Hypergeometric2F3

Notations

Traditional name

Generalized hypergeometric function ${}_2F_3$

Traditional notation

${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$

Mathematica StandardForm notation

`HypergeometricPFQ[{a1, a2}, {b1, b2, b3}, z]`

Primary definition

07.26.02.0001.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k z^k}{(b_1)_k (b_2)_k (b_3)_k k!}$$

For $a_i = -n$, $b_j = -m$ /; $m \geq n$ being nonpositive integers and $\nexists_{a_k} (a_k > -n \wedge a_k \in \mathbb{N}) \wedge \nexists_{b_k} (b_k > -m \wedge b_k \in \mathbb{N})$ the function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables a_i, b_j can approach nonpositive integers $-n$, $-m$; $m \geq n$ at different speeds. For the above conditions we define:

07.26.02.0002.01

$${}_2F_3\left(a_1, \ldots, a_i, \ldots, a_2; b_1, \ldots, b_j, \ldots, b_3; z\right) = \sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k z^k}{(b_1)_k (b_2)_k (b_3)_k k!} \text{ /; } a_i = -n \wedge b_j = -m \wedge m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge m \geq n$$

Specific values

Values at $z = 0$

07.26.03.0001.01

${}_2F_3(a_1, a_2; b_1, b_2, b_3; 0) = 1$

Specialized values

For fixed a_1, a_2, b_1, b_2, z

07.26.03.0002.01

${}_2F_3(a, b; c, d, b; z) = {}_1F_2(a; c, d; z)$

For fixed a_1, a_2, z

07.26.03.0003.01

$$ {}_2F_3\left(a, b; a+b, \frac{a+b}{2}, \frac{a+b+1}{2}; z\right) = {}_1F_1\left(a; a+b; 2\sqrt{z}\right) {}_1F_1\left(a; a+b; -2\sqrt{z}\right) $$

For fixed a_1, b_2, z

07.26.03.0004.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; 2a, d, 2a-d+1; z\right) = {}_1F_1\left(2a-d+\frac{1}{2}; 4a-2d+1; 2\sqrt{z}\right) {}_1F_1\left(d-\frac{1}{2}; 2d-1; -2\sqrt{z}\right) $$

07.26.03.0005.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; 2a, d, 2a-d+1; z\right) = {}_0F_1\left(; d; \frac{z}{4}\right) {}_0F_1\left(; 2a-d+1; \frac{z}{4}\right) $$

07.26.03.0006.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; 2a, d, 2a-d+1; z\right) = 2^{2a-1}\, \Gamma(d)\, \Gamma(2a-d+1)\, I_{d-1}\left(\sqrt{z}\right) z^{\frac{1}{2}-a} I_{2a-d}\left(\sqrt{z}\right) $$

07.26.03.0007.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{1}{2}, d, d+\frac{1}{2}; z\right) = \frac{1}{2}\left({}_1F_1\left(2a; 2d; -2\sqrt{z}\right) + {}_1F_1\left(2a; 2d; 2\sqrt{z}\right)\right) $$

07.26.03.0008.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{3}{2}, d, d+\frac{1}{2}; z\right) = \frac{2d-1}{4(2a-1)\sqrt{z}}\left({}_1F_1\left(2a-1; 2d-1; 2\sqrt{z}\right) - {}_1F_1\left(2a-1; 2d-1; -2\sqrt{z}\right)\right) $$

For fixed a_1, z

07.26.03.0009.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{1}{2}, 2a, 2a+\frac{1}{2}; z\right) = \cosh\left(\sqrt{z}\right) {}_0F_1\left(; 2a+\frac{1}{2}; \frac{z}{4}\right) $$

07.26.03.0010.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{1}{2}, 2a, 2a+\frac{1}{2}; z\right) = 2^{2a-\frac{1}{2}}\, \Gamma\left(2a+\frac{1}{2}\right) \cosh\left(\sqrt{z}\right) z^{\frac{1}{4}-a} I_{2a-\frac{1}{2}}\left(\sqrt{z}\right) $$

07.26.03.0011.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{3}{2}, 2a, 2a-\frac{1}{2}; z\right) = \frac{\sinh\left(\sqrt{z}\right)}{\sqrt{z}} {}_0F_1\left(; 2a-\frac{1}{2}; \frac{z}{4}\right) $$

07.26.03.0012.01

$$ {}_2F_3\left(a, a+\frac{1}{2}; \frac{3}{2}, 2a, 2a-\frac{1}{2}; z\right) = 2^{2a-\frac{3}{2}}\, \Gamma\left(2a-\frac{1}{2}\right) \sinh\left(\sqrt{z}\right) z^{\frac{1}{4}-a} I_{2a-\frac{3}{2}}\left(\sqrt{z}\right) $$

For fixed a_2, z

07.26.03.0013.01

$$ {}_2F_3\left(-n, b; b-n, \frac{b-n}{2}, \frac{b-n+1}{2}; z\right) = \frac{n!^2\, \Gamma(b-n)^2}{\Gamma(b)^2} L_n^{b-n-1}\left(-2\sqrt{z}\right) L_n^{b-n-1}\left(2\sqrt{z}\right) $$

For fixed z and integer parameters

07.26.03.0014.01

$$ {}_2F_3\left(n, n-\frac{3}{2}; 1, 2n-2, 2n-1; z\right) = (2n-3)!\, 2^{2n-3} z^{\frac{3}{2}-n} \left(I_0(\sqrt{z}) I_{2n-3}(\sqrt{z}) - \frac{2n-3}{2n-1} I_1(\sqrt{z}) I_{2n-2}(\sqrt{z}) - \frac{2\sqrt{z}}{4(n-1)^2-1} \left(I_1(\sqrt{z}) I_{2n-3}(\sqrt{z}) - I_0(\sqrt{z}) I_{2n-2}(\sqrt{z}) \right) \right) /; n-1 \in \mathbb{N}^+ $$

07.26.03.0015.01

$$ {}_2F_3\left(-n, \frac{1}{2}-n; \frac{3}{2}, -2n, -2n-\frac{1}{2}; z\right) = \frac{(2n)!\, 2^{2n-\frac{1}{2}} \sqrt{\pi}}{(4n+1)!} z^{n+\frac{1}{4}} \left(\cosh(\sqrt{z}) I_{2n+\frac{3}{2}}(\sqrt{z}) - \sinh(\sqrt{z}) I_{-2n-\frac{3}{2}}(\sqrt{z}) \right) /; n \in \mathbb{N}^+ $$

07.26.03.0016.01

$$ {}_2F_3\left(-n, -n-\frac{1}{2}; \frac{3}{2}, -2n-\frac{3}{2}, -2n-1; z\right) = \frac{2^{2n+\frac{3}{2}} (2n+2)!}{\sqrt{\pi}\, (4n+4)!} z^{n+\frac{3}{4}} \left(e^{\sqrt{z}} \pi I_{2n+\frac{5}{2}}(\sqrt{z}) + 2 K_{2n+\frac{5}{2}}(\sqrt{z}) \sinh(\sqrt{z}) \right) /; n \in \mathbb{N}^+ $$

07.26.03.0017.01

$$ {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}, -n, \frac{1}{2}-n; z\right) = \frac{2^{-n-\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} z^{\frac{2n+1}{4}} \left((-1)^n e^{-\sqrt{z}} \pi I_{n+\frac{1}{2}}(\sqrt{z}) + 2 K_{n+\frac{1}{2}}(\sqrt{z}) \cosh(\sqrt{z}) \right) /; n \in \mathbb{N}^+ $$

For fixed z

For fixed z and $a_1 = \frac{1}{4}, a_2 = \frac{3}{4}$

07.26.03.0018.01

$$ {}_2F_3\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, 1; z\right) = I_0(\sqrt{z}) \cosh(\sqrt{z}) $$

For fixed z and $a_1 = \frac{1}{2}, a_2 = 1$

07.26.03.0019.01

$$ {}_2F_3\left(\frac{1}{2}, 1; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; z\right) = \frac{\pi}{8\sqrt{z}} \operatorname{erf}(\sqrt[4]{4z}) \operatorname{erfi}(\sqrt[4]{4z}) $$

07.26.03.0020.01

$$ {}_2F_3\left(\frac{1}{2}, 1; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; z\right) = \frac{3\pi (4z)^{-3/4}}{4\sqrt{\pi}} \left(e^{-2\sqrt{z}} \operatorname{erfi}(\sqrt[4]{4z}) - e^{2\sqrt{z}} \operatorname{erf}(\sqrt[4]{4z}) \right) + \frac{3\pi}{8\sqrt{z}} \operatorname{erf}(\sqrt[4]{4z}) \operatorname{erfi}(\sqrt[4]{4z}) $$

For fixed z and $a_1 = \frac{3}{4}, a_2 = \frac{5}{4}$

07.26.03.0021.01

$$ {}_2F_3\left(\frac{3}{4}, \frac{5}{4}; \frac{1}{2}, \frac{3}{2}, 2; z\right) = \frac{2}{\sqrt{z}} I_1(\sqrt{z}) \cosh(\sqrt{z}) $$

07.26.03.0022.01

$$ {}_2F_3\left(\frac{3}{4}, \frac{5}{4}; 1, \frac{3}{2}, \frac{3}{2}; z\right) = \frac{1}{\sqrt{z}} I_0(\sqrt{z}) \sinh(\sqrt{z}) $$

For fixed z and $a_1 = 1, a_2 = 1$

07.26.03.0023.01

$${}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; z\right) = -\frac{1}{z}\left(-\text{Chi}\left(2\sqrt{z}\right) + \log(2) + \frac{\log(z)}{2} + \gamma\right)$$

For fixed z and $a_1 = \frac{5}{4}$, $a_2 = \frac{7}{4}$

07.26.03.0024.01

$${}_2F_3\left(\frac{5}{4}, \frac{7}{4}; \frac{3}{2}, 2, \frac{5}{2}; z\right) = \frac{2}{z} I_1\left(\sqrt{z}\right)\sinh\left(\sqrt{z}\right)$$

General characteristics

Domain and analyticity

${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ is an analytical function of a_1, a_2, b_1, b_2, b_3 and z which is defined in $\mathbb{C}^6$. For fixed a_1, a_2, b_1, b_2, b_3, it is an entire function of z. For fixed a_2, b_1, b_2, b_3, z, it is an entire function of a_1. For fixed a_1, b_1, b_2, b_3, z, it is an entire function of a_2. For negative integer $a_1$or a_2, ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ degenerates to a polynomial in z of order $-a_1$or $-a_2$.

