HypergeometricPFQRegularized

Notations

Traditional name

Regularized generalized hypergeometric function

Traditional notation

$${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$$

Mathematica StandardForm notation

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HypergeometricPFQRegularized[{a1, …, ap}, {b1, …, bq}, z]
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Primary definition

07.32.02.0001.01

$${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{k! \prod_{j=1}^{q} \Gamma(k+b_j)} \;/;$$

$$q \geq p \bigvee q = p-1 \land |z| < 1 \bigvee q = p-1 \bigwedge |z| = 1 \bigwedge \operatorname{Re}\left(\sum_{j=1}^{p-1} b_j - \sum_{j=1}^{p} a_j\right) > 0$$

In the cases $q < p-1$ the series above does not converge but it (together with symbol) can be used as asymptotical series, where, when needed a Borel summation is implicitly understood.

07.32.02.0002.01

$${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{n} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{k! \prod_{j=1}^{q} \Gamma(k+b_j)} \;/;\; \exists_{a_j}\, -a_j = n \in \mathbb{N}$$

For $a_i = -n,\, b_j = -m \;/;\; m \geq n$ being nonpositive integers and $\nexists_{a_k} (a_k > -n \land a_k \in \mathbb{N}) \bigwedge \nexists_{b_k} (b_k > -m \bigwedge b_k \in \mathbb{N})$ the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables a_i, b_j can approach nonpositive integers $-n, -m;\, m \geq n$ at different speeds. For the above conditions we define:

07.32.02.0003.01

$${}_p\tilde{F}_q(a_1, \ldots, a_i, \ldots, a_p; b_1, \ldots, b_j, \ldots, b_q; z) = \sum_{k=0}^{n} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{k! \prod_{j=1}^{q} \Gamma(k+b_j)} \;/;\; a_i = -n \land b_j = -m \land m \in \mathbb{N} \land n \in \mathbb{N} \land m \geq n$$

General characteristics

Some abbreviations

07.32.04.0001.01

$$\mathcal{NT}(\{a_1, \ldots, a_p\}) = \neg(-a_1 \in \mathbb{N} \bigvee \ldots \bigvee -a_p \in \mathbb{N})$$

Domain and analyticity

${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is an analytical function of $a_1, \ldots, a_p, b_1, \ldots, b_q$ and z which is defined in $\mathbb{C}^{p+q+1}$. In the cases $p \leq q$ it is an entire function of all variables. If parameters $a_k$include negative integers, the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ degenerates to a polynomial in z.

07.32.04.0002.01

$$(\{a_1 * \ldots * a_p\} * \{b_1 * \ldots * b_q\} * z) \longrightarrow {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) :: (\{\mathbb{C} \otimes \ldots \otimes \mathbb{C}\} \otimes \{\mathbb{C} \otimes \ldots \otimes \mathbb{C}\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.32.04.0003.02

$${}_p\tilde{F}_q(\overline{a_1}, \ldots, \overline{a_p}; \overline{b_1}, \ldots, \overline{b_q}; \bar{z}) = \overline{{}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)} \,/;\, \neg(z \in (1, \infty) \bigwedge p = q+1)$$

Permutation symmetry

07.32.04.0004.01

$${}_p\tilde{F}_q(a_1, a_2, \ldots, a_k, \ldots, a_j, \ldots, a_p; b_1, \ldots, b_q; z) = {}_p\tilde{F}_q(a_1, a_2, \ldots, a_j, \ldots, a_k, \ldots, a_p; b_1, \ldots, b_q; z) \,/;\, a_k \neq a_j \bigwedge k \neq j$$

07.32.04.0005.01

$${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, b_2, \ldots, b_k, \ldots, b_j, \ldots, b_q; z) = {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, b_2, \ldots, b_j, \ldots, b_k, \ldots, b_q; z) \,/;\, b_k \neq b_j \bigwedge k \neq j$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For $p = q+1$ and fixed a_l, b_j in nonpolynomial cases (when $\neg(-a_1 \in \mathbb{N} \bigvee \ldots \bigvee -a_p \in \mathbb{N})$), the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ does not have poles and essential singularities.

