GegenbauerC3

Notations

Traditional name

Gegenbauer polynomial

Traditional notation

$C_n^\lambda(z)$

Mathematica StandardForm notation

`GegenbauerC[n, λ, z]`

Primary definition

05.09.02.0001.01

$$C_n^\lambda(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (\lambda)_{n-k} (2z)^{n-2k}}{k! (n-2k)!} /; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n*, *λ*

05.09.03.0001.01

$$C_n^\lambda(0) = \frac{2^n \sqrt{\pi}\, \Gamma\left(\lambda + \frac{n}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right) \Gamma(n+1) \Gamma(\lambda)}$$

05.09.03.0002.01

$$C_n^\lambda(1) = \frac{\Gamma(2\lambda + n)}{\Gamma(2\lambda) \Gamma(n+1)}$$

05.09.03.0003.01

$$C_n^\lambda(-1) = \frac{\cos(\pi(\lambda+n)) \Gamma(2\lambda+n) \sec(\pi\lambda)}{\Gamma(2\lambda)\Gamma(n+1)} /; \operatorname{Re}(\lambda) < \frac{1}{2}$$

05.09.03.0004.01

$$C_n^\lambda(-1) = \tilde{\infty} /; \operatorname{Re}(\lambda) > \frac{1}{2}$$

For fixed *n*, *z*

05.09.03.0005.01

$C_n^0(z) == 0$

05.09.03.0006.01

$C_n^{-m}(z) == 0 \;/;\; m \in \mathbb{N}$

05.09.03.0007.01

$C_n^{\frac{1}{2}}(z) == P_n(z)$

05.09.03.0008.01

$C_n^1(z) == U_n(z)$

05.09.03.0009.01

$C_n^{-\frac{k+n}{2}}(z) == \tilde{\infty} \;/;\; k \in \mathbb{N}$

05.09.03.0023.01

$$C_n^{m+\frac{1}{2}}(z) = \frac{(2m-1)!!}{2^m \left(\frac{1}{2}\right)_m} \sum_{i_1=0}^{n} \dots \sum_{i_{2m+1}=0}^{n} \delta_{\sum_{j=1}^{2m+1} i_j, n} \prod_{j=1}^{2m+1} P_{i_j}(z) \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

05.09.03.0024.01

$$C_n^m(z) = \sum_{i_1=0}^{n} \dots \sum_{i_m=0}^{n} \delta_{\sum_{j=1}^{m} i_j, n} \prod_{j=1}^{m} U_{i_j}(z) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

For fixed λ, z

05.09.03.0010.01

$C_0^\lambda(z) == 1$

05.09.03.0011.01

$C_1^\lambda(z) == 2\lambda z$

05.09.03.0012.01

$C_2^\lambda(z) == 2\lambda(\lambda+1)z^2 - \lambda$

05.09.03.0013.01

$$C_3^\lambda(z) == \frac{4}{3}\lambda(\lambda+1)(\lambda+2)z^3 - 2\lambda(\lambda+1)z$$

05.09.03.0014.01

$$C_4^\lambda(z) == \frac{2}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)z^4 - 2\lambda(\lambda+1)(\lambda+2)z^2 + \frac{1}{2}\lambda(\lambda+1)$$

05.09.03.0015.01

$$C_5^\lambda(z) == \frac{4}{15}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^5 - \frac{4}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)z^3 + \lambda(\lambda+1)(\lambda+2)z$$

05.09.03.0016.01

$$C_6^\lambda(z) == \frac{4}{45}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)z^6 - \frac{2}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^4 + \lambda(\lambda+1)(\lambda+2)(\lambda+3)z^2 - \frac{1}{6}\lambda(\lambda+1)(\lambda+2)$$

05.09.03.0017.01

$$C_7^\lambda(z) = \frac{8}{315}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)z^7 - \frac{4}{15}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)z^5 + \frac{2}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^3 - \frac{1}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)z$$

05.09.03.0018.01

$$C_8^\lambda(z) = \frac{2}{315}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)z^8 - \frac{4}{45}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)z^6 + \frac{1}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)z^4 - \frac{1}{3}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^2 + \frac{1}{24}\lambda(\lambda+1)(\lambda+2)(\lambda+3)$$

05.09.03.0019.01

$$C_9^\lambda(z) = \frac{4\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)(\lambda+8)z^9}{2835} - \frac{8}{315}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)z^7 + \frac{2}{15}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)z^5 - \frac{2}{9}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)z^3 + \frac{1}{12}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z$$

05.09.03.0020.01

$$C_{10}^\lambda(z) = \frac{4\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)(\lambda+8)(\lambda+9)z^{10}}{14175} - \frac{2}{315}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)(\lambda+8)z^8 + \frac{2}{45}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)z^6 - \frac{1}{9}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)z^4 + \frac{1}{12}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)z^2 - \frac{1}{120}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)$$

05.09.03.0021.01

$$C_{-n}^\lambda(z) = 0\ /;\ n > 0$$

05.09.03.0022.01

$$C_n^\lambda(z) = \tilde{\infty}\ /;\ 2\lambda \in \mathbb{Z} \bigwedge 2\lambda \leq -n$$

Values at infinities

05.09.03.0025.01

$$C_n^\lambda(\infty) = (\lambda)_n\,\infty\ /;\ n > 0$$

05.09.03.0026.01

$$C_n^\lambda(-\infty) = (-1)^n\,(\lambda)_n\,\infty\ /;\ n > 0$$

General characteristics

Domain and analyticity

The function $C_n^\lambda(z)$ is defined over $\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}$. For fixed n, λ, the function $C_n^\lambda(z)$ is a polynomial in z of degree n. For fixed n, z, the function $C_n^\lambda(z)$ is a polynomial in λ of degree n.

05.09.04.0001.01

$$(n * \lambda * z) \longrightarrow C_n^\lambda(z) :: (\mathbb{N} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.09.04.0002.01

$$C_n^\lambda(-z) = (-1)^n C_n^\lambda(z)$$

Mirror symmetry

05.09.04.0003.01

$$C_n^{\bar{\lambda}}(\bar{z}) = \overline{C_n^\lambda(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed λ the function $C_n^\lambda(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.09.04.0004.01

$$\mathcal{Sing}_z\left(C_n^\lambda(z)\right) = \{\{\tilde{\infty}, n\}\}$$

With respect to λ

For fixed n, z, the function $C_n^\lambda(z)$ has an infinite set of singular points:

a) $\lambda = -\frac{n+j}{2} /; j \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^j 2^{j+\nu} \sqrt{\pi}}{\nu! \, j! \, \Gamma\left(-\frac{\nu+j}{2}\right)} \, {}_2\tilde{F}_1\left(-j, -\nu; \frac{1-\nu-j}{2}; \frac{1-z}{2}\right)$;

b) $\lambda = \tilde{\infty}$ is an essential singular point.

05.09.04.0005.01

$$\mathcal{Sing}_\lambda\left(C_n^\lambda(z)\right) = \left\{\left\{\left\{-\frac{n+j}{2}, 1\right\} /; j \in \mathbb{N}\right\}, \{\tilde{\infty}, \infty\}\right\}$$

05.09.04.0006.01

$$\operatorname{res}_\lambda\left(C_n^\lambda(z)\right)\left(-\frac{n+j}{2}\right) = \frac{(-1)^j 2^{j+n} \sqrt{\pi}}{n! \, j! \, \Gamma\left(-\frac{n+j}{2}\right)} \, {}_2\tilde{F}_1\left(-j, -n; \frac{1-n-j}{2}; \frac{1-z}{2}\right) /; j \in \mathbb{N}$$

Branch points

With respect to z

For fixed n, λ, the function $C_n^\lambda(z)$ does not have branch points.

05.09.04.0007.01

$$\mathcal{BP}_z\left(C_n^\lambda(z)\right) = \{\}$$

With respect to λ

For fixed n, z, the function $C_n^\lambda(z)$ does not have branch points.

05.09.04.0008.01

$$\mathcal{BP}_\lambda\left(C_n^\lambda(z)\right) = \{\}$$

Branch cuts

With respect to z

For fixed n, λ, the function $C_n^\lambda(z)$ does not have branch cuts.

05.09.04.0009.01

$$\mathcal{BC}_z\left(C_n^\lambda(z)\right) = \{\}$$

With respect to λ

For fixed n, z, the function $C_n^\lambda(z)$ does not have branch cuts.

05.09.04.0010.01

$$\mathcal{BC}_\lambda\left(C_n^\lambda(z)\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $\lambda = \lambda_0$

For the function itself

05.09.06.0019.01

$$C_n^\lambda(z) \propto \delta_n + \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2z)^{n-2s}}{s!\,(n-2s)!} \sum_{j=1}^{n-s} (-1)^{j+n} S_{n-s}^{(j)} \lambda_0^j \left(1 + \frac{j(\lambda-\lambda_0)}{\lambda_0} + \frac{(j-1)j(\lambda-\lambda_0)^2}{2\lambda_0^2} + \ldots\right) /; (\lambda \to \lambda_0)$$

05.09.06.0020.01

$$C_n^\lambda(z) \propto \delta_n + \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2z)^{n-2s}}{s!\,(n-2s)!} \sum_{j=1}^{n-s} (-1)^{j+n} S_{n-s}^{(j)} \lambda_0^j \left(1 + \frac{j(\lambda-\lambda_0)}{\lambda_0} + \frac{(j-1)j(\lambda-\lambda_0)^2}{2\lambda_0^2} + O\left((\lambda-\lambda_0)^3\right)\right)$$

05.09.06.0021.01

$$C_n^\lambda(z) = \delta_n + \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2z)^{n-2s}}{s!\,(n-2s)!} \sum_{j=1}^{n-s} (-1)^{j+n} S_{n-s}^{(j)} (j-k+1)_k \lambda_0^{j-k} (\lambda-\lambda_0)^k$$