07.26.04.0001.01

$$(\{a_1 * a_2\} * \{b_1 * b_2 * b_3\} * z) \longrightarrow {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) :: (\{\mathbb{C} \otimes \mathbb{C}\} \otimes \{\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.26.04.0002.01

$${}_2F_3\left(\overline{a_1}, \overline{a_2}; \overline{b_1}, \overline{b_2}, \overline{b_3}; \overline{z}\right) = \overline{{}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}$$

Permutation symmetry

07.26.04.0003.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = {}_2F_3(a_2, a_1; b_1, b_2, b_3; z)$$

07.26.04.0004.01

$${}_2F_3\left(a_1, a_2; b_1, b_2, \ldots, b_k, \ldots, b_j, \ldots, b_3; z\right) = {}_2F_3\left(a_1, a_2; b_1, b_2, \ldots, b_j, \ldots, b_k, \ldots, b_3; z\right) /; b_k \neq b_j \bigwedge k \neq j$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed a_1, a_2, b_1, b_2, b_3in nonpolynomial cases (when $\neg(-a_1 \in \mathbb{N} \bigvee -a_2 \in \mathbb{N})$), the function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

07.26.04.0005.01

$$\mathcal{S}ing_z({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) = \{\{\tilde{\infty}, \infty\}\} /; \neg(-a_1 \in \mathbb{N} \bigvee -a_2 \in \mathbb{N})$$

For negative integer a_1 or $a_2$and fixed b_1, b_2, b_3, the function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ is a polynomial and has pole of order $-a_1$or $-a_2$at $z = \tilde{\infty}$.

07.26.04.0006.01

$$\mathcal{S}ing_z({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\{\tilde{\infty}, -\alpha\}\} /; \\ (-a_1 \in \mathbb{N}^+ \wedge \alpha == a_1) \vee (-a_2 \in \mathbb{N}^+ \wedge \alpha == a_2) \vee (-a_1 \in \mathbb{N}^+ \wedge -a_2 \in \mathbb{N}^+ \wedge \alpha == \min(-a_1, -a_2))$$

With respect to a_j

For fixed a_1, b_1, b_2, b_3, z, the function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ has only one singular point at $a_2 = \tilde{\infty}$. It is an essential singular point.

For fixed a_2, b_1, b_2, b_3, z, the function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ has only one singular point at $a_1 = \tilde{\infty}$. It is an essential singular point.

07.26.04.0007.01

$$\mathcal{S}ing_{a_j}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\{\tilde{\infty}, \infty\}\} /; j \in \{1, 2\}$$

With respect to b_j

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ as a function of b_3 has an infinite set of singular points:

a) $b_3 == -k /; k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; b_1, b_2, -k; z)$;

b) $b_3 == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ as a function of b_2 has an infinite set of singular points:

a) $b_2 == -k /; k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; b_1, b_3, -k; z)$;

b) $b_2 == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ as a function of b_1 has an infinite set of singular points:

a) $b_1 == -k /; k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; b_2, b_3, -k; z)$;

b) $b_1 == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

07.26.04.0008.01

$$\mathcal{S}ing_{b_j}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\{\{-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\} /; j \in \{1, 2, 3\}$$

07.26.04.0009.01

$$\operatorname{res}_{b_3}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z))(-k) == \frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; b_1, b_2, -k; z) /; k \in \mathbb{N}$$

07.26.04.0010.01

$$\operatorname{res}_{b_2}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z))(-k) == \frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; b_1, -k, b_3; z) /; k \in \mathbb{N}$$

07.26.04.0011.01

$$\operatorname{res}_{b_1}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z))(-k) == \frac{(-1)^k}{k!}\, {}_2\tilde{F}_3(a_1, a_2; -k, b_2, b_3; z) /; k \in \mathbb{N}$$

Branch points

With respect to z

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch points with respect to z.

07.26.04.0012.01

$$\mathcal{BP}_z({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\}$$

With respect to a_k

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch points with respect to a_k.

07.26.04.0013.01

$$\mathcal{BP}_{a_k}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\} /; k \in \{1, 2\}$$

With respect to b_k

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch points with respect to b_k.

07.26.04.0014.01

$$\mathcal{BP}_{b_k}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\} /; k \in \{1, 2, 3\}$$

Branch cuts

With respect to z

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch cuts with respect to z.

07.26.04.0015.01

$$\mathcal{BC}_z({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\}$$

With respect to a_k

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch cuts with respect to a_k.

07.26.04.0016.01

$$\mathcal{BC}_{a_k}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\} /; k \in \{1, 2\}$$

With respect to b_k

The function ${}_2F_3(a_1, a_2; b_1, b_2, b_3; z)$ does not have branch cuts with respect to b_k.

07.26.04.0017.01

$$\mathcal{BC}_{b_k}({}_2F_3(a_1, a_2; b_1, b_2, b_3; z)) == \{\} /; k \in \{1, 2, 3\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.26.06.0014.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto {}_2F_3(a_1, a_2; b_1, b_2, b_3; z_0) + \frac{a_1 a_2}{b_1 b_2 b_3} {}_2F_3(a_1+1, a_2+1; b_1+1, b_2+1, b_3+1; z_0)\, (z-z_0) + \frac{a_1 (a_1+1) a_2 (a_2+1)}{2 b_1 (b_1+1) b_2 (b_2+1) b_3 (b_3+1)} {}_2F_3(a_1+2, a_2+2; b_1+2, b_2+2, b_3+2; z_0) (z-z_0)^2 + \ldots /; (z \to z_0) $$

07.26.06.0015.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto {}_2F_3(a_1, a_2; b_1, b_2, b_3; z_0) + \frac{a_1 a_2}{b_1 b_2 b_3} {}_2F_3(a_1+1, a_2+1; b_1+1, b_2+1, b_3+1; z_0)\, (z-z_0) + \frac{a_1 (a_1+1) a_2 (a_2+1)}{2 b_1 (b_1+1) b_2 (b_2+1) b_3 (b_3+1)} {}_2F_3(a_1+2, a_2+2; b_1+2, b_2+2, b_3+2; z_0) (z-z_0)^2 + O\left((z-z_0)^3\right) $$

07.26.06.0016.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{k! \left((b_1)_k (b_2)_k (b_3)_k\right)} {}_2F_3(k+a_1, k+a_2; k+b_1, k+b_2, k+b_3; z_0) (z-z_0)^k $$

07.26.06.0017.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = F_{3\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} a_1, a_2;;; \\ b_1, b_2, b_3;;; \end{matrix} z_0, z-z_0 \right) $$

07.26.06.0018.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto {}_2F_3(a_1, a_2; b_1, b_2, b_3; z_0) (1 + O(z-z_0)) $$

Expansions at $z = 0$

For the function itself

General case

07.26.06.0001.02

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto 1 + \frac{a_1 a_2}{b_1 b_2 b_3} z + \frac{a_1 (1+a_1) a_2 (1+a_2)}{2 b_1 (1+b_1) b_2 (1+b_2) b_3 (1+b_3)} z^2 + \ldots /; (z \to 0) $$

07.26.06.0019.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto 1 + \frac{a_1 a_2}{b_1 b_2 b_3} z + \frac{a_1 (1+a_1) a_2 (1+a_2)}{2 b_1 (1+b_1) b_2 (1+b_2) b_3 (1+b_3)} z^2 + O\left(z^3\right) $$

07.26.06.0002.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k z^k}{(b_1)_k (b_2)_k (b_3)_k k!} $$

07.26.06.0003.02

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto 1 + O(z) $$

07.26.06.0020.01

$$_2F_3(a_1, a_2; b_1, b_2, b_3; z) = F_\infty(z, a_1, a_2, b_1, b_2, b_3) /; \left(\left(F_n(z, a_1, a_2, b_1, b_2, b_3) = \sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k z^k}{(b_1)_k (b_2)_k (b_3)_k k!} = {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) - \left(\Gamma(n+a_1+1)\Gamma(n+a_2+1)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3) z^{n+1}\right) / (\Gamma(n+2)\Gamma(a_1)\Gamma(a_2)\Gamma(n+b_1+1)\Gamma(n+b_2+1)\Gamma(n+b_3+1)) \, {}_3F_4(1, n+a_1+1, n+a_2+1; n+2, n+b_1+1, n+b_2+1, n+b_3+1; z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at $z = \infty$ for polynomial cases

07.26.06.0004.01

$$_2F_3(-n, a_2; b_1, b_2, b_3; z) = \frac{(a_2)_n (-z)^n}{\prod_{k=1}^{3} (b_k)_n} \, {}_4F_1\left(-n, -n-b_1+1, -n-b_2+1, -n-b_3+1; -n-a_2+1; \frac{1}{z}\right) /; n \in \mathbb{N}^+$$

Asymptotic series expansions

Expansions for $|\mathrm{Arg}(z)| < \pi$

07.26.06.0005.01

$$_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{2\sqrt{\pi}\,\Gamma(a_1)\Gamma(a_2)} z^\chi e^{2\sqrt{z}} \left(1 + O\left(\frac{1}{\sqrt{z}}\right)\right) /; \chi = \frac{1}{2}\left(a_1 + a_2 - b_1 - b_2 - b_3 + \frac{1}{2}\right) \bigwedge |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

