sqrt{z^2}\right)==\frac{\left(z^2\right)^{3/2}}{z^3}\,S(z)$$

Complex characteristics

Real part

06.32.19.0001.01

$$\operatorname{Re}(S(x+i\,y)) = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{2k+1}\, \pi^{2k+\frac{1}{2}}\, x^{4k+3}}{(4k+3)!}\, \Gamma\!\left(2k+\frac{3}{2}\right)\left(1+\frac{y^2}{x^2}\right)^{2k+\frac{3}{2}} \cos\!\left((4k+3)\tan^{-1}\!\left(\frac{y}{x}\right)\right)$$

06.32.19.0002.01

$$\operatorname{Re}(S(x+i\,y)) = \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} \frac{(-1)^{j+k}\, 2^{2k+1}\, \pi^{2k+\frac{1}{2}}}{(2j)!\,(4k-2j+3)!}\, \Gamma\!\left(2k+\frac{3}{2}\right) x^{4k-2j+3}\, y^{2j}$$

06.32.19.0003.01

$$\operatorname{Re}(S(x+i\,y)) = \frac{1}{2}\left(S\!\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)+S\!\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Imaginary part

06.32.19.0004.01

$$\operatorname{Im}(S(x+i\,y)) = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{2k+1}\, \pi^{2k+\frac{1}{2}}\, x^{4k+3}}{(4k+3)!}\, \Gamma\!\left(2k+\frac{3}{2}\right)\left(1+\frac{y^2}{x^2}\right)^{2k+\frac{3}{2}} \sin\!\left((4k+3)\tan^{-1}\!\left(\frac{y}{x}\right)\right)$$

06.32.19.0005.01

$$\operatorname{Im}(S(x+i\,y)) = \sum_{k=0}^{\infty} \sum_{j=0}^{2k+1} \frac{(-1)^{j+k}\, 2^{2k+1}\, \pi^{2k+\frac{1}{2}}}{(2j+1)!\,(4k-2j+2)!}\, \Gamma\!\left(2k+\frac{3}{2}\right) x^{4k-2j+2}\, y^{2j+1}$$

06.32.19.0006.01

$$\operatorname{Im}(S(x+i\,y)) = \frac{x}{2y}\sqrt{-\frac{y^2}{x^2}}\left(S\!\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-S\!\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Absolute value

06.32.19.0007.01

$$|S(x+i\,y)| = \sqrt{S\!\left(x-x\sqrt{-\frac{y^2}{x^2}}\right) S\!\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)}$$

Argument

06.32.19.0008.01

$$\arg(S(x+i\,y)) = \tan^{-1}\!\left(\frac{1}{2}\left(S\!\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)+S\!\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right), \frac{x}{2y}\sqrt{-\frac{y^2}{x^2}}\left(S\!\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-S\!\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)\right)$$

Conjugate value

06.32.19.0009.01

$$\overline{S(x+i\,y)} = \frac{1}{2}\left(S\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)+S\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right)-\frac{i\,x}{2\,y}\sqrt{-\frac{y^2}{x^2}}\left(S\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-S\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Differentiation

Low-order differentiation

06.32.20.0001.01

$$\frac{\partial S(z)}{\partial z} = \sin\left(\frac{\pi z^2}{2}\right)$$

06.32.20.0002.01

$$\frac{\partial^2 S(z)}{\partial z^2} = \pi\, z \cos\left(\frac{\pi z^2}{2}\right)$$

Symbolic differentiation

06.32.20.0006.01

$$\frac{\partial^n S(z)}{\partial z^n} = S(z)\,\delta_n + \sum_{k=0}^{n-1} \frac{2^{k-n+1}\,\pi^k\, z^{2k-n+1}\,(n-1)!}{(2k-n+1)!\,(n-k-1)!} \sin\left(\frac{1}{2}\pi\left(z^2+k\right)\right) /;\, n \in \mathbb{N}$$

06.32.20.0003.01

$$\frac{\partial^n S(z)}{\partial z^n} = \delta_n\, S(z) + \sum_{k=0}^{n-1}\sum_{m=0}^{k} \frac{(-1)^m\, 2^{2m-k}\,\pi^k\, z^{2k-n+1}}{(k-m)!\,(2m-n+1)!} \left(\frac{1}{2}\right)_m \sin\left(\frac{1}{2}\pi\left(z^2-k\right)\right) /;\, n \in \mathbb{N}$$