05.09.06.0022.01

$$C_n^\lambda(z) \propto C_n^{\lambda_0}(z)\,(1 + O(\lambda-\lambda_0))$$

Expansions at generic point $z = z_0$

For the function itself

05.09.06.0023.01

$$C_n^\lambda(z) \propto C_n^\lambda(z_0) + 2\,\lambda\, C_{n-1}^{\lambda+1}(z_0)\,(z-z_0) + 2\,\lambda\,(\lambda+1)\, C_{n-2}^{\lambda+2}(z_0)\,(z-z_0)^2 + \ldots /; (z \to z_0)$$

05.09.06.0024.01

$$C_n^\lambda(z) \propto C_n^\lambda(z_0) + 2\,\lambda\, C_{n-1}^{\lambda+1}(z_0)\,(z-z_0) + 2\,\lambda\,(\lambda+1)\, C_{n-2}^{\lambda+2}(z_0)\,(z-z_0)^2 + O\big((z-z_0)^3\big)$$

05.09.06.0025.01

$$C_n^\lambda(z) = \sum_{k=0}^{\infty} \frac{2^k\,(\lambda)_k}{k!}\, C_{n-k}^{k+\lambda}(z_0)\,(z-z_0)^k$$

05.09.06.0026.01

$$C_n^\lambda(z) = \frac{2^{1-2\lambda}\sqrt{\pi}\;\Gamma(n+2\lambda)}{n!\,\Gamma(\lambda)} \sum_{k=0}^{\infty} \frac{(z_0-1)^{-k}}{k!}\, {}_3\tilde{F}_2\!\left(1, -n, n+2\lambda; 1-k, \lambda+\frac{1}{2}; \frac{1-z_0}{2}\right)(z-z_0)^k$$

05.09.06.0027.01

$$C_n^\lambda(z) \propto C_n^\lambda(z_0)\,(1+O(z-z_0))$$

Expansions at $z == 0$

For the function itself

05.09.06.0001.02

$$C_n^\lambda(z) \propto \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n}{2}+\lambda\right)}{\Gamma\!\left(\frac{1-n}{2}\right) n!\,\Gamma(\lambda)} + \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n+1}{2}+\lambda\right) z}{\Gamma\!\left(1-\frac{n}{2}\right)(n-1)!\,\Gamma(\lambda)} - \frac{2\cos\!\left(\frac{n\pi}{2}\right)\Gamma\!\left(\frac{n}{2}+\lambda+1\right) z^2}{\Gamma(\lambda)\,\Gamma\!\left(\frac{n}{2}\right)} - \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n+3}{2}+\lambda\right) z^3}{3\,\Gamma\!\left(1-\frac{n}{2}\right)(n-2)!\,\Gamma(\lambda)} + \ldots /; (z \to 0)$$

05.09.06.0028.01

$$C_n^\lambda(z) \propto \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n}{2}+\lambda\right)}{\Gamma\!\left(\frac{1-n}{2}\right) n!\,\Gamma(\lambda)} + \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n+1}{2}+\lambda\right) z}{\Gamma\!\left(1-\frac{n}{2}\right)(n-1)!\,\Gamma(\lambda)} - \frac{2\cos\!\left(\frac{n\pi}{2}\right)\Gamma\!\left(\frac{n}{2}+\lambda+1\right) z^2}{\Gamma(\lambda)\,\Gamma\!\left(\frac{n}{2}\right)} - \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\frac{n+3}{2}+\lambda\right) z^3}{3\,\Gamma\!\left(1-\frac{n}{2}\right)(n-2)!\,\Gamma(\lambda)} + O(z^4)$$

05.09.06.0002.02

$$C_n^\lambda(z) = \frac{2^{1-2\lambda}\sqrt{\pi}\;\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda)}\, \tilde{F}_{1\times0\times0}^{2\times0\times0}\!\left(\begin{matrix}-n, 2\lambda+n;;; \\ \lambda+\frac{1}{2};;;\end{matrix}\ \frac{1}{2}, -\frac{z}{2}\right)$$

05.09.06.0029.01

$$C_n^\lambda(z) = \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\lambda+\frac{n}{2}\right)}{\Gamma(\lambda)\,\Gamma\!\left(\frac{1-n}{2}\right) n!} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(-\frac{n}{2}\right)_k \left(\lambda+\frac{n}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} z^{2k} + \frac{2^n\sqrt{\pi}\; z\,\Gamma\!\left(\lambda+\frac{n+1}{2}\right)}{\Gamma(\lambda)\,\Gamma\!\left(1-\frac{n}{2}\right)(n-1)!} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(\frac{n+1}{2}+\lambda\right)_k}{\left(\frac{3}{2}\right)_k k!} z^{2k}$$

05.09.06.0030.01

$$C_n^\lambda(z) = \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\lambda+\frac{n}{2}\right)}{\Gamma(\lambda)\,\Gamma\!\left(\frac{1-n}{2}\right) n!}\, {}_2F_1\!\left(-\frac{n}{2}, \lambda+\frac{n}{2}; \frac{1}{2}; z^2\right) + \frac{2^n\sqrt{\pi}\; z\,\Gamma\!\left(\lambda+\frac{n+1}{2}\right)}{\Gamma(\lambda)\,\Gamma\!\left(1-\frac{n}{2}\right)(n-1)!}\, {}_2F_1\!\left(\frac{1-n}{2}, \frac{n+1}{2}+\lambda; \frac{3}{2}; z^2\right)$$

05.09.06.0003.02

$$C_n^\lambda(z) \propto \frac{2^n\sqrt{\pi}\;\Gamma\!\left(\lambda+\frac{n}{2}\right)}{\Gamma(\lambda)\,\Gamma\!\left(\frac{1-n}{2}\right) n!}\,(1+O(z))$$

05.09.06.0004.01

$$C_n^\lambda(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (\lambda)_{n-k} (2z)^{n-2k}}{k! (n-2k)!}$$

05.09.06.0005.02

$$C_n^\lambda(z) \propto \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} (\lambda)_{n-\left\lfloor \frac{n}{2} \right\rfloor} (2z)^{n-2\left\lfloor \frac{n}{2} \right\rfloor}}{\left\lfloor \frac{n}{2} \right\rfloor! \left(n-2\left\lfloor \frac{n}{2} \right\rfloor\right)!} \left(1+O\left(z^2\right)\right) /; n>0$$

Expansions at $z == 1$

For the function itself

05.09.06.0006.02

$$C_n^\lambda(z) \propto \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda)} \left(\frac{1}{\Gamma\left(\lambda+\frac{1}{2}\right)} + \frac{n(2\lambda+n)(z-1)}{2\,\Gamma\left(\lambda+\frac{3}{2}\right)} + \frac{-n(1-n)(2\lambda+n)(2\lambda+n+1)(z-1)^2}{8\,\Gamma\left(\lambda+\frac{5}{2}\right)} + \ldots\right) /; (z \to 1)$$

05.09.06.0031.01

$$C_n^\lambda(z) \propto \frac{(2\lambda)_n}{n!} \left(1 + \frac{n(n+2\lambda)}{1+2\lambda}(z-1) + \frac{(-1+n)\,n\,(n+2\lambda)(1+n+2\lambda)}{2(1+2\lambda)(3+2\lambda)}(z-1)^2 + \ldots\right) /; (z \to 1) \bigwedge -\lambda - \frac{1}{2} \notin \mathbb{N}$$

05.09.06.0032.01

$$C_n^\lambda(z) \propto \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda)} \left(\frac{1}{\Gamma\left(\lambda+\frac{1}{2}\right)} + \frac{n(2\lambda+n)(z-1)}{2\,\Gamma\left(\lambda+\frac{3}{2}\right)} + \frac{-n(1-n)(2\lambda+n)(2\lambda+n+1)(z-1)^2}{8\,\Gamma\left(\lambda+\frac{5}{2}\right)} + O\left((z-1)^3\right)\right)$$

05.09.06.0033.01

$$C_n^\lambda(z) \propto \frac{(2\lambda)_n}{n!} \left(1 + \frac{n(n+2\lambda)}{1+2\lambda}(z-1) + \frac{(-1+n)\,n\,(n+2\lambda)(1+n+2\lambda)}{2(1+2\lambda)(3+2\lambda)}(z-1)^2 + O\left((z-1)^3\right)\right) /; -\lambda - \frac{1}{2} \notin \mathbb{N}$$

05.09.06.0007.02

$$C_n^\lambda(z) = \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda)} \sum_{k=0}^{n} \frac{(-n)_k (2\lambda+n)_k}{\Gamma\left(k+\lambda+\frac{1}{2}\right) k!} \left(\frac{1-z}{2}\right)^k$$

05.09.06.0034.01

$$C_n^\lambda(z) = \frac{(2\lambda)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k /; -\lambda - \frac{1}{2} \notin \mathbb{N}$$

05.09.06.0008.02

$$C_n^\lambda(z) = \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda)} \, {}_2\tilde{F}_1\left(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{1-z}{2}\right)$$

05.09.06.0035.01

$$C_n^\lambda(z) = \frac{(2\lambda)_n}{n!} \, {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-z}{2}\right) /; -\lambda - \frac{1}{2} \notin \mathbb{N}$$

05.09.06.0009.02

$$C_n^\lambda(z) \propto \frac{\Gamma(2\lambda+n)}{\Gamma(2\lambda)\,\Gamma(n+1)} (1+O(z-1)) /; -\lambda - \frac{1}{2} \notin \mathbb{N}$$

Expansions at $z == -1$

For the function itself

05.09.06.0010.02

$$C_n^\lambda(z) \propto \frac{\cos(\pi(\lambda+n))\sec(\pi\lambda)(2\lambda)_n}{n!}\left(1-\frac{n(2\lambda+n)}{2\lambda+1}(z+1)-\frac{n(1-n)(2\lambda+n)(1+2\lambda+n)}{2(2\lambda+1)(2\lambda+3)}(z+1)^2-\ldots\right)/;$$
$$(z\to -1)\bigwedge \lambda+\frac{1}{2}\notin \mathbb{Z}$$

05.09.06.0036.01

$$C_n^\lambda(z) \propto \frac{\cos(\pi(\lambda+n))\sec(\pi\lambda)(2\lambda)_n}{n!}\left(1-\frac{n(2\lambda+n)}{2\lambda+1}(z+1)-\frac{n(1-n)(2\lambda+n)(1+2\lambda+n)}{2(2\lambda+1)(2\lambda+3)}(z+1)^2-O\left((z+1)^3\right)\right)/;\lambda+\frac{1}{2}\notin \mathbb{Z}$$