The general formulas

07.26.06.0006.01

$$_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto \prod_{j=1}^{3} \Gamma(b_j) \, \mathcal{A}_{\tilde{F}}\left(\begin{matrix} a_1, a_2; \\ b_1, b_2, b_3; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) /; (|z| \to \infty)$$

07.26.06.0007.01

$$_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto \prod_{j=1}^{3} \Gamma(b_j) \left(\mathcal{A}_{\tilde{F}}^{(\mathrm{power})}\left(\begin{matrix} a_1, a_2; \\ b_1, b_2, b_3; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\mathrm{trig})}\left(\begin{matrix} a_1, a_2; \\ b_1, b_2, b_3; \end{matrix} \{z, \tilde{\infty}, \infty\}\right)\right) /; (|z| \to \infty)$$

Expansions for any z in exponential form

Case of simple poles

07.26.06.0008.01

$$
{}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(a_2-a_1)}{\Gamma(b_1-a_1)\,\Gamma(b_2-a_1)\,\Gamma(b_3-a_1)\,\Gamma(a_2)} (-z)^{-a_1} \left(1 + \frac{a_1\,(a_1-b_1+1)\,(a_1-b_2+1)\,(a_1-b_3+1)}{(a_1-a_2+1)\,z} + (a_1\,(a_1+1)\,(a_1-b_1+1)\right.
$$

$$
\left.(a_1-b_1+2)\,(a_1-b_2+1)\,(a_1-b_2+2)\,(a_1-b_3+1)\,(a_1-b_3+2))\Big/\left(2\,(a_1-a_2+1)\,(a_1-a_2+2)\,z^2\right) + \ldots\right) +
$$

$$
\frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(b_2-a_2)\,\Gamma(b_3-a_2)\,\Gamma(a_1)} (-z)^{-a_2} \left(1 + \frac{a_2\,(a_2-b_1+1)\,(a_2-b_2+1)\,(a_2-b_3+1)}{(a_2-a_1+1)\,z} + \right.
$$

$$
(a_2\,(a_2+1)\,(a_2-b_1+1)\,(a_2-b_1+2)\,(a_2-b_2+1)\,(a_2-b_2+2)\,(a_2-b_3+1)\,(a_2-b_3+2))\Big/
$$

$$
\left.\left(2\,(a_2-a_1+1)\,(a_2-a_1+2)\,z^2\right) + \ldots\right) + \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)}{2\sqrt{\pi}\;\Gamma(a_1)\,\Gamma(a_2)} (-z)^{\frac{1}{2}\left(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\right)}
$$

$$
\left(e^{-i\left(\frac{1}{2}\pi\left(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\right)+2\sqrt{-z}\right)}\left(1 + \frac{d_1}{\sqrt{-z}} + \frac{d_2}{z} + \ldots\right) + e^{i\left(\frac{1}{2}\pi\left(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\right)+2\sqrt{-z}\right)}\left(1 - \frac{d_1}{\sqrt{-z}} + \frac{d_2}{z} + \ldots\right)\right) /;
$$

$$
(|z| \to \infty) \bigwedge d_1 == \frac{1}{16}\, i \left(12\,a_1^2 + 8\,(a_2-b_1-b_2-b_3-1)\,a_1 + 12\,a_2^2 - 4\,b_1^2 - 4\,b_2^2 - 4\,b_3^2 + 8\,b_1 + 8\,b_1\,b_2 + 8\,b_2 + \right.
$$

$$
\left. 8\,b_1\,b_3 + 8\,b_2\,b_3 + 8\,b_3 - 8\,a_2\,(b_1+b_2+b_3+1) - 3\right) \bigwedge d_2 == \frac{1}{512}\left(144\,a_1^4 + 64\,(3\,a_2 - 3\,b_1 - 3\,b_2 - 3\,b_3 - 7)\,a_1^3 + \right.
$$

$$
8\left(44\,a_2^2 - 8\,(5\,b_1 + 5\,b_2 + 5\,b_3 + 9)\,a_2 - 4\,b_1^2 - 4\,b_2^2 - 4\,b_3^2 + 72\,b_2 + 40\,b_2\,b_3 + 72\,b_3 + 8\,b_1\,(5\,b_2 + 5\,b_3 + 9) + 43\right) a_1^2 +
$$

$$
16\left(12\,a_2^3 - 4\,(5\,b_1 + 5\,b_2 + 5\,b_3 + 9)\,a_2^2 + \left(4\,b_1^2 + 8\,(3\,b_2 + 3\,b_3 + 5)\,b_1 + 4\,b_2^2 + 4\,b_3^2 + 40\,b_3 + 8\,b_2\,(3\,b_3 + 5) + 25\right) a_2 + \right.
$$

$$
4\,b_1^3 + 4\,b_2^3 + 4\,b_3^3 - 4\,b_2^2 - 4\,b_2\,b_3^2 - 4\,b_3^2 - 25\,b_2 - 4\,b_2^2\,b_3 - 40\,b_2\,b_3 - 25\,b_3 - 4\,b_1^2\,(b_2 + b_3 + 1) -
$$

$$
\left. b_1\left(4\,b_2^2 + 8\,(3\,b_3 + 5)\,b_2 + 4\,b_3^2 + 40\,b_3 + 25\right) - 1\right) a_1 + 144\,a_2^4 + 16\,b_1^4 + 16\,b_2^4 + 16\,b_3^4 - 64\,b_1^3 -
$$

$$
64\,b_1\,b_2^3 - 64\,b_2^3 - 64\,b_1\,b_3^3 - 64\,b_2\,b_3^3 - 64\,b_3^3 + 56\,b_1^2 + 96\,b_1^2\,b_2^2 + 64\,b_1\,b_2^2 + 56\,b_2^2 + 96\,b_1^2\,b_3^2 + 96\,b_2^2\,b_3^2 +
$$

$$
64\,b_1\,b_3^2 + 64\,b_1\,b_2\,b_3^2 + 64\,b_2\,b_3^2 + 56\,b_3^2 + 16\,b_1 - 64\,b_1^3\,b_2 + 64\,b_1^2\,b_2 + 400\,b_1\,b_2 + 16\,b_2 - 64\,b_1^3\,b_3 -
$$

$$
64\,b_2^3\,b_3 + 64\,b_1^2\,b_3 + 64\,b_1\,b_2^2\,b_3 + 64\,b_2^2\,b_3 + 400\,b_1\,b_3 + 64\,b_1^2\,b_2\,b_3 + 640\,b_1\,b_2\,b_3 + 400\,b_2\,b_3 + 16\,b_3 -
$$

$$
64\,a_2^3\,(3\,b_1 + 3\,b_2 + 3\,b_3 + 7) - 8\,a_2^2\left(4\,b_1^2 - 8\,(5\,b_2 + 5\,b_3 + 9)\,b_1 + 4\,b_2^2 + 4\,b_3^2 - 72\,b_3 - 8\,b_2\,(5\,b_3 + 9) - 43\right) +
$$

$$
16\,a_2\left(4\,b_1^3 - 4\,(b_2 + b_3 + 1)\,b_1^2 - \left(4\,b_2^2 + 8\,(3\,b_3 + 5)\,b_2 + 4\,b_3^2 + 40\,b_3 + 25\right) b_1 + \right.
$$

$$
\left.\left. 4\,b_2^3 + 4\,b_3^3 - 4\,b_3^2 - 25\,b_3 - 4\,b_2^2\,(b_3 + 1) - b_2\left(4\,b_3^2 + 40\,b_3 + 25\right) - 1\right) - 15\right)
$$

07.26.06.0009.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)}{2\sqrt{\pi}\;\Gamma(a_1)\,\Gamma(a_2)} (-z)^{\chi} \left(e^{i\left(\pi\chi+2\sqrt{-z}\right)} \sum_{k=0}^{\infty} (-i)^k\, 2^{-k}\, c_k\, (-z)^{-\frac{k}{2}} + e^{-i\left(\pi\chi+2\sqrt{-z}\right)} \sum_{k=0}^{\infty} i^k\, 2^{-k}\, c_k\, (-z)^{-\frac{k}{2}} \right) + $$
$$ \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(a_2-a_1)}{\Gamma(a_2)\,\Gamma(b_1-a_1)\,\Gamma(b_2-a_1)\,\Gamma(b_3-a_1)} (-z)^{-a_1}\, {}_4F_1\!\left(a_1, a_1-b_1+1, a_1-b_2+1, a_1-b_3+1; a_1-a_2+1; \frac{1}{z}\right) + $$
$$ \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(a_1-a_2)}{\Gamma(a_1)\,\Gamma(b_1-a_2)\,\Gamma(b_2-a_2)\,\Gamma(b_3-a_2)} (-z)^{-a_2}\, {}_4F_1\!\left(a_2, a_2-b_1+1, a_2-b_2+1, a_2-b_3+1; -a_1+a_2+1; \frac{1}{z}\right) /; $$
$$ (|z| \to \infty) \bigwedge \chi == \frac{1}{2}\left(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\right) \bigwedge c_0 == 1 \bigwedge c_1 == 2\left(\mathfrak{B}-\mathfrak{A}+\frac{1}{4}(3A_2+B_3-2)(A_2-B_3)-\frac{3}{16}\right) \bigwedge $$
$$ c_2 == \frac{c_1^2}{2} + \frac{1}{16}\left(-16(2A_2-3)(\mathfrak{B}-\mathfrak{A}) + 32\,\mathfrak{R} + 4\left(-8A_2^2+11A_2+8\mathfrak{A}+B_3-2\right)(A_2-B_3)-3\right) \bigwedge $$
$$ c_k == \frac{1}{2k}\Big((k-2\chi-3)(k-2\chi-2b_1-1)(k-2\chi-2b_2-1)(k-2\chi-2b_3-1)\,c_{k-3} - $$
$$ \left(4(k-1)^3 - 6(4\chi+B_3)(k-1)^2 + 2\left(24\chi^2+12B_3\chi+4\mathfrak{B}+B_3-1\right)(k-1) - 32\chi^3 - 24B_3\chi^2 - 4\mathfrak{B} - 8\mathfrak{R} - \right. $$
$$ \left. 4(4\mathfrak{B}+B_3-1)\chi + 2B_3 - 1\right) c_{k-2} + \left(5(k-1)^2 + 2(-10\chi+A_2-3B_3+3)(k-1) + 2c_1\right) c_{k-1}\Big) \bigwedge $$
$$ A_2 == a_1+a_2 \bigwedge B_3 == b_1+b_2+b_3 \bigwedge \mathfrak{A} == a_1 a_2 \bigwedge \mathfrak{B} == b_1 b_2 + b_3 b_2 + b_1 b_3 \bigwedge \mathfrak{R} == b_1 b_2 b_3 \bigwedge a_1 - a_2 \notin \mathbb{Z} $$