06.32.20.0004.02

$$\frac{\partial^n S(z)}{\partial z^n} = 2^{2n-\frac{11}{2}}\,\pi^{5/2}\, z^{3-n}\; {}_3\tilde{F}_4\left(\frac{3}{4}, 1, \frac{5}{4};\, 1-\frac{n}{4}, \frac{5-n}{4}, \frac{6-n}{4}, \frac{7-n}{4};\, -\frac{\pi^2 z^4}{16}\right) /;\, n \in \mathbb{N}^+$$

Fractional integro-differentiation

06.32.20.0005.02

$$\frac{\partial^\alpha S(z)}{\partial z^\alpha} = 2^{2\alpha-\frac{11}{2}}\,\pi^{5/2}\, z^{3-\alpha}\; {}_3\tilde{F}_4\left(\frac{3}{4}, 1, \frac{5}{4};\, 1-\frac{\alpha}{4}, \frac{5-\alpha}{4}, \frac{6-\alpha}{4}, \frac{7-\alpha}{4};\, -\frac{\pi^2 z^4}{16}\right)$$

Integration

Indefinite integration

Involving only one direct function

Linear arguments

06.32.21.0001.01

$$\int S(z)\, dz = z\, S(z) + \frac{1}{\pi}\cos\left(\frac{\pi z^2}{2}\right)$$

06.32.21.0002.01

$$\int S(a\,z)\,dz = \frac{\cos\left(\frac{1}{2}\,a^2\,\pi\,z^2\right)}{a\,\pi} + z\,S(a\,z)$$

Power arguments

06.32.21.0003.01

$$\int S\left(a\sqrt{z}\right)dz = \frac{\pi\,z\,S\left(a\sqrt{z}\right)a^2 + \sqrt{z}\,\cos\left(\frac{1}{2}\,a^2\,\pi\,z\right)a - C\left(a\sqrt{z}\right)}{a^2\,\pi}$$

Involving one direct function and a power function

Linear arguments

06.32.21.0004.01

$$\int z^{\alpha-1}\,S(z)\,dz = \frac{z^\alpha}{\alpha}\,S(z) + \frac{i}{\alpha}\,2^{\frac{\alpha-3}{2}}\,\pi^{-\frac{\alpha+1}{2}}\,z^{\alpha+1}\left(z^4\right)^{-\frac{\alpha+1}{4}}\left(\left(-i\,z^2\right)^{\frac{\alpha+1}{2}}\,\Gamma\left(\frac{\alpha+1}{2},\,\frac{i\,\pi}{2}\,z^2\right) - \left(i\,z^2\right)^{\frac{\alpha+1}{2}}\,\Gamma\left(\frac{\alpha+1}{2},\,-\frac{i\,\pi}{2}\,z^2\right)\right)$$

06.32.21.0005.01

$$\int z^n\,S(a\,z)\,dz = \frac{1}{2\,a\,(n+1)}\left(\pi^{-\frac{n}{2}-1}\,z^n\left(a^4\,z^4\right)^{-n/2}\right.$$
$$\left.\left(2\,a\,\pi^{\frac{n+2}{2}}\,z\,S(a\,z)\left(a^4\,z^4\right)^{n/2} + 2^{n/2}\left(\Gamma\left(\frac{n+2}{2},\,\frac{1}{2}\,i\,a^2\,\pi\,z^2\right)\left(-i\,a^2\,z^2\right)^{n/2} + \left(i\,a^2\,z^2\right)^{n/2}\,\Gamma\left(\frac{n+2}{2},\,-\frac{1}{2}\,i\,a^2\,\pi\,z^2\right)\right)\right)\right)$$

Power arguments

06.32.21.0006.01

$$\int \frac{S\left(a\sqrt{z}\right)}{\sqrt{z}}\,dz = \frac{2\cos\left(\frac{1}{2}\,a^2\,\pi\,z\right)}{a\,\pi} + 2\sqrt{z}\,S\left(a\sqrt{z}\right)$$