05.09.06.0011.02

$$C_n^\lambda(z) == \frac{\cos(\pi(\lambda+n))\sec(\pi\lambda)(2\lambda)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k k!}\left(\frac{z+1}{2}\right)^k /;\lambda+\frac{1}{2}\notin \mathbb{Z}$$

05.09.06.0012.02

$$C_n^\lambda(z) == \frac{\cos(\pi(\lambda+n))\sec(\pi\lambda)(2\lambda)_n}{n!}\,{}_2F_1\left(-n,2\lambda+n;\lambda+\frac{1}{2};\frac{z+1}{2}\right)/;\lambda+\frac{1}{2}\notin \mathbb{Z}$$

05.09.06.0013.02

$$C_n^\lambda(z) \propto \frac{\cos(\pi(n+\lambda))\sec(\pi\lambda)(2\lambda)_n}{n!}(1+O(z+1))/;(z\to -1)\bigwedge \lambda+\frac{1}{2}\notin \mathbb{Z}$$

05.09.06.0014.02

$$C_n^\lambda(z) \propto \frac{(-1)^n(2\lambda)_n}{n!}(1+O(z+1))$$

Expansions at $z == \infty$

For the function itself

Expansions in $1/z$

05.09.06.0037.01

$$C_n^\lambda(z) \propto \frac{2^n z^n(\lambda)_n}{n!}\left(1-\frac{n(1-n)}{4(1-n-\lambda)z^2}+\frac{n(1-n)(2-n)(3-n)}{32(1-n-\lambda)(2-n-\lambda)z^4}-\ldots\right)/;(|z|\to\infty)\bigwedge \lambda+n\notin \mathbb{N}^+$$

05.09.06.0038.01

$$C_n^\lambda(z) \propto \frac{2^n z^n(\lambda)_n}{n!}\left(1-\frac{n(1-n)}{4(1-n-\lambda)z^2}+\frac{n(1-n)(2-n)(3-n)}{32(1-n-\lambda)(2-n-\lambda)z^4}+O\left(\frac{1}{z^6}\right)\right)/;\lambda+n\notin \mathbb{N}^+$$

05.09.06.0039.01

$$C_n^\lambda(z) == \frac{2^n z^n(\lambda)_n}{n!}\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{\left(-\frac{n}{2}\right)_k\left(\frac{1-n}{2}\right)_k}{(1-n-\lambda)_k k!}z^{-2k}/;\lambda+n\notin \mathbb{N}^+$$

05.09.06.0040.01

$$C_n^\lambda(z) == \frac{2^n z^n (\lambda)_n}{n!}\, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n-\lambda; \frac{1}{z^2}\right) /; \lambda + n \notin \mathbb{N}^+$$

05.09.06.0041.01

$$C_n^\lambda(z) \propto \frac{2^n z^n (\lambda)_n}{n!} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; n \in \mathbb{N}^+$$

Expansions in $1/(1-z)$

05.09.06.0015.02

$$C_n^\lambda(z) \propto \frac{2^n (\lambda)_n}{n!} (z-1)^n \left(1 - \frac{n}{1-z} - \frac{n(1-n)(3-2\lambda-2n)}{4(1-\lambda-n)(1-z)^2} - \ldots\right) /; (|z| \to \infty) \bigwedge 2\lambda + 2n \notin \mathbb{Z}$$

05.09.06.0042.01

$$C_n^\lambda(z) \propto \frac{2^n (\lambda)_n}{n!} (z-1)^n \left(1 - \frac{n}{1-z} - \frac{n(1-n)(3-2\lambda-2n)}{4(1-\lambda-n)(1-z)^2} - O\left(\frac{1}{z^3}\right)\right) /; 2\lambda + 2n \notin \mathbb{Z}$$

05.09.06.0016.02

$$C_n^\lambda(z) == \frac{2^n (\lambda)_n}{n!} (z-1)^n \sum_{k=0}^{n} \frac{(-n)_k \left(\frac{1}{2}-\lambda-n\right)_k}{(1-2\lambda-2n)_k\, k!} \left(\frac{2}{1-z}\right)^k /; 2\lambda + 2n \notin \mathbb{Z}$$

05.09.06.0017.02

$$C_n^\lambda(z) == \frac{2^n (\lambda)_n}{n!} (z-1)^n\, {}_2F_1\left(-n, -\lambda-n+\frac{1}{2}; -2\lambda-2n+1; \frac{2}{1-z}\right) /; 2\lambda + 2n \notin \mathbb{Z}$$

05.09.06.0018.02

$$C_n^\lambda(z) \propto \frac{2^n (\lambda)_n}{n!} z^n \left(1 + O\left(\frac{1}{z}\right)\right) /; 2\lambda + 2n \notin \mathbb{Z}$$

Expansions at $\lambda == 0$

05.09.06.0043.01

$$C_n^\lambda(z) \propto \delta_n + \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2z)^{n-2k} (-1)^{n+1} S_{n-k}^{(1)}}{k!\,(n-2k)!}\right) \lambda + \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2z)^{n-2k} (-1)^n S_{n-k}^{(2)}}{k!\,(n-2k)!}\right) \lambda^2 + \ldots /; (\lambda \to 0)$$

05.09.06.0044.01

$$C_n^\lambda(z) \propto \delta_n + C_n^{(0)}(z)\, \lambda + \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^n (2z)^{n-2j}}{j!\,(n-2j)!} S_{n-j}^{(2)} \lambda^2 + \ldots /; (\lambda \to 0) \bigwedge n > 0$$

05.09.06.0045.01

$$C_n^\lambda(z) = \delta_n + C_n^{(0)}(z)\, \lambda + \sum_{k=2}^{n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+n} (2z)^{n-2j}}{j!\,(n-2j)!} S_{n-j}^{(k)} \lambda^k /; n > 0$$

05.09.06.0046.01

$$C_n^\lambda(z) \propto \delta_n + \sum_{k=1}^{n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+n} (2z)^{n-2j}}{j!\,(n-2j)!} S_{n-j}^{(k)} \lambda^k$$

05.09.06.0047.01

$$C_n^{\lambda}(z) = \sum_{k=0}^{n} \left(\theta\left(n - k - \left\lfloor \frac{n}{2} \right\rfloor - 1\right) \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+n} (j-n)_{n-j-k}}{j! (n-j-k)! (n-2j)!} (2z)^{n-2j} B_{n-j-k}^{(n-j+1)}(n-j) + \theta\left(k - n + \left\lfloor \frac{n}{2} \right\rfloor\right) \sum_{j=0}^{n-k} \frac{(-1)^{k+n} (j-n)_{n-j-k}}{j! (n-j-k)! (n-2j)!} (2z)^{n-2j} B_{n-j-k}^{(n-j+1)}(n-j) \right) \lambda^k$$

05.09.06.0048.01

$$C_n^{\lambda}(z) \propto \delta_n + C_n^{(0)}(z)\, \lambda\, (1 + O(\lambda)) \;/;\; n > 0$$

Expansions at $\lambda == \infty$

05.09.06.0049.01

$$C_n^{\lambda}(z) \propto \frac{(2z)^n \lambda^n}{n!} \left(1 + \frac{(n-1)\left(2z^2 - 1\right) n}{4 z^2 \lambda} + \frac{(n-2)(n-1)\left(3n\left(1 - 2z^2\right)^2 - \left(3 - 2z^2\right)^2\right) n}{96 z^4 \lambda^2} + \ldots\right) /; (|\lambda| \to \infty)$$

05.09.06.0050.01

$$C_n^{\lambda}(z) \propto \delta_n + \lambda^n \sum_{k=0}^{n-1} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (2z)^{n-2j}}{j! (n-2j)!} S_{n-j}^{(n-k)} \lambda^{-k}$$

05.09.06.0051.01

$$C_n^{\lambda}(z) = \lambda^n \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{k} \frac{(-1)^k (j-n)_{k-j}}{j! (k-j)! (n-2j)!} (2z)^{n-2j} B_{k-j}^{(-j+n+1)}(n-j) \lambda^{-k} + \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor + 1}^{n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (j-n)_{k-j}}{j! (k-j)! (n-2j)!} (2z)^{n-2j} B_{k-j}^{(-j+n+1)}(n-j) \lambda^{-k} \right)$$

05.09.06.0052.01

$$C_n^{\lambda}(z) = \lambda^n \sum_{k=0}^{n} \left(\theta\left(k - \left\lfloor \frac{n}{2} \right\rfloor - 1\right) \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (j-n)_{k-j} (2z)^{n-2j} B_{k-j}^{(n-j+1)}(n-j)}{j! (k-j)! (n-2j)!} + \theta\left(\left\lfloor \frac{n}{2} \right\rfloor - k\right) \sum_{j=0}^{k} \frac{(-1)^k (j-n)_{k-j} (2z)^{n-2j} B_{k-j}^{(n-j+1)}(n-j)}{j! (k-j)! (n-2j)!} \right) \lambda^{-k}$$

05.09.06.0053.01

$$C_n^{\lambda}(z) \propto \frac{(2z)^n \lambda^n}{n!} \left(1 + O\left(\frac{1}{\lambda}\right)\right)$$

Expansions at $n == \infty$

05.09.06.0054.01

$$C_n^{\lambda}(z) \propto \frac{2^{1-\lambda} n^{\lambda-1}}{\Gamma(\lambda)\left(1-z^2\right)^{\lambda/2}} \left(\cos\left(\frac{\pi\lambda}{2} - (n+\lambda)\cos^{-1}(z)\right) + \frac{(\lambda-1)\lambda}{2n}\left(\cos\left(\frac{\pi\lambda}{2} - (n+\lambda)\cos^{-1}(z)\right) + \frac{1}{\sqrt{1-z^2}}\sin\left(\frac{\pi\lambda}{2} - (n+\lambda-1)\cos^{-1}(z)\right)\right) + \frac{(1-\lambda)(2-\lambda)\lambda}{24n^2}\left((3\lambda-1)\cos\left(\frac{\pi\lambda}{2} - (n+\lambda)\cos^{-1}(z)\right) - \frac{6(\lambda-1)}{\sqrt{1-z^2}}\cos\left(\frac{1}{2}\pi(\lambda+1) - (n+\lambda-1)\cos^{-1}(z)\right) - \frac{3(\lambda+1)}{z^2-1}\cos\left(\frac{1}{2}\pi(\lambda+2) - (n+\lambda-2)\cos^{-1}(z)\right)\right) + \ldots\right) /; (n \to \infty)$$