07.26.06.0010.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; -z) \propto $$
$$ \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)}{\Gamma(a_1)\,\Gamma(a_2)} \left(\frac{\Gamma(a_1)\,\Gamma(a_2-a_1)}{\Gamma(b_1-a_1)\,\Gamma(b_2-a_1)\,\Gamma(b_3-a_1)} z^{-a_1}\left(1+O\left(\frac{1}{z}\right)\right) + \frac{\Gamma(a_2)\,\Gamma(a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(b_2-a_2)\,\Gamma(b_3-a_2)} z^{-a_2}\left(1+O\left(\frac{1}{z}\right)\right) + \right. $$
$$ \frac{(-z)^{\chi}}{\sqrt{\pi}} \left(\cos\!\left(\pi\chi+2\sqrt{-z}\right)\left(1+O\left(\frac{1}{z}\right)\right) + \frac{1}{16\sqrt{-z}} (2(3a_1+3a_2+b_1+b_2+b_3-2)(4\chi-1) + \right. $$
$$ \left.\left. 16(b_1 b_2 + b_1 b_3 + b_2 b_3 - a_1 a_2) - 3)\sin\!\left(\pi\chi+2\sqrt{-z}\right)\left(1+O\left(\frac{1}{z}\right)\right)\right)\right) /; $$
$$ \chi == \frac{1}{2}\left(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\right) \bigwedge a_1 \neq a_2 \bigwedge (|z| \to \infty) $$

Case of double poles

07.26.06.0011.01

$$ {}_2F_3(a_1, n+a_1; b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(n+a_1)\Gamma(b_1-a_1)\Gamma(b_2-a_1)\Gamma(b_3-a_1)} (-z)^{-n-a_1} \sum_{k=0}^{\infty} \frac{(a_1)_{k+n}(a_1-b_1+1)_{k+n}(a_1-b_2+1)_{k+n}(a_1-b_3+1)_{k+n}}{k!\,(k+n)!} (\psi(k+1)+\psi(k+n+1)-\psi(k+n+a_1)-\psi(b_1-a_1-n-k)-\psi(b_2-a_1-n-k)-\psi(b_3-a_1-n-k))\, z^{-k} + \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{2\sqrt{\pi}\,\Gamma(a_1)\Gamma(n+a_1)} (-z)^{\chi} \left(e^{i\left(\pi\chi+2\sqrt{-z}\right)} \sum_{k=0}^{\infty} (-i)^k 2^{-k} c_k (-z)^{-\frac{k}{2}} + e^{-i\left(\pi\chi+2\sqrt{-z}\right)} \sum_{k=0}^{\infty} i^k 2^{-k} c_k (-z)^{-\frac{k}{2}} \right) + \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{n!\,\Gamma(a_1)\Gamma(b_1-a_1-n)\Gamma(b_2-a_1-n)\Gamma(b_3-a_1-n)} z^{-n} (-z)^{-a_1} \log(-z)\; {}_4F_1\left(n+a_1, n+a_1-b_1+1, n+a_1-b_2+1, n+a_1-b_3+1; n+1; \frac{1}{z}\right) + \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(n+a_1)} (-z)^{-a_1} \sum_{k=0}^{n-1} \frac{\left((a_1)_k \Gamma(n-k)\right) z^{-k}}{\Gamma(-k-a_1+b_1)\Gamma(-k-a_1+b_2)\Gamma(-k-a_1+b_3)\, k!} \;/; $$

$$ (|z| \to \infty) \bigwedge n \in \mathbb{N} \bigwedge \chi = \frac{1}{2}\left(n + 2a_1 - b_1 - b_2 - b_3 + \frac{1}{2}\right) \bigwedge c_0 = 1 \bigwedge $$

$$ c_1 = 2\left(\mathfrak{B} - \mathfrak{A} + \frac{1}{4}(3A_2 + B_3 - 2)(A_2 - B_3) - \frac{3}{16}\right) \bigwedge $$

$$ c_2 = \frac{c_1^2}{2} + \frac{1}{16}\left(-16(2A_2-3)(\mathfrak{B}-\mathfrak{A}) + 32\,\mathfrak{R} + 4\left(-8A_2^2 + 11A_2 + 8\,\mathfrak{A} + B_3 - 2\right)(A_2 - B_3) - 3\right) \bigwedge $$

$$ c_k = \frac{1}{2k}\Big((k-2\chi-3)(k-2\chi-2b_1-1)(k-2\chi-2b_2-1)(k-2\chi-2b_3-1)\,c_{k-3} - \left(4(k-1)^3 - 6(4\chi+B_3)(k-1)^2 + 2\left(24\chi^2 + 12B_3\chi + 4\,\mathfrak{B} + B_3 - 1\right)(k-1) - 32\chi^3 - 24B_3\chi^2 - 4\,\mathfrak{B} - 8\,\mathfrak{R} - 4(4\,\mathfrak{B} + B_3 - 1)\chi + 2B_3 - 1\right)c_{k-2} + \left(5(k-1)^2 + 2(-10\chi + A_2 - 3B_3 + 3)(k-1) + 2c_1\right)c_{k-1}\Big) \bigwedge $$

$$ A_2 = n + 2a_1 \bigwedge B_3 = b_1 + b_2 + b_3 \bigwedge \mathfrak{A} = a_1(n+a_1) \bigwedge \mathfrak{B} = b_1 b_2 + b_3 b_2 + b_1 b_3 \bigwedge \mathfrak{R} = b_1 b_2 b_3 $$

Expansions for any *z* in trigonometric form

Case of double poles

07.26.06.0012.01

$$ {}_2F_3(a_1, a_1; b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\sqrt{\pi}\,\Gamma(a_1)^2} (-z)^{\chi} \left(\cos\left(\pi\chi + 2\sqrt{-z}\right)\left(1 + O\left(\frac{1}{z}\right)\right) + \frac{1}{16\sqrt{-z}}\left(2(6a_1 + b_1 + b_2 + b_3 - 2)(4\chi - 1) + 16\left(-a_1^2 + b_1 b_2 + b_1 b_3 + b_2 b_3\right) - 3\right) \sin\left(\pi\chi + 2\sqrt{-z}\right)\left(1 + O\left(\frac{1}{z}\right)\right)\right) + \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(b_2-a_1)\Gamma(b_3-a_1)} (-z)^{-a_1} \left(\log(-z)\left(1 + O\left(\frac{1}{z}\right)\right) - (\psi(b_1-a_1) + \psi(b_2-a_1) + \psi(b_3-a_1) + \psi(a_1) + 2\gamma)\left(1 + O\left(\frac{1}{z}\right)\right)\right) \;/; $$

$$ (|z| \to \infty) \bigwedge \chi = \frac{1}{2}\left(2a_1 - b_1 - b_2 - b_3 + \frac{1}{2}\right) $$

Residue representations

07.26.06.0013.01

$$ {}_2F_3(a_1, a_1; b_1, b_2, b_3; z) = \frac{\prod_{k=1}^{3}\Gamma(b_k)}{\prod_{k=1}^{2}\Gamma(a_k)} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\left(\frac{\prod_{k=1}^{2}\Gamma(a_k-s)}{\prod_{k=1}^{3}\Gamma(b_k-s)}(-z)^{-s}\right)\Gamma(s)\right)(-j) $$

Limit representations

07.26.09.0001.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \lim_{b\to\infty}\lim_{a\to\infty} {}_4F_3\left(a_1, a_2, a, b; b_1, b_2, b_3; \frac{z}{a\,b}\right) $$

07.26.09.0002.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \lim_{a\to\infty} {}_3F_3\left(a_1, a_2, a; b_1, b_2, b_3; \frac{z}{a}\right) $$

07.26.09.0003.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \lim_{p\to\infty} {}_2F_4(a_1, a_2; b_1, b_2, b_3, p; p\,z) $$

Continued fraction representations

07.26.10.0001.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = 1 + (a_1\,a_2\,z/(b_1\,b_2\,b_3)) \Bigg/ \left(1 + -\frac{z\,(1+a_1)\,(1+a_2)}{2\,(1+b_1)\,(1+b_2)\,(1+b_3)} \Bigg/ \left(1 + \frac{z\,(1+a_1)\,(1+a_2)}{2\,(1+b_1)\,(1+b_2)\,(1+b_3)} + \frac{-\frac{z\,(2+a_1)\,(2+a_2)}{3\,(2+b_1)\,(2+b_2)\,(2+b_3)}}{1 + \frac{z\,(2+a_1)\,(2+a_2)}{3\,(2+b_1)\,(2+b_2)\,(2+b_3)} + \ldots}\right)\right) $$