Involving direct function and other elementary functions

Involving exponential function and a power function

06.32.21.0007.01

$$\int \frac{e^{-b\sqrt{z}}\,S\left(a\sqrt{z}\right)}{\sqrt{z}}\,dz =$$
$$\frac{1}{b}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\left(-e^{-\frac{i\,b^2}{2\,a^2\,\pi}}\left(i\,\mathrm{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(i\,\pi\sqrt{z}\,a^2+b\right)}{a\sqrt{\pi}}\right) + e^{\frac{i\,b^2}{a^2\,\pi}}\,\mathrm{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\pi\sqrt{z}\,a^2+i\,b\right)}{a\sqrt{\pi}}\right)\right)\right) - (2-2\,i)\,e^{-b\sqrt{z}}\,S\left(a\sqrt{z}\right)\right)$$

Involving trigonometric functions

Involving sin

06.32.21.0008.01

$$\int \sin(b\,z)\,S\left(a\sqrt{z}\right)dz = \frac{1}{32\,b^3 - 8\,a^4\,b\,\pi^2} \left((1+i)\left(a\sqrt{\pi}\left(\pi\sqrt{a^2\,\pi-2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{a^2\,\pi-2\,b}\;\sqrt{z}\right)a^2+\pi\sqrt{\pi\,a^2+2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)a^2 - \sqrt{a^2\,\pi-2\,b}\;(\pi\,a^2+2\,b)\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{a^2\,\pi-2\,b}\;\sqrt{z}\right)+ (2\,b-a^2\,\pi)\sqrt{\pi\,a^2+2\,b}\;\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)+2\,b\sqrt{a^2\,\pi-2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{a^2\,\pi-2\,b}\;\sqrt{z}\right) - 2\,b\sqrt{\pi\,a^2+2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)\right)-(4-4\,i)\left(4\,b^2-a^4\,\pi^2\right)\cos(b\,z)\,S\left(a\sqrt{z}\right)\right)\right)$$

06.32.21.0009.01

$$\int \frac{\sin\left(b\sqrt{z}\right)}{\sqrt{z}}\,S\left(a\sqrt{z}\right)dz = \frac{1}{b}\left(\left(\frac{1}{4}+\frac{i}{4}\right)e^{-\frac{i\,b^2}{2\,a^2\,\pi}}\left(e^{\frac{i\,b^2}{a^2\,\pi}}\operatorname{erf}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(a^2\,\pi\sqrt{z}-b\right)}{a\sqrt{\pi}}\right)+e^{\frac{i\,b^2}{a^2\,\pi}}\operatorname{erf}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\pi\sqrt{z}\;a^2+b\right)}{a\sqrt{\pi}}\right)+\operatorname{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(b-a^2\,\pi\sqrt{z}\right)}{a\sqrt{\pi}}\right) - \operatorname{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\pi\sqrt{z}\;a^2+b\right)}{a\sqrt{\pi}}\right)-(4-4\,i)\,e^{\frac{i\,b^2}{2\,a^2\,\pi}}\cos\left(b\sqrt{z}\right)S\left(a\sqrt{z}\right)\right)\right)$$

Involving sin

06.32.21.0010.01

$$\int \cos(b\,z)\,S\left(a\sqrt{z}\right)dz = -\frac{1}{32\,b^3-8\,a^4\,b\,\pi^2} \left((1-i)\left(a\sqrt{\pi}\left(\pi\sqrt{2\,b-a^2\,\pi}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2\,b-a^2\,\pi}\;\sqrt{z}\right)a^2+\pi\sqrt{\pi\,a^2+2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)a^2 + \sqrt{2\,b-a^2\,\pi}\;(\pi\,a^2+2\,b)\operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2\,b-a^2\,\pi}\;\sqrt{z}\right)+\left(a^2\,\pi-2\,b\right)\sqrt{\pi\,a^2+2\,b}\; \operatorname{erf}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)+2\,b\sqrt{2\,b-a^2\,\pi}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{2\,b-a^2\,\pi}\;\sqrt{z}\right) - 2\,b\sqrt{\pi\,a^2+2\,b}\;\operatorname{erfi}\left(\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi\,a^2+2\,b}\;\sqrt{z}\right)\right)-(4+4\,i)\left(4\,b^2-a^4\,\pi^2\right)S\left(a\sqrt{z}\right)\sin(b\,z)\right)\right)$$