05.09.06.0055.01

$$C_n^{\lambda}(z) \propto \frac{2^{1-\lambda} n^{\lambda-1}}{\Gamma(\lambda)\left(1-z^2\right)^{\lambda/2}} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k} 2^{-j}\left(1-z^2\right)^{-\frac{j}{2}}(1-\lambda)_k(\lambda)_j}{j!(k-j)!} \cos\left((n-j+\lambda)\cos^{-1}(z) - \frac{\pi(j+\lambda)}{2}\right) B_{k-j}^{(\lambda-j)}(\lambda-j)\, n^{-k} /; (n \to \infty)$$

05.09.06.0056.01

$$C_n^{\lambda}(z) \propto \frac{2^{1-\lambda} n^{\lambda-1}}{\Gamma(\lambda)\left(1-z^2\right)^{\lambda/2}} \cos\left(\frac{\pi\lambda}{2} - (n+\lambda)\cos^{-1}(z)\right)(1+\ldots) /; (n \to \infty)$$

Integral representations

On the real axis

Of the direct function

05.09.07.0001.01

$$C_n^{\lambda}(z) = \frac{2^{1-2\lambda}\Gamma(n+2\lambda)}{n!\,\Gamma(\lambda)^2} \int_0^{\pi}\left(z+\sqrt{z^2-1}\cos(t)\right)^n \sin^{2\lambda-1}(t)\,dt /; \operatorname{Re}(\lambda) > 0 \land \operatorname{Re}(z) > 0$$

Integral representations of negative integer order

Rodrigues-type formula.

05.09.07.0002.01

$$C_n^{\lambda}(z) = \frac{(-1)^n \Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(n+2\lambda)\left(1-z^2\right)^{\frac{1}{2}-\lambda}}{n!\,2^n\,\Gamma(2\lambda)\,\Gamma\left(n+\lambda+\frac{1}{2}\right)} \frac{\partial^n\left(1-z^2\right)^{n+\lambda-\frac{1}{2}}}{\partial z^n}$$

Generating functions

05.09.11.0001.01

$$C_n^{\lambda}(z) = \left([t^n]\left(t^2-2zt+1\right)^{-\lambda}\right) /; -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.09.13.0001.01

$$\left(1-z^2\right) w''(z)-(2\lambda+1)\, z\, w'(z)+n\,(n+2\lambda)\, w(z)=0\ /;\ w(z)=c_1\, C_n^{\lambda}(z)\ +c_2\left(1-z^2\right)^{\frac{1}{4}(1-2\lambda)}\, Q_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda}(z)$$

05.09.13.0002.01

$$W_z\!\left(C_n^{\lambda}(z),\ \left(1-z^2\right)^{\frac{1}{4}(1-2\lambda)}\, Q_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda}(z)\right)=\frac{2^{\frac{1}{2}-\lambda}\sqrt{\pi}\,\left(1-z^2\right)^{-\lambda-\frac{1}{2}}}{\Gamma(\lambda)}$$

05.09.13.0003.01

$$w''(z)-\left(\frac{(2\lambda+1)\, g(z)\, g'(z)}{1-g(z)^2}+\frac{g''(z)}{g'(z)}\right) w'(z)+\frac{n\,(2\lambda+n)\, g'(z)^2}{1-g(z)^2}\, w(z)=0\ /;\ w(z)=\ c_1\, C_n^{\lambda}(g(z))+c_2\left(1-g(z)^2\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(g(z))$$

05.09.13.0004.01

$$W_z\!\left(C_n^{\lambda}(g(z)),\ \left(1-g(z)^2\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(g(z))\right)=\frac{2^{\frac{1}{2}-\lambda}\sqrt{\pi}\,\left(1-g(z)^2\right)^{-\lambda-\frac{1}{2}}\, g'(z)}{\Gamma(\lambda)}$$

05.09.13.0005.01

$$w''(z)-\left(\frac{(2\lambda+1)\, g(z)\, g'(z)}{1-g(z)^2}+\frac{2\, h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}\right) w'(z)+$$

$$\left(\frac{n\,(2\lambda+n)\, g'(z)^2}{1-g(z)^2}+\frac{(2\lambda+1)\, g(z)\, h'(z)\, g'(z)}{\left(1-g(z)^2\right) h(z)}+\frac{2\, h'(z)^2}{h(z)^2}+\frac{h'(z)\, g''(z)}{h(z)\, g'(z)}-\frac{h''(z)}{h(z)}\right) w(z)=0\ /;$$

$$w(z)=\ c_1\, h(z)\, C_n^{\lambda}(g(z))+c_2\, h(z)\left(1-g(z)^2\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(g(z))$$

05.09.13.0006.01

$$W_z\!\left(h(z)\, C_n^{\lambda}(g(z)),\ h(z)\left(1-g(z)^2\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(g(z))\right)=\frac{2^{\frac{1}{2}-\lambda}\sqrt{\pi}\,\left(1-g(z)^2\right)^{-\lambda-\frac{1}{2}}\, g'(z)\, h(z)^2}{\Gamma(\lambda)}$$

05.09.13.0007.01

$$w''(z)+\frac{-a^2\,(2s-2r\lambda-1)\, z^{2r}+r+2s-1}{z\left(a^2 z^{2r}-1\right)}\, w'(z)+\frac{a^2 z^{2r}\,(s+r\,n)\,(s-r\,(2\lambda+n))-s\,(r+s)}{z^2\left(a^2 z^{2r}-1\right)}\, w(z)=0\ /;$$

$$w(z)=c_1\, z^s\, C_n^{\lambda}(a\, z^r)+c_2\, z^s\left(1-a^2 z^{2r}\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(a\, z^r)$$

05.09.13.0008.01

$$W_z\!\left(z^s\, C_n^{\lambda}(a\, z^r),\ z^s\left(1-a^2 z^{2r}\right)^{\frac{1-2\lambda}{4}}\, Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(a\, z^r)\right)=\frac{2^{\frac{1}{2}-\lambda}\, a\sqrt{\pi}\ r\, z^{r+2s-1}\left(1-a^2 z^{2r}\right)^{-\lambda-\frac{1}{2}}}{\Gamma(\lambda)}$$

05.09.13.0009.01

$$w''(z)-\left(\frac{a^2\,(2\,\lambda+1)\log(r)\,r^{2\,z}}{1-a^2\,r^{2\,z}}+\log(r)+2\log(s)\right)w'(z)+\frac{a^2\,r^{2\,z}\,(n\log(r)+\log(s))\,(\log(s)-(2\,\lambda+n)\log(r))-\log(s)\,(\log(r)+\log(s))}{a^2\,r^{2\,z}-1}\,w(z)=0\;/;$$

$$w(z)=c_1\,s^z\,C_n^{\lambda}(a\,r^z)+c_2\,s^z\left(1-a^2\,r^{2\,z}\right)^{\frac{1-2\,\lambda}{4}}\,Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(a\,r^z)$$

05.09.13.0010.01

$$W_z\left(s^z\,C_n^{\lambda}(a\,r^z),\,s^z\left(1-a^2\,r^{2\,z}\right)^{\frac{1-2\,\lambda}{4}}\,Q_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(a\,r^z)\right)=\frac{2^{\frac{1}{2}-\lambda}\,a\,\sqrt{\pi}\,r^z\left(1-a^2\,r^{2\,z}\right)^{-\lambda-\frac{1}{2}}\,s^{2\,z}\log(r)}{\Gamma(\lambda)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.09.16.0001.01

$$C_n^{\lambda}(-z)=(-1)^n\,C_n^{\lambda}(z)$$

Products, sums, and powers of the direct function

Products of the direct function

05.09.16.0002.01

$$C_m^{\lambda}(z)\,C_n^{\lambda}(z)=\sum_{k=|m-n|}^{m+n}(1-(k+m+n)\bmod 2)\,(k+\lambda)\,k!\left(\frac{1}{2}\,(-k+m+n+2\,\lambda-2)\right)!\left(\frac{1}{2}\,(k+m-n+2\,\lambda-2)\right)!\left(\frac{1}{2}\,(k-m+n+2\,\lambda-2)\right)!\left(\frac{1}{2}\,(k+m+n+4\,\lambda-2)\right)!\Big/\left(\left(\frac{1}{2}\,(-k+m+n)\right)!\left(\frac{1}{2}\,(k+m-n)\right)!\left(\frac{1}{2}\,(k-m+n)\right)!\,(\lambda-1)!^2\left(\frac{1}{2}\,(k+m+n)+\lambda\right)!\,(k+2\,\lambda-1)!\right)C_k^{\lambda}(z)\;/;\;n\in\mathbb{N}\land m\in\mathbb{N}$$

05.09.16.0003.01

$$C_m^{\lambda}(z)\,C_n^{\lambda}(z)=\sum_{k=0}^{\mathrm{Min}(m,n)}\left(((-2\,k+m+n+\lambda)\,\Gamma(-2\,k+m+n+1)\,\Gamma(k+\lambda)\,\Gamma(-k+m+\lambda)\,\Gamma(-k+n+\lambda)\,\Gamma(-k+m+n+2\,\lambda))\,C_{-2\,k+m+n}^{\lambda}(z)\right)\Big/\left(\Gamma(k+1)\,\Gamma(-k+m+1)\,\Gamma(-k+n+1)\,\Gamma(\lambda)^2\,\Gamma(-k+m+n+\lambda+1)\,\Gamma(-2\,k+m+n+2\,\lambda)\right)\;/;\;n\in\mathbb{N}\land m\in\mathbb{N}$$