07.26.10.0002.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = 1 + \frac{a_1\,a_2\,z}{b_1\,b_2\,b_3\left(1 + \mathrm{K}_k\left(-\frac{(k+a_1)\,(k+a_2)\,z}{(k+1)\,(k+b_1)\,(k+b_2)\,(k+b_3)}, \frac{(k+a_1)\,(k+a_2)\,z}{(k+1)\,(k+b_1)\,(k+b_2)\,(k+b_3)} + 1\right)_1^{\infty}\right)} $$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.26.13.0002.01

$$ z^3\,w^{(4)}(z) + (b_1+b_2+b_3+3)\,z^2\,w^{(3)}(z) + (b_1 + b_1\,b_2 + b_2 + b_1\,b_3 + b_2\,b_3 + b_3 + 1 - z)\,z\,w''(z) + (b_1\,b_2\,b_3 - (a_1+a_2+1)\,z)\,w'(z) - a_1\,a_2\,w(z) = 0 \,/;\, w(z) = c_1\,{}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z) + c_2\left(G_{2,4}^{2,2}\left(z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_1, 1-b_2, 1-b_3\end{matrix}\right) + G_{2,4}^{2,2}\left(z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_2, 1-b_1, 1-b_3\end{matrix}\right) + G_{2,4}^{2,2}\left(z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_3, 1-b_1, 1-b_2\end{matrix}\right)\right) + c_3\left(G_{2,4}^{3,2}\left(-z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_1, 1-b_2, 1-b_3\end{matrix}\right) + G_{2,4}^{3,2}\left(-z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_1, 1-b_3, 1-b_2\end{matrix}\right) + G_{2,4}^{3,2}\left(-z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_2, 1-b_3, 1-b_1\end{matrix}\right)\right) + c_4\,G_{2,4}^{4,2}\left(z\,\middle|\,\begin{matrix}1-a_1, 1-a_2\\ 0, 1-b_1, 1-b_2, 1-b_3\end{matrix}\right) $$

07.26.13.0003.01

$$W_z\left({}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z) + G_{2,4}^{2,2}\left(z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_1, 1-b_2, 1-b_3 \end{array}\right.\right) + G_{2,4}^{2,2}\left(z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_2, 1-b_1, 1-b_3 \end{array}\right.\right) + G_{2,4}^{2,2}\left(z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_3, 1-b_1, 1-b_2 \end{array}\right.\right), G_{2,4}^{3,2}\left(-z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_1, 1-b_2, 1-b_3 \end{array}\right.\right) + G_{2,4}^{3,2}\left(-z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_1, 1-b_3, 1-b_2 \end{array}\right.\right) + G_{2,4}^{3,2}\left(-z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_2, 1-b_3, 1-b_1 \end{array}\right.\right), G_{2,4}^{4,2}\left(z \left| \begin{array}{c} 1-a_1, 1-a_2 \\ 0, 1-b_1, 1-b_2, 1-b_3 \end{array}\right.\right)\right) =$$
$$(-z)^{-b_1-b_2-b_3} z^{-b_1-b_2-b_3-3} \left(-(-z)^{b_2+b_3} (\csc(\pi(b_1-b_3))\sin(\pi(b_1-b_2)) + \csc(\pi(b_1-b_2))\sin(\pi(b_1-b_3))) z^{b_1} + (-z)^{b_1+b_3}\left(\csc^2(\pi(b_1-b_2)) + \csc^2(\pi(b_2-b_3))\right)\sin(\pi(b_1-b_2))\sin(\pi(b_2-b_3)) z^{b_2} - (-z)^{b_1+b_2}\left(\csc^2(\pi(b_1-b_3)) + \csc^2(\pi(b_2-b_3))\right)\sin(\pi(b_1-b_3))\sin(\pi(b_2-b_3)) z^{b_3} - 2\left((-z)^{b_2+b_3} z^{b_1} + (-z)^{b_1+b_3} z^{b_2} + (-z)^{b_1+b_2} z^{b_3}\right)\right)$$
$$\Gamma(a_1-b_1+1)\,\Gamma(a_2-b_1+1)\,\Gamma(a_1-b_2+1)\,\Gamma(a_2-b_2+1)\,\Gamma(a_1-b_3+1)\,\Gamma(a_2-b_3+1)$$

07.26.13.0004.01

$$z^3 w^{(4)}(z) + (b_1+b_2+b_3+3) z^2 w^{(3)}(z) + (b_1 + b_1 b_2 + b_2 + b_1 b_3 + b_2 b_3 + b_3 + 1 - z) z\, w''(z) + (b_1 b_2 b_3 - (a_1+a_2+1) z) w'(z) - a_1 a_2 w(z) = 0 /;$$
$$w(z) = c_1\, {}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z) + c_2 z^{1-b_1} {}_2\tilde{F}_3(a_1-b_1+1, a_2-b_1+1; 2-b_1, 1-b_1+b_2, 1-b_1+b_3; z) + c_3 z^{1-b_2} {}_2\tilde{F}_3(a_1-b_2+1, a_2-b_2+1; 2-b_2, b_1-b_2+1, 1-b_2+b_3; z) + c_4 z^{1-b_3} {}_2\tilde{F}_3(a_1-b_3+1, a_2-b_3+1; 2-b_3, b_1-b_3+1, b_2-b_3+1; z) \bigwedge$$
$$b_1 \notin \mathbb{Z} \bigwedge b_2 \notin \mathbb{Z} \wedge b_3 \notin \mathbb{Z} \bigwedge b_1 - b_2 \notin \mathbb{Z} \bigwedge b_1 - b_3 \notin \mathbb{Z} \bigwedge b_2 - b_3 \notin \mathbb{Z}$$

07.26.13.0005.01

$$W_z\left({}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z), z^{1-b_1} {}_2\tilde{F}_3(a_1-b_1+1, a_2-b_1+1; 2-b_1, -b_1+b_2+1, -b_1+b_3+1; z), z^{1-b_2} {}_2\tilde{F}_3(a_1-b_2+1, a_2-b_2+1; 2-b_2, b_1-b_2+1, -b_2+b_3+1; z), z^{1-b_3} {}_2\tilde{F}_3(a_1-b_3+1, a_2-b_3+1; 2-b_3, b_1-b_3+1, b_2-b_3+1; z)\right) =$$
$$\frac{z^{-b_1-b_2-b_3-3}\sin(\pi b_1)\sin(\pi(b_1-b_2))\sin(\pi b_2)\sin(\pi(b_1-b_3))\sin(\pi(b_2-b_3))\sin(\pi b_3)}{\pi^6}$$

07.26.13.0001.01

$$z^3 w^{(4)}(z) + (b_1+b_2+b_3+3) z^2 w^{(3)}(z) + (b_1 + b_1 b_2 + b_2 + b_1 b_3 + b_2 b_3 + b_3 + 1 - z) z\, w''(z) + (b_1 b_2 b_3 - (a_1+a_2+1) z) w'(z) - a_1 a_2 w(z) = 0 /;$$
$$w(z) = c_1\, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) + c_2 z^{1-b_1} {}_2F_3(a_1-b_1+1, a_2-b_1+1; 2-b_1, -b_1+b_2+1, -b_1+b_3+1; z) + c_3 z^{1-b_2} {}_2F_3(a_1-b_2+1, a_2-b_2+1; 2-b_2, b_1-b_2+1, -b_2+b_3+1; z) + c_4 z^{1-b_3} {}_2F_3(a_1-b_3+1, a_2-b_3+1; 2-b_3, b_1-b_3+1, b_2-b_3+1; z) \bigwedge$$
$$b_1 \notin \mathbb{Z} \bigwedge b_2 \notin \mathbb{Z} \wedge b_3 \notin \mathbb{Z} \bigwedge b_1 - b_2 \notin \mathbb{Z} \bigwedge b_1 - b_3 \notin \mathbb{Z} \bigwedge b_2 - b_3 \notin \mathbb{Z}$$

07.26.13.0006.01

$$W_z\left({}_2F_3(a_1, a_2; b_1, b_2, b_3; z), z^{1-b_1} {}_2F_3(a_1-b_1+1, a_2-b_1+1; 2-b_1, -b_1+b_2+1, -b_1+b_3+1; z), z^{1-b_2} {}_2F_3(a_1-b_2+1, a_2-b_2+1; 2-b_2, b_1-b_2+1, -b_2+b_3+1; z), z^{1-b_3} {}_2F_3(a_1-b_3+1, a_2-b_3+1; 2-b_3, b_1-b_3+1, b_2-b_3+1; z)\right) =$$
$$-(b_1-1)(b_2-1)(b_3-1)(b_1-b_2)(b_1-b_3)(b_2-b_3) z^{-b_1-b_2-b_3-3}$$

Transformations

Products, sums, and powers of the direct function

Products of the direct function

07.26.16.0001.01

$$ {}_2F_3(a_1,\, a_2;\, b_1,\, b_2,\, b_3;\, c\, z)\, {}_2F_3(\alpha_1,\, \alpha_2;\, \beta_1,\, \beta_2,\, \beta_3;\, d\, z) = \sum_{k=0}^{\infty} c_k\, z^k \; /; $$

$$ c_k = \frac{d^k\, (\alpha_1)_k\, (\alpha_2)_k}{k!\, \prod_{j=1}^{3} (\beta_j)_k}\, {}_6F_5\!\left(-k,\, 1-k-\beta_1,\, 1-k-\beta_2,\, 1-k-\beta_3,\, a_1,\, a_2;\, 1-k-\alpha_1,\, 1-k-\alpha_2,\, b_1,\, b_2,\, b_3;\, \frac{c}{d}\right) \bigvee $$

$$ c_k = \frac{c^k\, (a_1)_k\, (a_2)_k}{k!\, \prod_{j=1}^{3} (b_j)_k}\, {}_6F_5\!\left(-k,\, 1-k-b_1,\, 1-k-b_2,\, 1-k-b_3,\, \alpha_1,\, \alpha_2;\, 1-k-a_1,\, 1-k-a_2,\, \beta_1,\, \beta_2,\, \beta_3;\, \frac{d}{c}\right) $$