06.32.21.0011.01

$$\int \frac{\cos\left(b\sqrt{z}\right)S\left(a\sqrt{z}\right)}{\sqrt{z}}\,dz = -\frac{1}{b}\left(\left(\frac{1}{4}-\frac{i}{4}\right)e^{-\frac{i\,b^2}{2\,a^2\,\pi}}\left(e^{\frac{i\,b^2}{a^2\,\pi}}\operatorname{erf}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(a^2\,\pi\sqrt{z}-b\right)}{a\sqrt{\pi}}\right)-e^{\frac{i\,b^2}{a^2\,\pi}}\operatorname{erf}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\pi\sqrt{z}\;a^2+b\right)}{a\sqrt{\pi}}\right)-\operatorname{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(b-a^2\,\pi\sqrt{z}\right)}{a\sqrt{\pi}}\right) - \operatorname{erfi}\left(\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\pi\sqrt{z}\;a^2+b\right)}{a\sqrt{\pi}}\right)-(4+4\,i)\,e^{\frac{i\,b^2}{2\,a^2\,\pi}}\,S\left(a\sqrt{z}\right)\sin\left(b\sqrt{z}\right)\right)\right)$$

Involving logarithm and a power function

06.32.21.0012.01

$$\int \frac{\log(z)}{\sqrt{z}} S\left(a\sqrt{z}\right) dz = \frac{1}{a\pi}\left(2\left(\left(\cos\left(\frac{1}{2}a^2\pi z\right) + a\pi\sqrt{z}\, S\left(a\sqrt{z}\right)\right)(\log(z)-2) - \text{Ci}\left(\frac{1}{2}a^2\pi z\right)\right)\right)$$

Involving power of the direct function

06.32.21.0013.01

$$\int \frac{S\left(a\sqrt{z}\right)^2}{\sqrt{z}} dz = \frac{1}{a\pi}\left(2a\pi\sqrt{z}\, S\left(a\sqrt{z}\right)^2 + 4\cos\left(\frac{1}{2}a^2\pi z\right) S\left(a\sqrt{z}\right) - \sqrt{2}\, S\left(\sqrt{2}\, a\sqrt{z}\right)\right)$$

Involving products of the direct function

06.32.21.0014.01

$$\int \frac{S\left(a\sqrt{z}\right) S\left(b\sqrt{z}\right)}{\sqrt{z}} dz = \frac{1}{b\pi}\left(2 S\left(a\sqrt{z}\right)\left(\cos\left(\frac{1}{2}b^2\pi z\right) + b\pi\sqrt{z}\, S\left(b\sqrt{z}\right)\right) + \frac{1}{a}\left(2b\cos\left(\frac{1}{2}a^2\pi z\right) S\left(b\sqrt{z}\right) + \frac{1}{\sqrt{a^2+b^2}}\left(-\sqrt{a^2-b^2}\,\sqrt{a^2+b^2}\, S\left(\sqrt{a^2-b^2}\,\sqrt{z}\right) - \left(a^2+b^2\right) S\left(\sqrt{a^2+b^2}\,\sqrt{z}\right)\right)\right)\right)$$

Definite integration

For the direct function itself

06.32.21.0015.01

$$\int_0^\infty t^{\alpha-1} S(t)\, dt = \frac{1}{2\alpha}\left(\frac{\pi}{2}\right)^{-\frac{\alpha+1}{2}} \cos\left(\frac{\pi}{4}(\alpha+3)\right)\Gamma\left(\frac{\alpha+1}{2}\right) /; -3 < \text{Re}(\alpha) < 0$$

Involving the direct function

06.32.21.0016.01

$$\int_0^\infty e^{-zt} S(t)\, dt = -\frac{1}{2z}\left(\cos\left(\frac{z^2}{2\pi}\right)\left(2C\left(\frac{z}{\pi}\right) - 1\right) + \sin\left(\frac{z^2}{2\pi}\right)\left(2S\left(\frac{z}{\pi}\right) - 1\right)\right) /; \text{Re}(z) > 0$$