07.32.04.0006.01

$$\mathcal{Sing}_z({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\} \,/;\, p = q+1 \bigwedge \mathcal{NT}(\{a_1, \ldots, a_p\})$$

For $p \leq q$ and fixed a_l, b_j in nonpolynomial cases (when $\neg(-a_1 \in \mathbb{N} \bigvee \ldots \bigvee -a_p \in \mathbb{N})$), the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

07.32.04.0007.01

$$\mathcal{Sing}_z({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, \infty\}\} \,/;\, p \leq q \bigwedge \mathcal{NT}(\{a_1, \ldots, a_p\})$$

If parameters a_k include r negative integers α_k, the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is a polynomial and has pole of order $\min(-\alpha_1, \ldots, -\alpha_r\}$ at $z = \tilde{\infty}$.

07.32.04.0008.01

$$\mathcal{S}ing_z({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, -\alpha\}\} /; \neg\left(\mathcal{NT}(\{a_1, \ldots, a_p\})\right) \bigwedge \alpha = \min(-a_{s_1}, \ldots, -a_{s_r}\} \bigwedge -a_{s_k} \in \mathbb{N}^+$$

With respect to a_l

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of a_l, $1 \le l \le p$, has only one singular point at $a_l = \tilde{\infty}$. It is an essential singular point.

07.32.04.0009.01

$$\mathcal{S}ing_{a_l}({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, \infty\}\} /; 1 \le l \le p$$

With respect to b_j

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of b_j, $1 \le j \le q$, has only one singular point at $b_j = \tilde{\infty}$. It is an essential singular point.

07.32.04.0010.01

$$\mathcal{S}ing_{b_j}({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, \infty\}\} /; 1 \le j \le q$$

Branch points

With respect to z

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ does not have branch points for $p \le q$ and has two branch points: $z = 1$, $z = \tilde{\infty}$ for $p = q + 1$ in nonpolynomial case (when $\neg(-a_1 \in \mathbb{N} \vee \ldots \vee -a_p \in \mathbb{N})$)

07.32.04.0011.01

$$\mathcal{BP}_z({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\} /; p \le q$$

07.32.04.0012.01

$$\mathcal{BP}_z({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)) = \{1, \tilde{\infty}\} /; \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.32.04.0013.01

$$\mathcal{R}_z({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), 1) = \log /; \psi_q \in \mathbb{Z} \bigvee \psi_q \notin \mathbb{Q} \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.32.04.0014.01

$$\mathcal{R}_z({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), 1) = s /;$$
$$\psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j = \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) = 1 \bigwedge \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.32.04.0015.01

$$\mathcal{R}_z({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), \tilde{\infty}) = \log /;$$
$$\exists_{a_i, a_j} (a_i - a_j \in \mathbb{Z} \wedge 1 \le i \le q + 1 \wedge 1 \le j \le q + 1 \wedge i \ne j) \bigwedge (a_1 \notin \mathbb{Q} \vee \ldots \vee a_{q+1} \notin \mathbb{Q})$$

07.32.04.0016.01

$$\mathcal{R}_z\left({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), \tilde{\infty}\right) = \text{lcm}\left(s_1, \ldots, s_{q+1}\right) /; a_l = \frac{r_l}{s_l} \bigwedge \{r_l, s_l\} \in \mathbb{Z} \bigwedge s_l > 1 \bigwedge \gcd(r_l, s_l) = 1 \bigwedge 1 \leq l \leq q+1 \bigwedge \mathcal{NT}\left(\{a_1, \ldots, a_{q+1}\}\right)$$

With respect to a_l

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of a_l, $1 \leq l \leq p$, does not have branch points.

07.32.04.0017.01

$$\mathcal{BP}_{a_l}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \{\} /; 1 \leq l \leq p$$

With respect to b_j

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of b_j, $1 \leq j \leq q$, does not have branch points.

07.32.04.0018.01

$$\mathcal{BP}_{b_j}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \{\} /; 1 \leq j \leq q$$

Branch cuts

With respect to z

For all nonnegative integer a_k, the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ in the cases $p = q+1$ is a single-valued function on the z-plane cut along the interval $(1, \infty)$, where it is continuous from below. In the cases $p \leq q$ this function does not have branch cuts.