07.26.16.0002.01

$$ {}_2F_3(a_1,\, a_2;\, b_1,\, b_2,\, b_3;\, c\, z)\, {}_2F_3(\alpha_1,\, \alpha_2;\, \beta_1,\, \beta_2,\, \beta_3;\, d\, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(a_1)_m\, (a_2)_m\, (\alpha_1)_{k-m}\, (\alpha_2)_{k-m}\, c^m\, d^{k-m}\, z^k}{\left(\prod_{j=1}^{3} (b_j)_m\, m!\right) \prod_{j=1}^{3} (\beta_j)_{k-m}\, (k-m)!} $$

07.26.16.0003.01

$$ {}_2F_3(a_1,\, a_2;\, b_1,\, b_2,\, b_3;\, c\, z)\, {}_2F_3(\alpha_1,\, \alpha_2;\, \beta_1,\, \beta_2,\, \beta_3;\, d\, z) = F_{0:3;3}^{0:2;2}\!\left(\begin{matrix} :a_1,\, a_2;\, \alpha_1,\, \alpha_2; \\ :b_1,\, b_2,\, b_3;\, \beta_1,\, \beta_2,\, \beta_3; \end{matrix}\; c\, z,\, d\, z\right) $$

Identities

Recurrence identities

Consecutive neighbors

07.26.17.0001.01

$$ {}_2F_3(a,\, a_2;\, b_1,\, b_2,\, b_3;\, z) = (B_1 + C_1\, z)\, {}_2F_3(a+1,\, a_2;\, b_1,\, b_2,\, b_3;\, z) + (B_2 + C_2\, z)\, {}_2F_3(a+2,\, a_2;\, b_1,\, b_2,\, b_3;\, z) + B_3\, {}_2F_3(a+3,\, a_2;\, b_1,\, b_2,\, b_3;\, z) + B_4\, {}_2F_3(a+4,\, a_2;\, b_1,\, b_2,\, b_3;\, z) \; /; $$

$$ B_1 = \left((a+1)\left(4\, a^2 + 11\, a - (3\, a+4)(b_1+b_2+b_3) + 2\,(b_1\, b_2 + b_3\, b_2 + b_1\, b_3) + 8\right) - b_1\, b_2\, b_3\right) / ((a-b_1+1)(a-b_2+1)(a-b_3+1)) \bigwedge C_1 = -\frac{a-a_2+1}{(a-b_1+1)(a-b_2+1)(a-b_3+1)} \bigwedge $$

$$ B_2 = -\left((a+1)\left(6\, a^2 + 21\, a + b_1\, b_2 + b_1\, b_3 + b_2\, b_3 - (3\, a+5)(b_1+b_2+b_3) + 19\right)\right) / ((a-b_1+1)(a-b_2+1)(a-b_3+1)) \bigwedge $$

$$ C_2 = \frac{a+1}{(a-b_1+1)(a-b_2+1)(a-b_3+1)} \bigwedge $$

$$ B_3 = \frac{(a+1)(a+2)(4\, a - b_1 - b_2 - b_3 + 9)}{(a-b_1+1)(a-b_2+1)(a-b_3+1)} \bigwedge B_4 = -\frac{(a+1)(a+2)(a+3)}{(a-b_1+1)(a-b_2+1)(a-b_3+1)} $$

07.26.17.0002.01

$$_2F_3(a, a_2; b_1, b_2, b_3; z) = B_1\, _2F_3(a-1, a_2; b_1, b_2, b_3; z) + (B_2 + C_2 z)\, _2F_3(a-2, a_2; b_1, b_2, b_3; z) + (B_3 + C_3 z)\, _2F_3(a-3, a_2; b_1, b_2, b_3; z) + B_4\, _2F_3(a-4, a_2; b_1, b_2, b_3; z) \;/;\; B_1 = \frac{b_1 + b_2 + b_3 - 4a + 7}{1-a} \bigwedge B_2 = \frac{1}{(a-1)(a-2)} \left((3a-7)(b_1+b_2+b_3) - 6a^2 + 27a - b_1 b_2 - b_1 b_3 - b_2 b_3 - 31\right) \bigwedge C_2 = \frac{1}{(a-1)(a-2)} \bigwedge B_3 = \frac{1}{(a-1)(a-2)(a-3)} \left((a-3)\left(4a^2 - 21a + (8-3a)(b_1+b_2+b_3) + 2(b_1 b_2 + b_3 b_2 + b_1 b_3) + 28\right) - b_1 b_2 b_3\right) \bigwedge C_3 = \frac{a_2 - a + 3}{(a-1)(a-2)(a-3)} \bigwedge B_4 = -\frac{(a-b_1-3)(a-b_2-3)(a-b_3-3)}{(a-1)(a-2)(a-3)}$$

07.26.17.0003.01

$$_2F_3(a_1, a_2; b, b_2, b_3; z) = B_1\, _2F_3(a_1, a_2; b+1, b_2, b_3; z) + (B_2 + C_2 z)\, _2F_3(a_1, a_2; b+2, b_2, b_3; z) + (B_3 + C_3 z)\, _2F_3(a_1, a_2; b+3, b_2, b_3; z) + C_4 z\, _2F_3(a_1, a_2; b+4, b_2, b_3; z) \;/;\; B_1 = \frac{3b - b_2 - b_3 + 5}{b} \bigwedge B_2 = \frac{(b+2)(2b_2 + 2b_3 - 3b - 7) - b_2 b_3}{b(b+1)} \bigwedge C_2 = \frac{1}{b(b+1)} \bigwedge B_3 = \frac{(b - b_2 + 3)(b - b_3 + 3)}{b(b+1)} \bigwedge C_3 = \frac{a_1 + a_2 - 2b - 5}{b(b+1)(b+2)} \bigwedge C_4 = \frac{(b - a_1 + 3)(b - a_2 + 3)}{b(b+1)(b+2)(b+3)}$$

07.26.17.0004.01

$$_2F_3(a_1, a_2; b, b_2, b_3; z) = \frac{B_1 + C_1 z}{z}\, _2F_3(a_1, a_2; b-1, b_2, b_3; z) + \frac{B_2 + C_2 z}{z}\, _2F_3(a_1, a_2; b-2, b_2, b_3; z) + \frac{B_3}{z}\, _2F_3(a_1, a_2; b-3, b_2, b_3; z) + \frac{B_4}{z}\, _2F_3(a_1, a_2; b-4, b_2, b_3; z) \;/;\; B_1 = -\frac{(b-1)(b-2)(b-b_2-1)(b-b_3-1)}{(b-a_1-1)(b-a_2-1)} \bigwedge C_1 = \frac{(b-1)(2b-a_1-a_2-3)}{(b-a_1-1)(b-a_2-1)} \bigwedge B_2 = \frac{(b-1)(b-2)((3b-2b_2-2b_3-5)(b-2)+b_2 b_3)}{(b-a_1-1)(b-a_2-1)} \bigwedge C_2 = -\frac{(b-1)(b-2)}{(b-a_1-1)(b-a_2-1)} \bigwedge B_3 = \frac{(b-1)(b-2)(b-3)(b_2+b_3-3b+7)}{(b-a_1-1)(b-a_2-1)} \bigwedge B_4 = \frac{(b-1)(b-2)(b-3)(b-4)}{(b-a_1-1)(b-a_2-1)}$$

Functional identities

Relations between contiguous functions

07.26.17.0005.01

$$b\, _2F_3(a, b+1; b_1, b_2, b_3; z) - a\, _2F_3(a+1, b; b_1, b_2, b_3; z) + (a-b)\, _2F_3(a, b; b_1, b_2, b_3; z) = 0$$

07.26.17.0006.01

$$c\, _2F_3(a, a_2; c, b_2, b_3; z) - a\, _2F_3(a+1, a_2; c+1, b_2, b_3; z) + (a-c)\, _2F_3(a, a_2; c+1, b_2, b_3; z) = 0$$

07.26.17.0007.01

$$(c-d)\, _2F_3(a_1, a_2; c+1, d+1, b_3; z) + d\, _2F_3(a_1, a_2; c+1, d, b_3; z) - c\, _2F_3(a_1, a_2; c, d+1, b_3; z) = 0$$

07.26.17.0008.01

$$(a-b)\, c\, _2F_3(a, b; c, b_2, b_3; z) - a(c-b)\, _2F_3(a+1, b; c+1, b_2, b_3; z) + (c-a)\, b\, _2F_3(a, b+1; c+1, b_2, b_3; z) = 0$$

07.26.17.0009.01

$$a(c-d)\, _2F_3(a+1, a_2; c+1, d+1, b_3; z) - d(c-a)\, _2F_3(a, a_2; c+1, d, b_3; z) + (d-a)\, c\, _2F_3(a, a_2; c, d+1, b_3; z) = 0$$

07.26.17.0010.01

$$\left(\prod_{k=1}^{3} b_k\right)\left({}_2F_3(a, a_2; b_1, b_2, b_3; z) - {}_2F_3(a+1, a_2; b_1, b_2, b_3; z)\right) + a_2\, z\, {}_2F_3(a+1, a_2+1; b_1+1, b_2+1, b_3+1; z) = 0$$

07.26.17.0011.01

$$b_2\, b_3\, c\,(c+1)\left({}_2F_3(a_1, a_2; c, b_2, b_3; z) - {}_2F_3(a_1, a_2; c+1, b_2, b_3; z)\right) - a_1\, a_2\, z\, {}_2F_3(a_1+1, a_2+1; c+2, b_2+1, b_3+1; z) = 0$$

07.26.17.0012.01

$$(b-a)\, z\, {}_2F_3(a+1, b+1; b_1+1, b_2+1, b_3+1; z) + \left(\prod_{k=1}^{3} b_k\right)\left({}_2F_3(a, b+1; b_1, b_2, b_3; z) - {}_2F_3(a+1, b; b_1, b_2, b_3; z)\right) = 0$$

07.26.17.0013.01

$$(c-a)\, z\, {}_2F_3(a+1, a_2+1; c+2, b_2+1, b_3+1; z)\, a_2 + (c+1)\, c\left({}_2F_3(a, a_2; c, b_2, b_3; z) - {}_2F_3(a+1, a_2; c+1, b_2, b_3; z)\right) b_2\, b_3 = 0$$