06.32.21.0017.01

$$\int_0^\infty t^{\alpha-1} e^{-zt} S(t)\, dt = \frac{1}{2} z^{-\alpha}\,\Gamma(\alpha) + \frac{2^{\alpha/2}\pi^{-\frac{\alpha}{2}-1} z}{\alpha+1}\cos\left(\frac{\pi\alpha}{4}\right)\Gamma\left(\frac{\alpha}{2}+1\right) {}_3F_4\left(\frac{\alpha+1}{4}, \frac{\alpha+2}{4}, \frac{\alpha}{4}+1; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{\alpha+5}{4}; -\frac{z^4}{16\pi^2}\right) + \frac{1}{2\alpha}\left(\frac{\pi}{2}\right)^{-\frac{\alpha+1}{2}}\cos\left(\frac{\pi}{4}(\alpha+3)\right)\Gamma\left(\frac{\alpha+1}{2}\right) {}_3F_4\left(\frac{\alpha+1}{4}, \frac{\alpha+3}{4}, \frac{\alpha}{4}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{\alpha}{4}+1; -\frac{z^4}{16\pi^2}\right) + \frac{1}{\alpha+2} 2^{\frac{\alpha-1}{2}}\pi^{-\frac{\alpha+3}{2}} z^2\cos\left(\frac{\pi}{4}(\alpha+5)\right)\Gamma\left(\frac{\alpha+3}{2}\right) {}_3F_4\left(\frac{\alpha+2}{4}, \frac{\alpha+3}{4}, \frac{\alpha+5}{4}; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{\alpha+6}{4}; -\frac{z^4}{16\pi^2}\right) - \frac{2^{\alpha/2}\pi^{-\frac{\alpha}{2}-2} z^3}{3(\alpha+3)}\Gamma\left(\frac{\alpha}{2}+2\right)\sin\left(\frac{\pi\alpha}{4}\right) {}_3F_4\left(\frac{\alpha+3}{4}, \frac{\alpha}{4}+1, \frac{\alpha+6}{4}; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \frac{\alpha+7}{4}; -\frac{z^4}{16\pi^2}\right) /; \text{Re}(z) > 0 \land \text{Re}(\alpha) > -3$$

Integral transforms

Laplace transforms

06.32.22.0001.01

$$\mathcal{L}_t[S(t)](z) = -\frac{1}{2z}\left(\cos\left(\frac{z^2}{2\pi}\right)\left(2C\left(\frac{z}{\pi}\right)-1\right)+\sin\left(\frac{z^2}{2\pi}\right)\left(2S\left(\frac{z}{\pi}\right)-1\right)\right) /; \operatorname{Re}(z)>0$$

Mellin transforms

06.32.22.0002.01

$$\mathcal{M}_t[S(t)](z) = \frac{1}{2z}\left(\frac{\pi}{2}\right)^{-\frac{z+1}{2}}\cos\left(\frac{\pi}{4}(z+3)\right)\Gamma\left(\frac{z+1}{2}\right) /; -3<\operatorname{Re}(z)<0$$

Representations through more general functions

Through hypergeometric functions

Involving $_pF_q$

06.32.26.0001.01

$$S(z) = \frac{\pi z^3}{6}\,{}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}; -\frac{\pi^2 z^4}{16}\right)$$

Involving hypergeometric U

06.32.26.0002.01

$$S(z) = \frac{i z}{2\sqrt{2}}\left(\frac{1}{\sqrt{i z^2}}\left(1-\frac{1}{\sqrt{\pi}}e^{-\frac{i\pi}{2}z^2}U\left(\frac{1}{2}, \frac{1}{2}, \frac{i\pi}{2}z^2\right)\right)-\frac{1}{\sqrt{-i z^2}}\left(1-\frac{1}{\sqrt{\pi}}e^{\frac{i\pi}{2}z^2}U\left(\frac{1}{2}, \frac{1}{2}, -\frac{i\pi}{2}z^2\right)\right)\right)$$

Through Meijer G

Classical cases for the direct function itself

06.32.26.0003.01

$$S(z) = \frac{\pi}{\sqrt{2}\,(z^2)^{3/4}(-z)^{3/4}} z^{9/4} G_{1,3}^{1,0}\left(-\frac{\pi^2 z^4}{16}\,\middle|\, \begin{matrix}1\\ \frac{3}{4}, \frac{1}{4}, 0\end{matrix}\right)$$

06.32.26.0004.01

$$S(z) = \frac{\pi}{\sqrt{2}} e^{-\frac{3i\pi}{4}} G_{1,3}^{1,0}\left(-\frac{\pi^2 z^4}{16}\,\middle|\, \begin{matrix}1\\ \frac{3}{4}, \frac{1}{4}, 0\end{matrix}\right) /; -\frac{\pi}{2} < \arg(z) \le 0$$