Addition formulas

05.09.16.0004.01

$$C_n^{\alpha}(\cos(\theta_0))=\sum_{k=0}^{n}\frac{(\alpha)_k\,(n-k)!}{(\alpha-1/2)_k\,(2\,k+2\,\alpha)_{n-k}}\sin^k(\theta)\sin^k(\vartheta)\,C_{n-k}^{\alpha+k}(\cos(\theta))\,C_{n-k}^{\alpha+k}(\cos(\vartheta))\,C_k^{\alpha-1/2}(\cos(\phi))\;/;\;n\in\mathbb{N}\bigwedge\cos(\theta_0)=\cos(\theta)\cos(\vartheta)+\cos(\phi)\sin(\theta)\sin(\vartheta)$$

Identities

Recurrence identities

Consecutive neighbors

With respect to *n*

05.09.17.0001.01

$$C_n^\lambda(z) == \frac{2(\lambda+n+1)z}{2\lambda+n} C_{n+1}^\lambda(z) - \frac{n+2}{2\lambda+n} C_{n+2}^\lambda(z)$$

05.09.17.0002.01

$$C_n^\lambda(z) == \frac{2(\lambda+n-1)z}{n} C_{n-1}^\lambda(z) - \frac{2\lambda+n-2}{n} C_{n-2}^\lambda(z)$$

With respect to λ

05.09.17.0007.01

$$C_n^\lambda(z) == \frac{2\lambda\left(2\lambda-2\left(z^2-1\right)(\lambda+n+1)+1\right)}{(2\lambda+n)(2\lambda+n+1)} C_n^{\lambda+1}(z) + \frac{4\lambda(\lambda+1)\left(z^2-1\right)}{(2\lambda+n)(2\lambda+n+1)} C_n^{\lambda+2}(z)$$

05.09.17.0008.01

$$C_n^\lambda(z) = \frac{2(\lambda-1)z^2+2n\left(z^2-1\right)-4\lambda+5}{2\left(z^2-1\right)(\lambda-1)} C_n^{\lambda-1}(z) + \frac{(n+2\lambda-4)(n+2\lambda-3)}{4\left(z^2-1\right)((\lambda-3)\lambda+2)} C_n^{\lambda-2}(z)$$

Distant neighbors

With respect to *n*

05.09.17.0009.01

$$C_n^\lambda(z) == \mathcal{C}_m(n,\lambda,z)\, C_{m+n}^\lambda(z) - \frac{m+n+1}{m+n+2\lambda-1} \mathcal{C}_{m-1}(n,\lambda,z)\, C_{m+n+1}^\lambda(z) \;/;\; \mathcal{C}_0(n,\lambda,z) == 1 \bigwedge$$
$$\mathcal{C}_1(n,\lambda,z) == \frac{2(n+\lambda+1)z}{n+2\lambda} \bigwedge \mathcal{C}_m(n,\lambda,z) == \frac{2z(m+n+\lambda)}{m+n+2\lambda-1} \mathcal{C}_{m-1}(n,\lambda,z) - \frac{m+n}{m+n+2\lambda-2} \mathcal{C}_{m-2}(n,\lambda,z) \bigwedge m \in \mathbb{N}^+$$

05.09.17.0010.01

$$C_n^\lambda(z) == \mathcal{C}_m(n,\lambda,z)\, C_{n-m}^\lambda(z) - \frac{n-m+2\lambda-1}{n-m+1} \mathcal{C}_{m-1}(n,\lambda,z)\, C_{n-m-1}^\lambda(z) \;/;\; \mathcal{C}_0(n,\lambda,z) == 1 \bigwedge$$
$$\mathcal{C}_1(n,\lambda,z) == \frac{2(n+\lambda-1)z}{n} \bigwedge \mathcal{C}_m(n,\lambda,z) == \frac{2z(n-m+\lambda)}{n-m+1} \mathcal{C}_{m-1}(n,\lambda,z) - \frac{n-m+2\lambda}{n-m+2} \mathcal{C}_{m-2}(n,\lambda,z) \bigwedge n \in \mathbb{N}^+$$

With respect to λ

Functional identities

Relations between contiguous functions

Recurrence relations

05.09.17.0003.01

$$(2\lambda + n - 1)\, C_{n-1}^{\lambda}(z) + (n+1)\, C_{n+1}^{\lambda}(z) == 2\,(\lambda + n)\, z\, C_n^{\lambda}(z)$$

05.09.17.0004.01

$$C_n^{\lambda}(z) == \frac{1}{2\,(\lambda + n)\, z}\left((2\lambda + n - 1)\, C_{n-1}^{\lambda}(z) + (n+1)\, C_{n+1}^{\lambda}(z)\right)$$

05.09.17.0011.01

$$C_n^{\lambda}(z) == z\, C_{n-1}^{\lambda}(z) + \frac{2\lambda + n - 2}{2\lambda - 2}\, C_n^{\lambda-1}(z)$$

05.09.17.0012.01

$$C_n^{\lambda}(z) == \frac{(n+1)\, z}{2\lambda + n}\, C_{n+1}^{\lambda}(z) - \frac{2\lambda\left(z^2 - 1\right)}{2\lambda + n}\, C_n^{\lambda+1}(z)$$

Normalized recurrence relation

05.09.17.0005.01

$$z\, p(n, z) == \frac{n\,(n + 2\lambda - 1)}{4\,(n + \lambda - 1)\,(n + \lambda)}\, p(n-1, z) + p(n+1, z) \,/;\, p(n, z) == \frac{n!}{2^n\,(\lambda)_n}\, C_n^{\lambda}(z) \bigwedge n > 0$$

Relations of special kind

05.09.17.0006.01

$$(2\lambda + n)\, n\, C_n^{\lambda}(z) - 2\, z\, \lambda\,(2\lambda + 1)\, C_{n-1}^{\lambda+1}(z) - 4\left(z^2 - 1\right)\lambda\,(\lambda + 1)\, C_{n-2}^{\lambda+2}(z) == 0$$

Complex characteristics

Real part

05.09.19.0001.01

$$\operatorname{Re}\left(C_n^{\lambda}(x + i\, y)\right) == \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\,(\lambda)_{2j}}{(2j)!}\, C_{n-2j}^{2j+\lambda}(x)\, y^{2j} \,/;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R}$$

Imaginary part

05.09.19.0002.01

$$\operatorname{Im}\left(C_n^{\lambda}(x + i\, y)\right) == \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1}\,(\lambda)_{2j+1}}{(2j+1)!}\, C_{n-2j-1}^{2j+\lambda+1}(x)\, y^{2j+1} \,/;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R}$$

Argument

05.09.19.0003.01

$$\arg\left(C_n^{\lambda}(x + i\, y)\right) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\,(\lambda)_{2j}}{(2j)!}\, C_{n-2j}^{2j+\lambda}(x)\, y^{2j},\, \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1}\,(\lambda)_{2j+1}}{(2j+1)!}\, C_{n-2j-1}^{2j+\lambda+1}(x)\, y^{2j+1}\right) /;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R}$$

Conjugate value

05.09.19.0004.01

$$\overline{C_n^\lambda(x+i\,y)} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, (\lambda)_{2j}}{(2j)!}\, C_{n-2j}^{2j+\lambda}(x)\, y^{2j} - i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1}\, (\lambda)_{2j+1}}{(2j+1)!}\, C_{n-2j-1}^{2j+\lambda+1}(x)\, y^{2j+1} \,/;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge \lambda \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to λ

05.09.20.0001.01

$$\frac{\partial C_n^\lambda(z)}{\partial \lambda} = \frac{2\,n\,(z-1)\,\Gamma(2\lambda+n)}{(2\lambda+1)^2\,\Gamma(n+1)\,\Gamma(2\lambda)} \left((2\lambda+1)\, F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix} 1-n,\, 2\lambda+n+1;\, 1;\, 1,\, 2\lambda+n; \\ 2,\, \lambda+\frac{3}{2};;\, 2\lambda+n+1; \end{matrix} \frac{1-z}{2},\, \frac{1-z}{2} \right) - (2\lambda+n)\, F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix} 1-n,\, 2\lambda+n+1;\, 1;\, 1,\, \lambda+\frac{1}{2}; \\ 2,\, \lambda+\frac{3}{2};;\, \lambda+\frac{3}{2}; \end{matrix} \frac{1-z}{2},\, \frac{1-z}{2} \right) \right) - \frac{2^{2-2\lambda}\sqrt{\pi}\, \Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(\lambda)} (\psi(2\lambda)-\psi(2\lambda+n))\, {}_2\tilde{F}_1\left(-n,\, 2\lambda+n;\, \lambda+\frac{1}{2};\, \frac{1-z}{2}\right)$$

05.09.20.0009.01

$$\frac{\partial C_n^\lambda(z)}{\partial \lambda} = \sum_{k=0}^{n-1} \left(\frac{2\left(1+(-1)^{n-k}\right)(k+\lambda)}{(k+n+2\lambda)\,(n-k)}\, C_k^\lambda(z) + \left(\frac{2\,(k+1)}{(k+2\lambda)\,(2k+2\lambda+1)} + \frac{2}{k+n+2\lambda} \right) C_n^\lambda(z) \right)$$

05.09.20.0010.01

$$\frac{\partial^2 C_n^\lambda(z)}{\partial \lambda^2} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2z)^{n-2k}}{k!\,(n-2k)!} \sum_{j=0}^{n-k-2} (-1)^{j+n}\,(j+1)\,(j+2)\, S_{n-k}^{(j+2)}\, \lambda^j$$

With respect to z

Forward shift operator:

05.09.20.0002.01

$$\frac{\partial C_n^\lambda(z)}{\partial z} = 2\lambda\, C_{n-1}^{\lambda+1}(z)$$

05.09.20.0003.01

$$\frac{\partial^2 C_n^\lambda(z)}{\partial z^2} = 4\lambda\,(\lambda+1)\, C_{n-2}^{\lambda+2}(z)$$

Backward shift operator:

05.09.20.0004.01

$$\left(1-z^2\right) \frac{\partial C_n^\lambda(z)}{\partial z} + z\,(1-2\lambda)\, C_n^\lambda(z) = -\frac{(n+1)\,(n+2\lambda-1)}{2\,(\lambda-1)}\, C_{n+1}^{\lambda-1}(z)$$

05.09.20.0005.01

$$\frac{\partial\left(\left(1-z^2\right)^{\lambda-\frac{1}{2}} C_n^{\lambda}(z)\right)}{\partial z} = -\frac{(n+1)(n+2\lambda-1)}{2(\lambda-1)}\left(1-z^2\right)^{\lambda-\frac{3}{2}} C_{n+1}^{\lambda-1}(z)$$

Symbolic differentiation

With respect to λ

05.09.20.0011.01

$$\frac{\partial^m C_n^{\lambda}(z)}{\partial \lambda^m} = \delta_m \delta_n + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2z)^{n-2k}}{k!(n-2k)!} \sum_{j=1}^{n-k} (-1)^{j+n} S_{n-k}^{(j)} (j-m+1)_m \lambda^{j-m} /; m \in \mathbb{N}$$

With respect to z

05.09.20.0006.02

$$\frac{\partial^m C_n^{\lambda}(z)}{\partial z^m} = 2^m (\lambda)_m C_{n-m}^{m+\lambda}(z) /; m \in \mathbb{N}$$

05.09.20.0007.02

$$\frac{\partial^m C_n^{\lambda}(z)}{\partial z^m} = \frac{2^{1-2\lambda}\sqrt{\pi}(z-1)^{-m}\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(\lambda)} {}_3\tilde{F}_2\left(1, -n, 2\lambda+n; 1-m, \lambda+\frac{1}{2}; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to λ

05.09.20.0012.01

$$\frac{\partial^\alpha C_n^{\lambda}(z)}{\partial \lambda^\alpha} = \frac{\delta_n \lambda^{a-\alpha}}{\Gamma(1-\alpha)} + \sum_{k=1}^{n} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+n} k!(2z)^{n-2j} S_{n-j}^{(k)} \lambda^{k-\alpha}}{j!(n-2j)!\Gamma(k-\alpha+1)}$$

With respect to z

05.09.20.0008.01

$$\frac{\partial^\alpha C_n^{\lambda}(z)}{\partial z^\alpha} = \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(\lambda)} z^{-\alpha} \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -n, 2\lambda+n; 1; ; \\ \lambda+\frac{1}{2}; 1-\alpha; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

05.09.21.0001.01

$$\int C_n^{\lambda}(z)\,dz = \frac{1}{2(\lambda-1)} C_{n+1}^{\lambda-1}(z)$$

05.09.21.0002.01

$$\int C_n^{\frac{3}{2}}(z)\,dz = P_{n+1}(z)$$

05.09.21.0003.01

$$\int C_n^2(z)\,dz = \frac{1}{2}\,U_{n+1}(z)$$

Involving one direct function and elementary functions

Involving power function

05.09.21.0004.01

$$\int z^{\alpha-1}\,C_n^{\lambda}(z)\,dz = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\,(\lambda)_{n-k}\,2^{n-2k}\,z^{n-2k+\alpha}}{k!\,(n-2k)!\,(n-2k+\alpha)}$$

Involving algebraic functions

05.09.21.0005.01

$$\int \left(1-z^2\right)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(z)\,dz = -\frac{2\left(1-z^2\right)^{\lambda+\frac{1}{2}}\,\lambda}{n\,(n+2\lambda)}\,C_{n-1}^{\lambda+1}(z)$$

05.09.21.0006.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-n-3)}\,C_n^{\lambda}(z)\,dz = \frac{\left(1-z^2\right)^{\frac{1}{2}(-n-1)}}{n+2\lambda}\,C_{n+1}^{\lambda}(z)$$

05.09.21.0007.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(n+2\lambda-3)}\,C_n^{\lambda}(z)\,dz = -\frac{\left(1-z^2\right)^{\frac{1}{2}(n+2\lambda-1)}}{n}\,C_{n-1}^{\lambda}(z)$$

Definite integration

Involving the direct function

Orthogonality:

05.09.21.0008.01

$$\int_{-1}^{1}\left(1-t^2\right)^{\lambda-\frac{1}{2}}\,C_m^{\lambda}(t)\,C_n^{\lambda}(t)\,dt = \frac{\pi\,2^{1-2\lambda}\,\Gamma(n+2\lambda)}{n!\,(n+\lambda)\,\Gamma(\lambda)^2}\,\delta_{m,n}\ /;\ \mathrm{Re}(\lambda) > -\frac{1}{2}\bigwedge \lambda \neq 0$$

Summation

Finite summation

05.09.23.0001.01

$$\sum_{k=0}^{n} \frac{(-1)^k\,4^k\,\Gamma(n-k+1)\,\Gamma(k+\lambda)^2\,(2k+2\lambda-1)}{\Gamma(k+n+2\lambda)}\left(z_1^2-1\right)^{k/2}\left(z_2^2-1\right)^{k/2}C_{n-k}^{k+\lambda}(z_1)\,C_{n-k}^{k+\lambda}(z_2)\,C_k^{\lambda-\frac{1}{2}}(\alpha) = \frac{4^{1-\lambda}\sqrt{\pi}\,\Gamma(\lambda)}{\Gamma\left(\lambda-\frac{1}{2}\right)}\,C_n^{\lambda}\left(z_1\,z_2-\sqrt{z_1^2-1}\,\sqrt{z_2^2-1}\,\alpha\right)$$

Infinite summation

05.09.23.0002.01

$$\sum_{n=0}^{\infty} C_n^{\lambda}(z)\, w^n = \left(w^2 - 2\, z\, w + 1\right)^{-\lambda} \;/;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0003.01

$$\sum_{n=0}^{\infty} \frac{\left(\lambda + \frac{1}{2}\right)_n}{(2\,\lambda)_n}\, C_n^{\lambda}(z)\, w^n = \frac{2^{\lambda - \frac{1}{2}}}{\sqrt{w^2 - 2\, z\, w + 1}} \left(1 - w\, z + \sqrt{w^2 - 2\, z\, w + 1}\right)^{\frac{1}{2} - \lambda} \;/;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0004.01

$$\sum_{n=0}^{\infty} \frac{C_n^{\lambda}(z)\, w^n}{(2\,\lambda)_n} = e^{z\, w}\, {}_0F_1\!\left(; \lambda + \frac{1}{2}; \frac{1}{4}\left(z^2 - 1\right) w^2\right) /;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0005.01

$$\sum_{n=0}^{\infty} \frac{1}{(2\,\lambda)_n \left(\lambda + \frac{1}{2}\right)_n}\, C_n^{\lambda}(z)\, w^n = {}_0F_1\!\left(; \lambda + \frac{1}{2}; \frac{1}{2}(z - 1)\, w\right) {}_0F_1\!\left(; \lambda + \frac{1}{2}; \frac{1}{2}(z + 1)\, w\right) /;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, (2\,\lambda - \gamma)_n}{(2\,\lambda)_n \left(\lambda + \frac{1}{2}\right)_n}\, C_n^{\lambda}(z)\, w^n = {}_2F_1\!\left(\gamma, 2\,\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} - w\right)\right)$$

$${}_2F_1\!\left(\gamma, 2\,\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\, z\, w + 1} + w\right)\right) /;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0007.01

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(2\,\lambda)_n}\, C_n^{\lambda}(z)\, w^n = (1 - w\, z)^{\lambda}\, {}_2F_1\!\left(\frac{\lambda}{2}, \frac{\lambda + 1}{2}; \lambda + \frac{1}{2}; \frac{\left(z^2 - 1\right) w^2}{(1 - w\, z)^2}\right) /;\; -1 < z < 1 \land |w| < 1$$

05.09.23.0008.01

$$\sum_{n=0}^{\infty} \frac{n!\, (n + \lambda)}{\Gamma(n + 2\,\lambda)}\, C_n^{\lambda}(x)\, C_n^{\lambda}(y) = \frac{\pi\, 2^{1 - 2\,\lambda}}{\Gamma(\lambda)^2} \left(1 - x^2\right)^{\frac{1 - 2\,\lambda}{4}} \left(1 - y^2\right)^{\frac{1 - 2\,\lambda}{4}} \delta(x - y) \;/;\; \operatorname{Re}(\lambda) > -\frac{1}{2} \bigwedge \lambda \neq 0 \bigwedge -1 < x < 1 \bigwedge -1 < y < 1$$

Operations

Limit operation

05.09.25.0001.01

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\, C_n^{\lambda}(z) = C_n^{(0)}(z)$$

05.09.25.0002.01

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\, C_n^{\lambda}(z) = \frac{2}{n}\, T_n(z)$$

05.09.25.0003.01

$$\lim_{\lambda \to \infty} \lambda^{-\frac{n}{2}}\, C_n^{\lambda}\!\left(\frac{z}{\sqrt{\lambda}}\right) = \frac{2}{n!}\, H_n(z) \;/;\; |z| < 1$$

05.09.25.0004.01

$$\lim_{z\to\infty} (2\,z)^{-n}\, C_n^{\lambda}(z) == \frac{(\lambda)_n}{n!}$$

Orthogonality, completeness, and Fourier expansions

The set of functions $C_n^{\lambda}(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{n!\,(n+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(n+2\lambda)} \left(1-x^2\right)^{\lambda-\frac{1}{2}}$) system on the interval $(-1, 1)$.