07.26.17.0014.01

$$a\, z\, {}_2F_3(a+1, b+1; c+1, b_2+1, b_3+1; z) + \left(c\, {}_2F_3(a, b; c, b_2, b_3; z) - a\, {}_2F_3(a+1, b+1; c+1, b_2, b_3; z) - (c-a)\, {}_2F_3(a, b+1; c+1, b_2, b_3; z)\right) b_2\, b_3 = 0$$

07.26.17.0015.01

$${}_2F_3(a+1, b+1; c+1, d+1, e+1; z) - \frac{d\, e\,(a-c)\,(b-c)}{a\, b\,(d-c)\,(e-c)}\, {}_2F_3(a, b; c+1, d, e; z) - \frac{c\, e\,(a-d)\,(b-d)}{a\, b\,(c-d)\,(e-d)}\, {}_2F_3(a, b; c, d+1, e; z) - \frac{c\, d\,(a-e)\,(b-e)}{a\, b\,(c-e)\,(d-e)}\, {}_2F_3(a, b; c, d, e+1; z) = 0$$

Relations of special kind

07.26.17.0016.01

$${}_2F_3(a_1, a_2; -c, c+1, b_3; z) + {}_2F_3(a_1, a_2; c, 1-c, b_3; z) = 2\, {}_2F_3(a_1, a_2; c+1, 1-c, b_3; z)$$

07.26.17.0017.01

$${}_2F_3(a, a_2; -a, a+1, b_3; z) - 2\, {}_2F_3(a, a_2; 1-a, a+1, b_3; z) = -{}_1F_2(a_2; 1-a, b_3; z)$$

07.26.17.0018.01

$${}_2F_3(-a, a_2; 1-a, b_2, b_3; z) + {}_2F_3(a, a_2; a+1, b_2, b_3; z) = 2\, {}_3F_4(a, -a, a_2; a+1, 1-a, b_2, b_3; z)$$

07.26.17.0019.01

$${}_2F_3(-a, a+1; b_1, b_2, b_3; z) + {}_2F_3(a, 1-a; b_1, b_2, b_3; z) = 2\, {}_2F_3(a, -a; b_1, b_2, b_3; z)$$

Division on even and odd parts and generalization

07.26.17.0020.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = A^+(z) + A^-(z) /; A^+(z) = \frac{1}{2}\left({}_2F_3(a_1, a_2; b_1, b_2, b_3; z) + {}_2F_3(a_1, a_2; b_1, b_2, b_3; -z)\right) \bigwedge A^-(z) = \frac{1}{2}\left({}_2F_3(a_1, a_2; b_1, b_2, b_3; z) - {}_2F_3(a_1, a_2; b_1, b_2, b_3; -z)\right)$$

07.26.17.0021.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = A^-(z) + A^+(z) /; A^+(z) = {}_4F_7\left(\frac{a_1}{2}, \frac{a_2}{2}, \frac{a_1+1}{2}, \frac{a_2+1}{2}; \frac{1}{2}, \frac{b_1}{2}, \frac{b_2}{2}, \frac{b_3}{2}, \frac{b_1+1}{2}, \frac{b_2+1}{2}, \frac{b_3+1}{2}; \frac{z^2}{16}\right) \bigwedge A^-(z) = \frac{z\, a_1\, a_2}{\prod_{j=1}^{3} b_j}\, {}_4F_7\left(\frac{a_1+1}{2}, \frac{a_2+1}{2}, \frac{a_1+2}{2}, \frac{a_2+2}{2}; \frac{3}{2}, \frac{b_1+1}{2}, \frac{b_2+1}{2}, \frac{b_3+1}{2}, \frac{b_1+2}{2}, \frac{b_2+2}{2}, \frac{b_3+2}{2}; \frac{z^2}{16}\right)$$

07.26.17.0022.01

$$ {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{n-1} \frac{z^k (a_1)_k (a_2)_k}{k! \prod_{j=1}^{3} (b_j)_k} \, {}_{2n+1}F_{4n}\left(1, \frac{k+a_1}{n}, \ldots, \frac{k+n+a_1-1}{n}, \frac{k+a_2}{n}, \ldots, \frac{k+n+a_2-1}{n}; \frac{k+1}{n}, \ldots, \frac{k+n}{n}, \frac{k+b_1}{n}, \ldots, \frac{k+n+b_1-1}{n}, \frac{k+b_2}{n}, \ldots, \frac{k+n+b_2-1}{n}, \frac{k+b_3}{n}, \ldots, \frac{k+n+b_3-1}{n}; n^{-2n} z^n\right) $$

Differentiation

Low-order differentiation

With respect to a_1

07.26.20.0001.01

$$ {}_2F_3^{(\{1,0\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \psi(k+a_1) z^k}{(b_1)_k (b_2)_k (b_3)_k k!} - \psi(a_1)\, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) $$

07.26.20.0002.01

$$ {}_2F_3^{(\{1,0\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \frac{z a_2}{b_1 b_2 b_3} F_{4\times0\times1}^{2\times1\times2}\left(\begin{matrix} a_1+1, a_2+1; 1; 1, a_1; \\ 2, b_1+1, b_2+1, b_3+1;; a_1+1; \end{matrix} z, z \right) $$

With respect to a_2

07.26.20.0003.01

$$ {}_2F_3^{(\{0,1\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \psi(k+a_2) z^k}{(b_1)_k (b_2)_k (b_3)_k k!} - \psi(a_2)\, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) $$

07.26.20.0004.01

$$ {}_2F_3^{(\{0,1\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \frac{z a_1}{b_1 b_2 b_3} F_{4\times0\times1}^{2\times1\times2}\left(\begin{matrix} a_1+1, a_2+1; 1; 1, a_2; \\ 2, b_1+1, b_2+1, b_3+1;; a_2+1; \end{matrix} z, z \right) $$

With respect to b_1

07.26.20.0005.01

$$ {}_2F_3^{(\{0,0\},\{1,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \psi(b_1)\, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) - \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \psi(k+b_1) z^k}{(b_1)_k (b_2)_k (b_3)_k k!} $$

07.26.20.0006.01

$$ {}_2F_3^{(\{0,0\},\{1,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = -\frac{z a_1 a_2}{b_1^2 b_2 b_3} F_{4\times0\times1}^{2\times1\times2}\left(\begin{matrix} a_1+1, a_2+1; 1; 1, b_1; \\ 2, b_1+1, b_2+1, b_3+1;; b_1+1; \end{matrix} z, z \right) $$

With respect to b_2

07.26.20.0007.01

$$ {}_2F_3^{(\{0,0\},\{0,1,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \psi(b_2)\, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) - \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \psi(k+b_2) z^k}{(b_1)_k (b_2)_k (b_3)_k k!} $$

07.26.20.0008.01

$$ {}_2F_3^{(\{0,0\},\{0,1,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = -\frac{z a_1 a_2}{b_2^2 b_1 b_3} F_{4\times0\times1}^{2\times1\times2}\left(\begin{matrix} a_1+1, a_2+1; 1; 1, b_2; \\ 2, b_1+1, b_2+1, b_3+1;; b_2+1; \end{matrix} z, z \right) $$

With respect to b_3

07.26.20.0009.01

$${}_2F_3^{(\{0,0\},\{0,0,1\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \psi(b_3)\,{}_2F_3(a_1, a_2; b_1, b_2, b_3; z) - \sum_{k=0}^{\infty} \frac{(a_1)_k\,(a_2)_k\,\psi(k+b_3)\,z^k}{(b_1)_k\,(b_2)_k\,(b_3)_k\,k!}$$

07.26.20.0010.01

$${}_2F_3^{(\{0,0\},\{0,0,1\},0)}(a_1, a_2; b_1, b_2, b_3; z) = -\frac{z\,a_1\,a_2}{b_3^2\,b_1\,b_2} F_{4\times0\times1}^{2\times1\times2}\left(\begin{matrix} a_1+1, a_2+1; 1; 1, b_3; \\ 2, b_1+1, b_2+1, b_3+1;; b_3+1; \end{matrix} z, z\right)$$

With respect to element of parameters ||| With respect to element of parameters

07.26.20.0011.01

$$\frac{\partial\,{}_2F_3(a, a_2; a+1, b_2, b_3; z)}{\partial a} = \frac{a_2\,z}{b_2\,b_3\,(a+1)^2}\,{}_3F_4(a+1, a+1, a_2+1; a+2, a+2, b_2+1, b_3+1; z)$$

07.26.20.0012.01

$$\frac{\partial\,{}_2F_3(a+1, a_2; a, b_2, b_3; z)}{\partial a} = -\frac{a_2\,z}{b_2\,b_3\,a^2}\,{}_1F_2(a_2+1; b_2+1, b_3+1; z)$$

With respect to z

07.26.20.0013.01

$$\frac{\partial\,{}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\partial z} = \frac{a_1\,a_2}{b_1\,b_2\,b_3}\,{}_2F_3(a_1+1, a_2+1; b_1+1, b_2+1, b_3+1; z)$$

07.26.20.0014.01

$$\frac{\partial^2\,{}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\partial z^2} = \frac{a_1\,(a_1+1)\,a_2\,(a_2+1)}{b_1\,(b_1+1)\,b_2\,(b_2+1)\,b_3\,(b_3+1)}\,{}_2F_3(a_1+2, a_2+2; b_1+2, b_2+2, b_3+2; z)$$

Symbolic differentiation

With respect to a_1

07.26.20.0015.02

$${}_2F_3^{(\{n,0\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_2)_k}{(b_1)_k\,(b_2)_k\,(b_3)_k\,k!} \frac{\partial^n (a_1)_k}{\partial a_1^n} z^k \;/;\; n \in \mathbb{N}$$