06.32.26.0017.01

$$S\left(\sqrt[4]{z}\right) = \frac{1}{2} G_{1,3}^{1,1}\left(\frac{\pi^2 z}{16}\,\middle|\, \begin{matrix}1\\ \frac{3}{4}, 0, \frac{1}{4}\end{matrix}\right)$$

06.32.26.0005.01

$$S\left(\sqrt[4]{z}\right) = \frac{\pi z^{3/4}}{\sqrt{2}\,(-z)^{3/4}} G_{1,3}^{1,0}\left(-\frac{\pi^2 z}{16}\,\middle|\, \begin{matrix}1\\ \frac{3}{4}, \frac{1}{4}, 0\end{matrix}\right)$$

06.32.26.0006.01

$$S\left(\sqrt[4]{z}\right) = \frac{1}{2} - \frac{1}{2} G_{1,3}^{2,0}\left(\frac{\pi^2 z}{16} \middle| \begin{matrix} 1 \\ 0, \frac{3}{4}, \frac{1}{4} \end{matrix}\right)$$

06.32.26.0007.01

$$S\left((1+i)\sqrt{\frac{2}{\pi}} \sqrt[4]{z}\right) = \frac{i-1}{2} \pi G_{1,3}^{1,0}\left(z \middle| \begin{matrix} 1 \\ \frac{3}{4}, 0, \frac{1}{4} \end{matrix}\right)$$

Classical cases involving powers of Fresnel C, S

06.32.26.0018.01

$$C\left(\sqrt[4]{z}\right)^2 + S\left(\sqrt[4]{z}\right)^2 = \sqrt{\frac{1}{2}} G_{2,4}^{1,2}\left(\frac{\pi^2 z}{16} \middle| \begin{matrix} \frac{1}{2}, 1 \\ \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, 0 \end{matrix}\right)$$

Generalized cases for the direct function itself

06.32.26.0019.01

$$S(z) = \frac{1}{2} G_{1,3}^{1,1}\left(\frac{\sqrt{\pi} z}{2}, \frac{1}{4} \middle| \begin{matrix} 1 \\ \frac{3}{4}, 0, \frac{1}{4} \end{matrix}\right)$$

06.32.26.0008.01

$$S(z) = \frac{\pi}{\sqrt{2}} e^{-\frac{3\pi i}{4}} G_{1,3}^{1,0}\left(\frac{1}{2} e^{\frac{\pi i}{4}} \sqrt{\pi} z, \frac{1}{4} \middle| \begin{matrix} 1 \\ \frac{3}{4}, \frac{1}{4}, 0 \end{matrix}\right)$$

Generalized cases involving cos, sin and Fresnel C

06.32.26.0009.01

$$\cos\left(\frac{\pi z^2}{2}\right) C(z) + \sin\left(\frac{\pi z^2}{2}\right) S(z) = \sqrt{\frac{\pi}{2}} G_{1,3}^{1,1}\left(\frac{\sqrt{\pi} z}{2}, \frac{1}{4} \middle| \begin{matrix} \frac{1}{4} \\ \frac{1}{4}, 0, \frac{1}{2} \end{matrix}\right)$$

06.32.26.0010.01

$$\cos\left(\frac{\pi z^2}{2}\right) S(z) - \sin\left(\frac{\pi z^2}{2}\right) C(z) = -\sqrt{\frac{\pi}{2}} G_{1,3}^{1,1}\left(\frac{\sqrt{\pi} z}{2}, \frac{1}{4} \middle| \begin{matrix} \frac{3}{4} \\ \frac{3}{4}, 0, \frac{1}{2} \end{matrix}\right)$$

06.32.26.0011.01

$$\cos\left(\frac{\pi z^2}{2}\right)\left(\frac{1}{2} - C(z)\right) + \sin\left(\frac{\pi z^2}{2}\right)\left(\frac{1}{2} - S(z)\right) = (2\pi)^{-3/2} G_{1,3}^{3,1}\left(\frac{\sqrt{\pi} z}{2}, \frac{1}{4} \middle| \begin{matrix} \frac{1}{4} \\ 0, \frac{1}{4}, \frac{1}{2} \end{matrix}\right)$$

06.32.26.0012.01

$$\cos\left(\frac{\pi z^2}{2}\right)\left(\frac{1}{2} - C(z)\right) + \sin\left(\frac{\pi z^2}{2}\right)\left(\frac{1}{2} - S(z)\right) = (2\pi)^{-3/2} G_{1,3}^{3,1}\left(\frac{\sqrt{\pi} z}{2}, \frac{1}{4} \middle| \begin{matrix} \frac{3}{4} \\ 0, \frac{1}{2}, \frac{3}{4} \end{matrix}\right)$$