07.32.04.0019.01

$$\mathcal{BC}_z\left({}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)\right) = \{\{(1, \infty), i\}\} /; \mathcal{NT}\left(\{a_1, \ldots, a_{q+1}\}\right)$$

07.32.04.0020.01

$$\mathcal{BC}_z\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \{\} /; p \leq q$$

07.32.04.0021.01

$$\lim_{\epsilon \to +0} {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x - i\epsilon) = {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x) /; x > 1$$

07.32.04.0025.01

$$\lim_{\epsilon \to +0} {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x + i\epsilon) = \frac{1}{\prod_{k=1}^{q+1} \Gamma(a_k)} G_{q+1,q+1}^{q+1,1}\left(e^{\pi i} \frac{1}{x} \middle| \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_{q+1} \end{matrix}\right) /; x > 1$$

07.32.04.0022.02

$$\lim_{\epsilon \to +0} {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x + i\epsilon) = \frac{\pi^q}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j \neq k}^{q+1} \csc\left(\pi (a_j - a_k)\right)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} e^{\pi i a_k} x^{-a_k}$$
$${}_{q+1}\tilde{F}_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{x}\right) /;$$
$$\forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq q+1 \land 1 \leq k \leq q+1} \left(a_j - a_k \notin \mathbb{Z}\right) \land x > 1$$

With respect to a_l

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of a_l, $1 \leq l \leq p$, does not have branch cuts.

07.32.04.0023.01

$$\mathcal{BC}_{a_l}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \{\} \,/;\, 1 \leq l \leq p$$

With respect to b_j

The function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of b_j, $1 \leq j \leq q$, does not have branch cuts.

07.32.04.0024.01

$$\mathcal{BC}_{b_j}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) = \{\} \,/;\, 1 \leq j \leq q$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.32.06.0045.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!} \, {}_{p+1}F_{q+1}(1, a_1, \ldots, a_p; 1-k, b_1, \ldots, b_q; z_0)\,(z-z_0)^k \,/; $$
$$ p \neq q+1 \lor p == q+1 \land z_0 \notin (1, \infty) $$

07.32.06.0046.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == $$
$$ \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j-a_k)}{\prod_{j=1}^{q}\Gamma(b_j-a_k)} \left(-\frac{1}{z_0}\right)^{-a_k \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} (-z_0)^{-a_k \left(\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor + 1\right)} \sum_{j=0}^{\infty} \frac{\Gamma(a_k+j)\,(-z_0)^{-j}}{j!} \, {}_{q+1}\tilde{F}_q $$
$$ \left(a_k - b_1 + 1, \ldots, a_k - b_q + 1, \, j + a_k; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z_0}\right)(z-z_0)^j \,/; $$
$$ |z_0| > 1 \land \forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq q+1 \land 1 \leq k \leq q+1} \left(a_j - a_k \notin \mathbb{Z}\right) $$

Expansions on branch cuts for $p == q + 1$

For the function itself

07.32.06.0047.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=0}^{\infty} \frac{1}{k!} G_{q+1,q+1}^{1,q+1}\left(-x\, e^{2\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \,\middle|\, \begin{matrix} 1-a_1-k, \ldots, 1-a_{q+1}-k \\ 0, 1-b_1-k, \ldots, 1-b_q-k \end{matrix}\right)(z-x)^k \,/;\, x > 1 $$

07.32.06.0048.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=0}^{\infty} \frac{1}{k!} G_{q+1,q+1}^{q+1,1}\left(e^{-\pi i \left(1 + 2\left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor\right)} \frac{1}{x} \,\middle|\, \begin{matrix} 1, k+b_1, \ldots, k+b_q \\ k+a_1, \ldots, k+a_{q+1} \end{matrix}\right)(z-x)^k \,/;\, x > 1 $$