With respect to a_2

07.26.20.0016.02

$${}_2F_3^{(\{0,n\},\{0,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{(b_1)_k\,(b_2)_k\,(b_3)_k\,k!} \frac{\partial^n (a_2)_k}{\partial a_2^n} z^k \;/;\; n \in \mathbb{N}$$

With respect to b_1

07.26.20.0017.02

$${}_2F_3^{(\{0,0\},\{n,0,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\,(a_2)_k}{(b_2)_k\,(b_3)_k\,k!} \frac{\partial^n \frac{1}{(b_1)_k}}{\partial b_1^n} z^k \;/;\; n \in \mathbb{N}$$

With respect to b_2

07.26.20.0018.02

$${}_2F_3^{(\{0,0\},\{0,n,0\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k (b_3)_k k!} \frac{\partial^n \frac{1}{(b_2)_k}}{\partial b_2^n} z^k /; n \in \mathbb{N}$$

With respect to b_3

07.26.20.0019.02

$${}_2F_3^{(\{0,0\},\{0,0,n\},0)}(a_1, a_2; b_1, b_2, b_3; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k (b_2)_k k!} \frac{\partial^n \frac{1}{(b_3)_k}}{\partial b_3^n} z^k /; n \in \mathbb{N}$$

With respect to element of parameters ||| With respect to element of parameters

07.26.20.0030.02

$$\frac{\partial^n {}_2F_3(a, a_2; a+1, b_2, b_3; z)}{\partial a^n} = \frac{(-1)^{n-1} n! \, z \, a_2}{(a+1)^{n+1} b_2 b_3} {}_{n+2}F_{n+3}(a+1, \ldots, a+1, a_2+1; a+2, \ldots, a+2, b_2+1, b_3+1; z) /; n \in \mathbb{N}$$

07.26.20.0031.02

$$\frac{\partial^n {}_2F_3(a+1, a_2; a, b_2, b_3; z)}{\partial a^n} = \frac{(-1)^n n!}{a^{n+1}} \left(a \, {}_1F_2(a_2; b_2, b_3; z) + \frac{z \, a_2}{b_2 b_3} {}_1F_2(a_2+1; b_2+1, b_3+1; z) \right) + \frac{(-1)^{n-1} n!}{a^n} {}_1F_2(a_2; b_2, b_3; z) /; n \in \mathbb{N}$$

With respect to z

07.26.20.0020.02

$$\frac{\partial^n {}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\partial z^n} = \frac{\prod_{j=1}^{2} (a_j)_n}{\prod_{j=1}^{3} (b_j)_n} {}_2F_3(n+a_1, n+a_2; n+b_1, n+b_2, n+b_3; z) /; n \in \mathbb{N}$$

07.26.20.0021.02

$$\frac{\partial^n {}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\partial z^n} = z^{-n} \prod_{j=1}^{3} \Gamma(b_j) \, {}_3\tilde{F}_4(1, a_1, a_2; 1-n, b_1, b_2, b_3; z) /; n \in \mathbb{N}$$

07.26.20.0022.02

$$\frac{\partial^n (z^\alpha \, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z))}{\partial z^n} = (-1)^n (-\alpha)_n z^{\alpha-n} \, {}_3F_4(\alpha+1, a_1, a_2; 1-n+\alpha, b_1, b_2, b_3; z) /; n \in \mathbb{N}$$

07.26.20.0023.02

$$\frac{\partial^n \left(z^{a+n-1} \, {}_2F_3(a, a_2; b_1, b_2, b_3; z)\right)}{\partial z^n} = (a)_n z^{a-1} \, {}_2F_3(a+n, a_2; b_1, b_2, b_3; z) /; n \in \mathbb{N}$$

07.26.20.0024.02

$$\frac{\partial^n \left(z^{c-1} \, {}_2F_3(a_1, a_2; c, b_2, b_3; z)\right)}{\partial z^n} = (c-n)_n z^{c-n-1} \, {}_2F_3(a_1, a_2; c-n, b_2, b_3; z) /; n \in \mathbb{N}$$

07.26.20.0025.02

$$\frac{\partial^n \left(z^n \, {}_2F_3\left(-n, a_2; \frac{1}{2}, b_2, b_3; z\right)\right)}{\partial z^n} = n! \, {}_3F_4\left(-n, n+1, a_2; \frac{1}{2}, 1, b_2, b_3; z\right) /; n \in \mathbb{N}$$

07.26.20.0026.02

$$\frac{\partial^n \left(z^\alpha \, {}_2F_3(-n, a_2; b_1, b_2, b_3; z)\right)}{\partial z^n} = (-1)^n \, (-\alpha)_n \, z^{\alpha-n} \, {}_3F_4(-n, \alpha+1, a_2; 1-n+\alpha, b_1, b_2, b_3; z) \,/;\, n \in \mathbb{N}$$

07.26.20.0027.02

$$\frac{\partial^n \left(z^\alpha \, {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; b_1, b_2, b_3; z^m\right)\right)}{\partial z^n} = (-1)^n \, (-\alpha)_n \, z^{\alpha-n}$$

$${}_{m+2}F_{m+3}\left(-\frac{n}{2}, \frac{1-n}{2}, \frac{\alpha+1}{m}, \frac{\alpha+2}{m}, \ldots, \frac{\alpha+m}{m}; \frac{\alpha-n+1}{m}, \frac{\alpha-n+2}{m}, \ldots, \frac{\alpha-n+m}{m}, b_1, b_2, b_3; z^m\right) \,/;\, m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

07.26.20.0028.02

$$\frac{\partial^n \left(e^{-z} \, {}_2F_3(-n, a_2; b_1, b_2, b_3; z)\right)}{\partial z^n} = (-1)^n \, e^{-z} \sum_{k=0}^{n} \frac{(-n)_k \, z^k}{k! \prod_{j=1}^{3} (b_j)_k} \, {}_3F_3(-n, k-n, k+a_2; k+b_1, k+b_2, k+b_3; z) \,/;\, n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.26.20.0029.01

$$\frac{\partial^\alpha \, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\partial z^\alpha} = z^{-\alpha} \prod_{j=1}^{3} \Gamma(b_j) \, {}_3\tilde{F}_4(1, a_1, a_2; 1-\alpha, b_1, b_2, b_3; z)$$

Integration

Indefinite integration

Involving only one direct function

07.26.21.0001.01

$$\int {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \, dz = \frac{(b_1-1)(b_2-1)(b_3-1)}{(a_1-1)(a_2-1)} \, {}_2F_3(a_1-1, a_2-1; b_1-1, b_2-1, b_3-1; z)$$

Involving one direct function and elementary functions

Involving power function

07.26.21.0002.01

$$\int z^{\alpha-1} \, {}_2F_3(a_1, a_2; b_1, b_2, b_3; z) \, dz = \frac{z^\alpha}{\alpha} \, {}_3F_4(\alpha, a_1, a_2; \alpha+1, b_1, b_2, b_3; z)$$

Definite integration

For the direct function itself

07.26.21.0003.01

$$\int_0^\infty t^{\alpha-1} \, {}_2F_3(a_1, a_2; b_1, b_2, b_3; -t) \, dt = \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,\Gamma(\alpha)\,\Gamma(a_1-\alpha)\,\Gamma(a_2-\alpha)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(b_1-\alpha)\,\Gamma(b_2-\alpha)\,\Gamma(b_3-\alpha)} \,/;$$

$$0 < \mathrm{Re}(\alpha) < \min\left(\mathrm{Re}(a_1), \mathrm{Re}(a_2), \frac{1}{4} - \frac{1}{2} \mathrm{Re}\left(\sum_{j=1}^{2} a_1 - \sum_{k=1}^{3} b_k\right)\right)$$

Involving the direct function

07.26.21.0004.01

$$\int_0^\infty t^{\alpha-1} e^{-c\,t}\,{}_2F_3(a_1, a_2; b_1, b_2, b_3; -t)\,dt = c^{-\alpha}\,\Gamma(\alpha)\,{}_3F_3\left(\alpha, a_1, a_2; b_1, b_2, b_3; -\frac{1}{c}\right) /; \operatorname{Re}(\alpha) > 0 \land \operatorname{Re}(c) > 0$$

Integral transforms

Laplace transforms

07.26.22.0001.01

$$\mathcal{L}_t[{}_2F_3(a_1, a_2; b_1, b_2, b_3; -t)](z) = \frac{1}{z}\,{}_3F_3\left(1, a_1, a_2; b_1, b_2, b_3; -\frac{1}{z}\right) /; \operatorname{Re}(z) > 0$$

Operations

Limit operation

07.26.25.0001.01

$$\lim_{b_1 \to -n} \frac{{}_2F_3(a_1, a_2; b_1, b_2, b_3; z)}{\Gamma(b_1)} = z^{n+1}\,(a_1)_{n+1}\,(a_2)_{n+1}\,{}_2\tilde{F}_3(n + a_1 + 1, n + a_2 + 1; n + 2, n + b_2 + 1, n + b_3 + 1; z) /; n \in \mathbb{N}$$

07.26.25.0002.01

$$\lim_{q \to \infty} \lim_{p \to \infty} {}_2F_3(a_1, a_2; b_1, p, q; p\,q\,z) = {}_2F_1(a_1, a_2; b_1; z)$$

07.26.25.0003.01

$$\lim_{p \to \infty} {}_2F_3(a_1, a_2; b_1, b_2, p; p\,z) = {}_2F_2(a_1, a_2; b_1, b_2; z)$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

07.26.26.0001.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)\,{}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z)$$

Through Meijer *G*

Classical cases for the direct function itself

07.26.26.0002.01

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \frac{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(b_3)}{\Gamma(a_1)\,\Gamma(a_2)}\,G_{2,4}^{1,2}\left(-z \,\middle|\, \begin{matrix} 1 - a_1, 1 - a_2 \\ 0, 1 - b_1, 1 - b_2, 1 - b_3 \end{matrix}\right)$$

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