06.32.26.0020.01

$$\cos(z)\, C\left(\sqrt{\frac{2z}{\pi}}\right) + \sin(z)\, S\left(\sqrt{\frac{2z}{\pi}}\right) = \sqrt{\frac{\pi}{2}} G_{1,3}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{1}{4} \\ \frac{1}{4}, 0, \frac{1}{2} \end{matrix}\right)$$

06.32.26.0021.01

$$\cos(z)\, S\left(\sqrt{\frac{2z}{\pi}}\right) - \sin(z)\, C\left(\sqrt{\frac{2z}{\pi}}\right) = -\sqrt{\frac{\pi}{2}} G_{1,3}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{3}{4} \\ \frac{3}{4}, 0, \frac{1}{2} \end{matrix}\right)$$

06.32.26.0022.01

$$\cos(z)\left(\frac{1}{2}-C\left(\sqrt{\frac{2z}{\pi}}\right)\right)+\sin(z)\left(\frac{1}{2}-S\left(\sqrt{\frac{2z}{\pi}}\right)\right)=(2\pi)^{-3/2}\,G_{1,3}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4}\\ 0,\frac{1}{4},\frac{1}{2}\end{matrix}\right)$$

06.32.26.0023.01

$$\cos(z)\left(\frac{1}{2}-S\left(\sqrt{\frac{2z}{\pi}}\right)\right)-\sin(z)\left(\frac{1}{2}-C\left(\sqrt{\frac{2z}{\pi}}\right)\right)=(2\pi)^{-3/2}\,G_{1,3}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}\frac{3}{4}\\ 0,\frac{1}{2},\frac{3}{4}\end{matrix}\right)$$

Generalized cases involving powers of Fresnel *C*, *S*

06.32.26.0013.01

$$C(z)^2+S(z)^2=\sqrt{\frac{1}{2}}\;G_{2,4}^{1,2}\left(\frac{\sqrt{\pi}\,z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{2},1\\ \frac{1}{2},\frac{3}{4},\frac{1}{4},0\end{matrix}\right)$$

Through other functions

06.32.26.0014.01

$$S(z)=\frac{i\,z}{2\sqrt{2}}\left(\frac{1}{\sqrt{i\,z^2}}\left(1-\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2},\frac{i\,\pi}{2}\,z^2\right)\right)-\frac{1}{\sqrt{-i\,z^2}}\left(1-\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2},-\frac{i\,\pi}{2}\,z^2\right)\right)\right)$$

06.32.26.0015.01

$$S(z)=\frac{i\,z}{2\sqrt{2}}\left(\frac{1}{\sqrt{i\,z^2}}\left(1-Q\left(\frac{1}{2},\frac{i\,\pi}{2}\,z^2\right)\right)-\frac{1}{\sqrt{-i\,z^2}}\left(1-Q\left(\frac{1}{2},-\frac{i\,\pi}{2}\,z^2\right)\right)\right)$$

06.32.26.0016.01

$$S(z)=\frac{\sqrt{-i\,z^2}+\sqrt{i\,z^2}}{2\sqrt{2}\,z}-\frac{i\,z}{4}\left(E_{\frac{1}{2}}\left(\frac{i\,\pi}{2}\,z^2\right)-E_{\frac{1}{2}}\left(-\frac{i\,\pi}{2}\,z^2\right)\right)$$

Representations through equivalent functions

With related functions

06.32.27.0001.01

$$S(z)=\frac{1+i}{4}\left(\operatorname{erf}\left(\frac{1+i}{2}\sqrt{\pi}\,z\right)-i\operatorname{erf}\left(\frac{1-i}{2}\sqrt{\pi}\,z\right)\right)$$

Zeros

06.32.30.0001.01

$$S(z)=0\;/;\;z=0$$

Theorems

Cornu spirals

Freeway exits will often have the shape of clothoids (Cornu spirals), which are curves with the parametrization $\{S(t), C(t)\}$.

History

– A.J. Fresnel (1798, 1818, 1826)
– K. W. Knochenhauer (1839) found a series representaion
– N. Nielsen (1906)

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