07.32.06.0049.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \frac{\pi^q}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \sum_{u=0}^{\infty} \frac{(-x)^{-u}\, \Gamma(u+a_k)}{u!\, \prod_{j=1}^{q} \Gamma(b_j - a_k)} \left(\prod_{\substack{j=1 \\ j \neq k}}^{q+1} \csc(\pi (a_j - a_k)) \right) e^{-\pi i a_k \left(1 + 2\,\mathrm{Floor}\left[\frac{\arg[x-z]}{2\pi}\right]\right)} x^{-a_k} $$

$$ {}_{q+1}\tilde{F}_q\left(u + a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{x}\right) $$

$$ (z - x)^u \text{ /; } \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \wedge j \neq k \wedge 1 \leq j \leq q+1 \wedge 1 \leq k \leq q+1} (a_j - a_k \notin \mathbb{Z}) \wedge x > 1 $$

Expansions at $z == 0$

07.32.06.0001.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \frac{1}{\prod_{j=1}^{q} \Gamma(b_j)} \left(1 + \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} z + \frac{\left(\prod_{j=1}^{p} a_j (a_j + 1)\right)}{2 \prod_{j=1}^{q} b_j (b_j + 1)} z^2 + \ldots \right) \text{ /; } q == p - 1 \wedge |z| < 1 \vee q \geq p $$

07.32.06.0002.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \sum_{k=0}^{\infty} \frac{\left(\prod_{j=1}^{p} (a_j)_k\right) z^k}{\left(\prod_{j=1}^{q} \Gamma(b_j + k)\right) k!} \text{ /; } q == p - 1 \wedge |z| < 1 \vee q \geq p $$

07.32.06.0003.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \frac{1}{\prod_{j=1}^{q} \Gamma(b_j)} (1 + O(z)) $$

Expansions at $z == 1$ for $p == q + 1$

The point $z == 1$ is the end point of the branch cut for the function ${}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$, where it has a rather complicated behavior. The corresponding general formula (for noninteger $\psi_q == \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j$) includes two major terms - regular and singular which are an analytical functions. Moreover, the singular term has representation of the form const $(1 - z)^{\psi_q} (1 + O(z - 1))$ and regular term is bounded near point $z == 1$. A more detailed description of this behavior is presented below.

At the singular point $z == 1$ the function ${}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$ is continuous for $\mathrm{Re}(\psi_q) > 0$, bounded for $\mathrm{Re}(\psi_q) == 0$, $\psi_q \neq 0$ and has, in general, a logarithmic singularity for $\psi_q == 0$ while for $\mathrm{Re}(\psi_q) < 0$ it has a power singularity of order $-\psi_q$ to which for integer ψ_qa logarithmic singularity can also occur.

The general formulas

07.32.06.0004.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) == \mathcal{A}_{\tilde{F}}\begin{pmatrix} a_1, \ldots, a_{q+1}; & \\ b_1, \ldots, b_q; & \{z, 1, \infty\} \end{pmatrix} $$

07.32.06.0005.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{k=0}^{\infty} g_k(\psi_q) (1-z)^k + \sum_{k=0}^{\infty} g_k(0) (1-z)^k \right) /; $$

$$ |1-z| < 1 \bigwedge q > 1 \bigwedge g_k(r) = \frac{(-1)^k \Gamma(k+r+a_1) \Gamma(k+r+a_2) \Gamma(\psi_q - 2r - k)}{k!} $$

$$ \sum_{j=0}^{\infty} \frac{(\psi_q - r - k)_j \mathcal{E}_j^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(j + a_1 + \psi_q) \Gamma(j + a_2 + \psi_q)} = \frac{(-1)^k \Gamma(\psi_q - 2r - k)}{k! \prod_{j=1}^{q} \Gamma(k + r + b_j)} \left(\prod_{j=1}^{q+1} \Gamma(k + r + a_j) \right) \lim_{m \to \infty} \frac{1}{\Gamma(\psi_q - r - k)} $$

$$ {}_{q+1}F_q(k + r + a_1, k + r + a_2, \ldots, n + r + a_{q+1}, -m; k + r + b_1, k + r + b_2, \ldots, n + r + b_q, k - m + r - \psi_q + 1; 1) \bigwedge $$