05.09.25.0005.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{n!\,(n+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(n+2\lambda)}} \left(1-x^2\right)^{\frac{2\lambda-1}{4}} C_n^{\lambda}(x) \right) \left(\sqrt{\frac{n!\,(n+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(n+2\lambda)}} \left(1-y^2\right)^{\frac{2\lambda-1}{4}} C_n^{\lambda}(y) \right) == \delta(x-y) \,/;$$

$$\mathrm{Re}(\lambda) > -\frac{1}{2} \bigwedge \lambda \neq 0 \bigwedge -1 < x < 1 \bigwedge -1 < y < 1$$

05.09.25.0006.01

$$\int_{-1}^{1} \left(\sqrt{\frac{m!\,(m+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(m+2\lambda)}} \left(1-t^2\right)^{\frac{2\lambda-1}{4}} C_m^{\lambda}(t) \right) \left(\sqrt{\frac{n!\,(n+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(n+2\lambda)}} \left(1-t^2\right)^{\frac{2\lambda-1}{4}} C_n^{\lambda}(t) \right) dt == \delta_{m,n} \,/;\, \mathrm{Re}(\lambda) > -\frac{1}{2} \bigwedge \lambda \neq 0$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\left\{C_n^{\lambda}(x)\right\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.09.25.0007.01

$$f(x) == \sum_{n=0}^{\infty} c_n\, \psi_n(x) \,/;\, c_n == \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) == \sqrt{\frac{n!\,(n+\lambda)\,\Gamma(\lambda)^2}{\pi\, 2^{1-2\lambda}\,\Gamma(n+2\lambda)}} \left(1-x^2\right)^{\frac{2\lambda-1}{4}} C_n^{\lambda}(x) \bigwedge -1 < x < 1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_0\tilde{F}_1$

05.09.26.0001.01

$$C_n^{\lambda}(z) == \frac{2^{1-2\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(\lambda)}\, {}_2\tilde{F}_1\!\left(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{1-z}{2}\right)$$

Involving ${}_2F_1$

05.09.26.0002.01

$$C_n^{\lambda}(z) == \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)\,\Gamma(n+1)}\, {}_2F_1\!\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-z}{2}\right) /;\, -\lambda-\frac{1}{2} \notin \mathbb{N}$$

05.09.26.0003.01

$$C_n^{\lambda}(z) == \frac{\cos(\pi(\lambda+n))\sec(\pi\lambda)\,\Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(2\lambda)}\, {}_2F_1\!\left(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{z+1}{2}\right) /;\, \lambda+\frac{1}{2} \notin \mathbb{Z}$$

05.09.26.0004.01

$$C_n^{\lambda}(z) == \frac{2^n\,\Gamma(\lambda+n)}{\Gamma(n+1)\,\Gamma(\lambda)}\,(z-1)^n\,{}_2F_1\left(-n,\,-\lambda-n+\frac{1}{2};\,-2\,\lambda-2\,n+1;\,\frac{2}{1-z}\right)/;\,2\,\lambda+2\,n\notin\mathbb{Z}$$

Through hypergeometric functions of two variables

05.09.26.0005.01

$$C_n^{\lambda}(z) == \frac{2^{1-2\lambda}\,\sqrt{\pi}\,\Gamma(2\,\lambda+n)}{\Gamma(n+1)\,\Gamma(\lambda)}\,\tilde{F}_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix}-n,\,2\,\lambda+n;;; \\ \lambda+\frac{1}{2};;;\end{matrix}\;\frac{1}{2},\,-\frac{z}{2}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

05.09.26.0006.01

$$C_n^{\lambda}(z) == -\frac{2^{1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)}\lim_{m\to n}\sin(\pi\,m)\,G_{2,2}^{1,2}\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}m+1,\,-m-2\,\lambda+1 \\ 0,\,\frac{1}{2}-\lambda\end{matrix}\right)$$

Classical cases involving algebraic functions

05.09.26.0007.01

$$(z+1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(2\,z+1) == \frac{\cos((\lambda+n)\,\pi)\,(2\,\lambda)_n}{\cos(\lambda\,\pi)\,\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(n+1)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}-\lambda-n+\frac{1}{2},\,\lambda+n+\frac{1}{2} \\ 0,\,\frac{1}{2}-\lambda\end{matrix}\right)$$

05.09.26.0008.01

$$(z+1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(1+\frac{2}{z}\right) == \frac{\cos((\lambda+n)\,\pi)\,(2\,\lambda)_n}{\cos(\lambda\,\pi)\,\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\lambda+\frac{1}{2},\,2\,\lambda \\ -n,\,2\,\lambda+n\end{matrix}\right)/;\,z\notin(-1,\,0)$$

05.09.26.0009.01

$$(z+1)^{-2\lambda-n}\,C_n^{\lambda}\left(\frac{1-z}{1+z}\right) == \frac{1}{\Gamma(2\,\lambda)\left(\lambda+\frac{1}{2}\right)_n\Gamma(n+1)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}1-2\,\lambda-n,\,\frac{1}{2}-\lambda-n \\ 0,\,\frac{1}{2}-\lambda\end{matrix}\right)/;\,z\notin(-\infty,\,-1)$$

05.09.26.0010.01

$$(z+1)^{-2\lambda-n}\,C_n^{\lambda}\left(\frac{z-1}{z+1}\right) == \frac{1}{\Gamma(2\,\lambda)\left(\lambda+\frac{1}{2}\right)_n\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}1-2\,\lambda-n,\,\frac{1}{2}-\lambda-n \\ 0,\,\frac{1}{2}-\lambda\end{matrix}\right)/;\,z\notin(-1,\,0)$$

05.09.26.0011.01

$$(z+1)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\frac{1}{\sqrt{z+1}}\right) == \frac{2^n}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}\frac{1-n}{2}-\lambda,\,1-\lambda-\frac{n}{2} \\ 0,\,\frac{1}{2}-\lambda\end{matrix}\right)$$

05.09.26.0012.01

$$(z+1)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\sqrt{\frac{z}{z+1}}\right) == \frac{2^n}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}1-\lambda-\frac{n}{2},\,\frac{1-n}{2} \\ 0,\,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,\,0)$$

05.09.26.0013.01

$$(z+1)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\frac{z+2}{2\sqrt{z+1}}\right) == \frac{1}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}1-\lambda,\,-2\,\lambda-n+1 \\ 0,\,1-2\,\lambda\end{matrix}\right)$$

05.09.26.0014.01

$$(z+1)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\frac{2z+1}{2\sqrt{z}\,\sqrt{z+1}}\right)=\frac{1}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{array}{c}1-\lambda-\frac{n}{2},\,\lambda-\frac{n}{2}\\-\frac{n}{2},\,\lambda+\frac{n}{2}\end{array}\right)/;\,z\notin(-1,0)$$

Classical cases involving unit step θ

05.09.26.0015.01

$$\theta(1-|z|)\,(1-z)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(2z-1)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\lambda+n+\frac{1}{2},\,-\lambda-n+\frac{1}{2}\\0,\,\frac{1}{2}-\lambda\end{array}\right)/;\,z\notin(-1,0)$$

05.09.26.0016.01

$$\theta(|z|-1)\,(z-1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(2z-1)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\lambda+n+\frac{1}{2},\,-\lambda-n+\frac{1}{2}\\0,\,\frac{1}{2}-\lambda\end{array}\right)$$

05.09.26.0017.01

$$\theta(1-|z|)\,(1-z)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{2}{z}-1\right)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\lambda+\frac{1}{2},\,2\lambda\\2\lambda+n,\,-n\end{array}\right)$$

05.09.26.0018.01

$$\theta(|z|-1)\,(z-1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{2}{z}-1\right)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\lambda+\frac{1}{2},\,2\lambda\\2\lambda+n,\,-n\end{array}\right)/;\,z\notin(-\infty,-1)$$

05.09.26.0019.01

$$\theta(1-|z|)\,(1-z)^{\lambda-\frac{1}{2}}\,C_n^{(\lambda)}\left(\sqrt{z}\right)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{n+1}{2}+\lambda,\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{array}\right)/;\,z\notin(-1,0)$$

05.09.26.0020.01

$$\theta(|z|-1)\,(z-1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\sqrt{z}\right)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)(2\lambda)_n}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1-n}{2},\,\lambda+\frac{n+1}{2}\\0,\,\frac{1}{2}\end{array}\right)$$

05.09.26.0021.01

$$\theta(1-|z|)\,(1-z)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{1}{\sqrt{z}}\right)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)(2\lambda)_n}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\lambda,\,\lambda+\frac{1}{2}\\\lambda+\frac{n}{2},\,-\frac{n}{2}\end{array}\right)/;\,z\notin(-1,0)$$

05.09.26.0022.01

$$\theta(|z|-1)\,(z-1)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{1}{\sqrt{z}}\right)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\lambda+\frac{1}{2},\,\lambda\\-\frac{n}{2},\,\lambda+\frac{n}{2}\end{array}\right)$$

05.09.26.0023.01

$$\theta(1-|z|)\,(1-z)^{-\frac{n}{2}-\lambda}\,C_n^{\lambda}\left(\frac{1}{\sqrt{1-z}}\right)=\frac{(-2)^n\,\Gamma(1-\lambda)}{n!}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1-n}{2}-\lambda,\,1-\frac{n}{2}-\lambda\\0,\,\frac{1}{2}-\lambda\end{array}\right)$$

05.09.26.0024.01

$$\theta(|z|-1)\,(z-1)^{-\frac{n}{2}-\lambda}\,C_n^{\lambda}\left(\sqrt{\frac{z}{z-1}}\right)=\frac{(-2)^n\,\Gamma(1-\lambda)}{n!}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}-\frac{n}{2}-\lambda+1,\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{array}\right)$$

05.09.26.0025.01

$$\theta(1-|z|)\,(1-z)^{-\frac{n}{2}-\lambda}\,C_n^{\lambda}\left(\frac{2-z}{2\sqrt{1-z}}\right)=\frac{(-1)^n\,\Gamma(1-\lambda)}{n!}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}-n-2\lambda+1,\,1-\lambda\\0,\,1-2\lambda\end{matrix}\right)$$

05.09.26.0026.01

$$\theta(|z|-1)\,(z-1)^{-\frac{n}{2}-\lambda}\,C_n^{\lambda}\left(\frac{2z-1}{2\sqrt{z}\,\sqrt{z-1}}\right)=\frac{(-1)^n\,\Gamma(1-\lambda)}{n!}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}1-\frac{n}{2}-\lambda,\,\lambda-\frac{n}{2}\\\frac{n}{2}+\lambda,\,-\frac{n}{2}\end{matrix}\right)$$