$$ g_0(\psi_q) = \Gamma(-\psi_q) \bigwedge \mathcal{E}_{k_1}^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) = \frac{(b_1 - a_3)_{k_1}}{k_1!} \left(\sum_{j=2}^{q} b_j - \sum_{j=3}^{q+1} a_j \right)_{k_1} $$

$$ \sum_{k_2=0}^{k_1} \frac{(-k_1)_{k_2}}{\left(\sum_{j=2}^{q} b_j - \sum_{j=3}^{q+1} a_j\right)_{k_2} (a_3 - b_1 - k_1 + 1)_{k_2}} \frac{(b_2 - a_4)_{k_2}}{k_2!} \left(\sum_{j=3}^{q} b_j - \sum_{j=4}^{q+1} a_j \right)_{k_2} $$

$$ \sum_{k_3=0}^{k_2} \frac{(-k_2)_{k_3}}{\left(\sum_{j=3}^{q} b_j - \sum_{j=4}^{q+1} a_j\right)_{k_3} (a_4 - b_2 - k_2 + 1)_{k_3}} \cdots \frac{(b_{q-2} - a_q)_{k_{q-2}}}{k_{q-2}!} \left(\sum_{j=q-1}^{q} b_j - \sum_{j=q}^{q+1} a_j \right)_{k_{q-2}} $$

$$ \sum_{k_{q-1}=0}^{k_{q-2}} \frac{(-k_{q-2})_{k_{q-1}}}{\left(\sum_{j=q-1}^{q} b_j - \sum_{j=q}^{q+1} a_j\right)_{k_{q-1}} (a_q - b_{q-2} - k_{q-2} + 1)_{k_{q-1}}} \frac{(b_q - a_{q+1})_{k_{q-1}} (b_{q-1} - a_{q+1})_{k_{q-1}}}{k_{q-2}!} \bigwedge $$

$$ \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q \notin \mathbb{Z} \bigwedge \operatorname{Re}(a_3) > 0 \bigwedge \ldots \bigwedge \operatorname{Re}(a_{q+1}) > 0 $$

The logarithmic cases

07.32.06.0006.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left(\sum_{j=0}^{\psi_q - 1} k_j\, (1-j)^j + (1-z)^{\psi_q} \sum_{j=0}^{\infty} \left(p_j + q_j \log(1-z)\right) (1-z)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge $$

$$ \left(k_j = \frac{(-1)^j\, \Gamma(j+a_1)\, \Gamma(j+a_2)}{j!} \sum_{k=0}^{\infty} \frac{(k-j+\psi_q-1)!\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k+a_1+\psi_q)\, \Gamma(k+a_2+\psi_q)} /; \operatorname{Re}(a_3) > -j \land \ldots \land \right. $$

$$ \left. \operatorname{Re}(a_{p+1}) > -j \right) \bigwedge \left(p_j = \frac{(-1)^{\psi_q}\, (a_1+\psi_q)_j\, (a_2+\psi_q)_j}{j!\, (j+\psi_q)!} \left((-1)^j\, j! \sum_{k=j+1}^{\infty} \frac{(k-j-1)!\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k\, (a_2+\psi_q)_k} + \right. \right. $$

$$ \left. \sum_{k=0}^{j} \frac{(-j)_k\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k\, (a_2+\psi_q)_k} \left(\psi(j-k+1) + \psi(j+\psi_q+1) - \psi(j+a_1+\psi_q) - \psi(j+a_2+\psi_q)\right) \right) /; $$

$$ \left. \operatorname{Re}(a_3) > -j-\psi_q \land \ldots \land \operatorname{Re}(a_{p+1}) > -j-\psi_q \right) \bigwedge q_j = \frac{(-1)^{\psi_q-1}\, (a_1+\psi_q)_j\, (a_2+\psi_q)_j}{j!\, (j+\psi_q)!} $$

$$ \sum_{k=0}^{j} \frac{(-j)_k\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k\, (a_2+\psi_q)_k} \bigwedge \psi_2 \in \mathbb{N} $$

07.32.06.0007.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = $$

$$ \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left((z-1)^{\psi_q} \sum_{j=0}^{\infty} \frac{(a_1+\psi_q)_j\, (a_2+\psi_q)_j}{j!\, (j+\psi_q)!} \left(\sum_{k=j+1}^{\infty} \frac{(-1)^j\, j!\, (k-j-1)!}{(a_1+\psi_q)_k\, (a_2+\psi_q)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) + \right. \right. $$

$$ \sum_{k=0}^{j} \frac{(-j)_k}{(a_1+\psi_q)_k\, (a_2+\psi_q)_k} \left(-\log(1-z) + \psi(j-k+1) + \psi(j+\psi_q+1) - \psi(j+a_1+\psi_q) - \psi(j+a_2+\psi_q)\right) $$

$$ \left. \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) \right) (1-z)^j + $$

$$ \left. \sum_{j=0}^{\psi_q-1} \frac{\Gamma(j+a_1)\, \Gamma(j+a_2)}{j!} \sum_{k=0}^{\infty} \frac{(k-j+\psi_q-1)!\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k+a_1+\psi_q)\, \Gamma(k+a_2+\psi_q)} (z-1)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge $$

$$ \psi_q \in \mathbb{N} $$

07.32.06.0008.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{j=0}^{-\psi_q-1} h_j (1-j)^j + \sum_{j=0}^{\infty} \left(u_j + v_j \log(1-z)\right)(1-z)^j \right) /; |1-z| < 1 \bigwedge $$

$$ \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge h_j = \frac{(-1)^j (a_1+\psi_q)_j (a_2+\psi_q)_j (-j-\psi_q-1)!}{j!} \sum_{k=0}^{j} \frac{(-j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \bigwedge $$

$$ \left(u_j = \frac{(a_1+\psi_q)_{j-\psi_q} (a_2+\psi_q)_{j-\psi_q}}{j!\,(j-\psi_q)!} \left((-1)^j (j-\psi_q)! \sum_{k=j-\psi_q+1}^{\infty} \frac{(k-j+\psi_q-1)!\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} + \right.\right. $$

$$ \left. (-1)^{\psi_q} \sum_{k=0}^{j-\psi_q} \frac{(\psi_q-j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \left(\psi(j+1) - \psi(j+a_1) - \psi(j+a_2) + \psi(j-k-\psi_q+1)\right) \right) /; $$

$$ \left. \operatorname{Re}(a_3) > -j \land \ldots \land \operatorname{Re}(a_{q+1}) > -j \right) \bigwedge v_j = \frac{(-1)^{\psi_q-1} (a_1+\psi_q)_{j-\psi_q} (a_2+\psi_q)_{j-\psi_q}}{j!\,(j-\psi_q)!} $$

$$ \sum_{k=0}^{j-\psi_q} \frac{(\psi_q-j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \bigwedge -\psi_q \in \mathbb{N} $$

07.32.06.0009.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = $$

$$ \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{j=0}^{-\psi_q-1} (-j-\psi_q-1)!\,(a_1+\psi_q)_j (a_2+\psi_q)_j \sum_{k=0}^{j} \frac{(-1)^k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})(z-1)^j}{(j-k)!\,(a_1+\psi_q)_k (a_2+\psi_q)_k} + \right. $$

$$ (-1)^{-\psi_q} \sum_{j=\psi_q}^{\infty} \frac{1}{(j-\psi_q)!} \sum_{k=0}^{\infty} \frac{\Gamma(j+k+a_1)\,\Gamma(j+k+a_2)}{(j+k)!\,\Gamma(k+a_1+\psi_q)\,\Gamma(k+a_2+\psi_q)} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) $$

$$ \left(-\log(1-z) + \psi(j+k+1) + \psi(j-\psi_q+1) - \psi(j+k+a_1) - \psi(j+k+a_2)\right)(1-z)^j + $$

$$ \left. (-1)^{-\psi_q-1} \sum_{k=0}^{\infty} \sum_{j=-k}^{\psi_q-1} \frac{\Gamma(j+k+a_1)\,\Gamma(j+k+a_2)}{(j+k)!\,\Gamma(k+a_1+\psi_q)\,\Gamma(k+a_2+\psi_q)} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})(1-z)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge -\psi_q \in \mathbb{N}^+ $$