05.09.26.0027.01

$$\theta(1-|z|)\,(1-z)^{2\lambda-1}\,C_n^{\lambda}\left(\frac{z+1}{2\sqrt{z}}\right)=\frac{\Gamma(2\lambda+n)}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\lambda-\frac{n}{2},\,2\lambda+\frac{n}{2}\\\lambda+\frac{n}{2},\,-\frac{n}{2}\end{matrix}\right)\,/;\,z\notin(-1,\,0)$$

05.09.26.0028.01

$$\theta(|z|-1)\,(z-1)^{2\lambda-1}\,C_n^{\lambda}\left(\frac{z+1}{2\sqrt{z}}\right)=\frac{\Gamma(2\lambda+n)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\lambda-\frac{n}{2},\,2\lambda+\frac{n}{2}\\\lambda+\frac{n}{2},\,-\frac{n}{2}\end{matrix}\right)$$

Generalized cases involving algebraic functions

05.09.26.0029.01

$$\left(z^2+1\right)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\frac{z}{\sqrt{z^2+1}}\right)=\frac{2^n}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}1-\lambda-\frac{n}{2},\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{matrix}\right)\,/;\,\mathrm{Re}(z)>0$$

05.09.26.0030.01

$$\left(z^2+1\right)^{-\lambda-\frac{n}{2}}\,C_n^{\lambda}\left(\frac{2z^2+1}{2z\sqrt{z^2+1}}\right)=\frac{1}{\Gamma(\lambda)\,\Gamma(n+1)}\,G_{2,2}^{2,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}1-\lambda-\frac{n}{2},\,\lambda-\frac{n}{2}\\-\frac{n}{2},\,\lambda+\frac{n}{2}\end{matrix}\right)\,/;\,\mathrm{Re}(z)>0$$

Generalized cases involving unit step θ

05.09.26.0031.01

$$\theta(1-|z|)\left(1-z^2\right)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(z)=\frac{(2\lambda)_n}{\Gamma(n+1)}\,\Gamma\left(\lambda+\frac{1}{2}\right)G_{2,2}^{2,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{n+1}{2}+\lambda,\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{matrix}\right)$$

05.09.26.0032.01

$$\theta(|z|-1)\left(z^2-1\right)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}(z)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)(2\lambda)_n}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-n}{2},\,\lambda+\frac{n+1}{2}\\0,\,\frac{1}{2}\end{matrix}\right)\,/;\,\mathrm{Re}(z)>0$$

05.09.26.0033.01

$$\theta(1-|z|)\left(1-z^2\right)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{1}{z}\right)=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)(2\lambda)_n}{\Gamma(n+1)}\,G_{2,2}^{2,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\lambda,\,\lambda+\frac{1}{2}\\\lambda+\frac{n}{2},\,-\frac{n}{2}\end{matrix}\right)$$

05.09.26.0034.01

$$\theta(|z|-1)\left(z^2-1\right)^{\lambda-\frac{1}{2}}\,C_n^{\lambda}\left(\frac{1}{z}\right)=\frac{(2\lambda)_n\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(n+1)}\,G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\lambda+\frac{1}{2},\,\lambda\\-\frac{n}{2},\,\lambda+\frac{n}{2}\end{matrix}\right)\,/;\,\mathrm{Re}(z)>0$$

05.09.26.0035.01

$$\theta(|z|-1)\left(z^2-1\right)^{-\frac{n}{2}-\lambda}\,C_n^{\lambda}\left(\frac{z}{\sqrt{z^2-1}}\right)=\frac{(-2)^n\,\Gamma(1-\lambda)}{n!}\,G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}1-\frac{n}{2}-\lambda,\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{matrix}\right)\,/;\,\mathrm{Re}(z)>0$$

05.09.26.0036.01

$$\theta(|z|-1)\left(z^2-1\right)^{-\frac{n}{2}-\lambda} C_n^\lambda\left(\frac{2z^2-1}{2z\sqrt{z^2-1}}\right) = \frac{(-1)^n\,\Gamma(1-\lambda)}{n!}\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 1-\frac{n}{2}-\lambda, \lambda-\frac{n}{2} \\ \frac{n}{2}+\lambda, -\frac{n}{2} \end{array}\right) /; \mathrm{Re}(z)>0$$

05.09.26.0037.01

$$\theta(1-|z|)\left(1-z^2\right)^{2\lambda-1} C_n^\lambda\left(\frac{z^2+1}{2z}\right) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)}\, G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \lambda-\frac{n}{2}, 2\lambda+\frac{n}{2} \\ \lambda+\frac{n}{2}, -\frac{n}{2} \end{array}\right)$$

05.09.26.0038.01

$$\theta(|z|-1)\left(z^2-1\right)^{2\lambda-1} C_n^\lambda\left(\frac{z^2+1}{2z}\right) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)}\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \lambda-\frac{n}{2}, 2\lambda+\frac{n}{2} \\ \lambda+\frac{n}{2}, -\frac{n}{2} \end{array}\right) /; \mathrm{Re}(z)>0$$

Through other functions

Involving some hypergeometric-type functions

05.09.26.0039.01

$$C_n^\lambda(z) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(z)$$

05.09.26.0040.01

$$C_{2n}^\lambda(z) = \frac{(\lambda)_n}{\left(\frac{1}{2}\right)_n} P_n^{\left(\lambda-\frac{1}{2},-\frac{1}{2}\right)}\left(2z^2-1\right)$$

05.09.26.0041.01

$$C_{2n+1}^\lambda(z) = \frac{(\lambda)_{n+1}\, z}{\left(\frac{1}{2}\right)_{n+1}} P_n^{\left(\lambda-\frac{1}{2},\frac{1}{2}\right)}\left(2z^2-1\right)$$

Representations through equivalent functions

With related functions

05.09.27.0001.01

$$C_n^\lambda(z) = \frac{2^{\frac{1}{2}-\lambda}\sqrt{\pi}\,\Gamma(n+2\lambda)}{\Gamma(n+1)\,\Gamma(\lambda)}\left(1-z^2\right)^{\frac{2\lambda-1}{4}} P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(z)$$

05.09.27.0002.01

$$C_n^{(\lambda)}(z) = \frac{2^{1/2-\lambda}\sqrt{\pi}\,\Gamma(2\lambda+n)}{\Gamma(n+1)\,\Gamma(\lambda)}(z+1)^{\frac{1-2\lambda}{4}}(z-1)^{\frac{1-2\lambda}{4}}\,\mathfrak{P}_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda}(z)$$

05.09.27.0003.01

$$C_n^\lambda(z) = \frac{\pi\, 2^{1-\lambda}}{(\lambda+n)\,\Gamma(\lambda)}\sqrt{\frac{(\lambda+n)\,\Gamma(2\lambda+n)}{\Gamma(n+1)}}\left(1-z^2\right)^{\frac{1-2\lambda}{4}} Y_{\lambda+\mathrm{nn}-1/2}^{1/2-\lambda}\left(\cos^{-1}(z), 0\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) == \sum_{k=0}^{\infty} c_k\, \psi_k(x) \,/;\quad c_k == \int_{-1}^{1} f(t)\, \psi_k(t)\, dt,\ \ \psi_k(x) == \frac{2^{1/2-\lambda} \sqrt{\pi\, \Gamma(2\lambda + n)}}{\Gamma(\lambda) \sqrt{n!\, (\lambda + n)}} \left(1 - x^2\right)^{\frac{2\lambda-1}{4}} C_k^{\lambda}(x),\ \ k \in \mathbb{N}.$$

Eigenfunctions of the angular part of a *d*-dimensional Laplace operator

The eigenfunctions of the angular part $L^2 == -\sum_{i>j}^{d} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right)^2$ of a d-dimensional Laplace operator $\Delta == \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + \frac{L^2}{r^2}$, where **u, u′** are two unit vectors in $\mathbb{R}^d$, has the representation $L^2\, C_n^{d/2-1}(\mathbf{u}.\mathbf{u}') == n(n + d - 2)\, C_n^{d/2-1}(\mathbf{u}.\mathbf{u}')$.

Removing Gibbs oscillations from Fourier series

Let $f(x)$ be a doubly periodic function with $f(-1) \neq f(1)$. Let $\hat{f}_k$ be its Fourier components $\hat{f}_k = \frac{1}{2} \int_{-1}^{1} f(x)\, e^{-i k \pi x}\, dx$. Then the Fourier sum $\sum_{k=-n}^{n} \hat{f}_k\, e^{i k \pi x}$ exhibits Gibbs oscillations.

It is possible to recover the original function $f(x)$ as a sum without such Gibbs oscillations in the following manner: $\sum_{k=0}^{n} g_k^n\, C_k^{\lambda}(x)$, where $g_k^n = \delta_{0\,k}\, \hat{f}_0 + \Gamma(\lambda)\, i^k\, (k + \lambda) \sum_{l=-n}^{n} (1 - \delta_{0\,l})\, J_{k+\lambda}(\pi\, l) \left(\frac{2}{l\, \pi}\right)^{\lambda} \hat{f}_l \,/;\, \lambda = \left\lfloor \frac{2\, \pi\, e\, n}{27} \right\rfloor$.

This sum converges pointwise to $f(x)$.

Quantum mechanical eigenfunctions of the hydrogen atom

The quantum mechanical eigenfunctions $\psi_{n\,m\,l}(p, \theta, \phi)$ of the hydrogen atom in the momentum representation are:

$$\psi_{n\,m\,l}(p, \theta, \phi) == 16\, \pi \sqrt{m}\ \kappa_n^2\, 2^l\, l! \sqrt{\frac{(n - l - 1)!}{(n + l)!}}\ \frac{1}{\left(\kappa_n^2 + p^2\right)^2} \left(\frac{\kappa_n\, p}{\kappa_n^2 + p^2}\right)^l C_{n-l-1}^{l+1}\!\left(\frac{\kappa_n^2 - p^2}{\kappa_n^2 + p^2}\right) Y_l^m(\theta, \phi) \,/;\, \kappa_n = \frac{\tau}{n} \,/;\, \tau > 0,\ n, l \in \mathbb{N},\ l \leq n,\ m \in \mathbb{Z},\ |m| \leq l.$$

History

– L. Gegenbauer (1893)

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