07.32.06.0010.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} $$

$$ \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j}{j!^2} \left(\sum_{k=0}^{j} \frac{(-j)_k (-\log(1-z) + \psi(j+1) - \psi(j+a_1) - \psi(j+a_2) + \psi(j-k+1))}{(a_1)_k (a_2)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) + \right. $$

$$ \left. (-1)^j j! \sum_{k=j+1}^{\infty} \frac{(k-j-1)!}{(a_1)_k (a_2)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) \right) (1-z)^j /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge \psi_q = 0 $$

The major terms in the general formula for expansions of function ${}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$ at $z == 1$

07.32.06.0011.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1)\,(1 + O(z-1)) + \frac{\Gamma(-\psi_q)}{\prod_{k=1}^{q+1}\Gamma(a_k)}\,(1-z)^{\psi_q}\,(1 + O(z-1))\;/; $$

$$ (z \to 1) \bigwedge \psi_q == \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q \notin \mathbb{Z} $$

07.32.06.0012.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto $$

$$ \frac{\Gamma(\psi_q)}{\prod_{k=3}^{q+1}\Gamma(a_k)} \sum_{k=0}^{\infty} \frac{(\psi_q)_k\, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k + a_1 + \psi_q)\,\Gamma(k + a_2 + \psi_q)}\,(1 + O(z-1)) + \frac{\Gamma(-\psi_q)}{\prod_{k=3}^{q+1}\Gamma(a_k)}\,(1-z)^{\psi_q}\,(1 + O(z-1))\;/; $$

$$ (z \to 1) \bigwedge \psi_q == \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \mathrm{Re}(\psi_q) > 0 \bigwedge \mathrm{Re}(a_3) > 0 \bigwedge \ldots \bigwedge \mathrm{Re}(a_{q+1}) > 0 $$

07.32.06.0013.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1)\,(1 + O(z-1)) - \frac{1}{\prod_{j=1}^{q+1}\Gamma(a_j)}\,\log(1-z)\,(1 + O(z-1))\;/; $$

$$ (z \to 1) \bigwedge \psi_q == \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q == 0 $$

Expansions at $z == \infty$ for $p == q + 1$

The general formulas

07.32.06.0014.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) == \mathcal{A}_{\tilde{F}}\begin{pmatrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{pmatrix} \{z, \tilde{\infty}, \infty\}\;/; z \notin (0, 1) $$

07.32.06.0015.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) == \mathcal{A}_{\tilde{F}}^{(\mathrm{power})}\begin{pmatrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{pmatrix} \{z, \tilde{\infty}, \infty\}\;/; z \notin (0, 1) $$

Case of simple poles

07.32.06.0016.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) == $$

$$ \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j - a_k)}{\prod_{j=1}^{q}\Gamma(b_j - a_k)}\,(-z)^{-a_k} \left(1 + \frac{a_k \prod_{j=1}^{q}(a_k - b_j + 1)}{\prod_{j=1, j\neq k}^{q+1}(a_k - a_j + 1)\, z} + \frac{a_k(a_k+1)\prod_{j=1}^{q}(a_k - b_j + 1)(a_k - b_j + 2)}{2\prod_{j=1, j\neq k}^{q+1}\left((a_k - a_j + 1)(a_k - a_j + 2)\right) z^2} + \ldots\right)\;/; $$

$$ |z| > 1 \land \forall_{\{j,k\},\{j,k\}\in\mathbb{Z} \land j\neq k \land 1\leq j\leq q+1 \land 1\leq k\leq q+1}\,(a_j - a_k \notin \mathbb{Z}) $$

07.32.06.0017.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j - a_k)}{\prod_{j=1}^{q}\Gamma(b_j - a_k)} (-z)^{-a_k} \sum_{i=0}^{\infty} \frac{(a_k)_i \prod_{j=1}^{q}(a_k - b_j + 1)_i}{i! \prod_{j=1, j\neq k}^{q+1}(a_k - a_j + 1)_i z^i} \;/; $$

$$ |z| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

07.32.06.0018.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1}\sum_{i=0}^{\infty} \Gamma\text{Res}\left(\begin{matrix} 0; & 1-a_1, \ldots, 1-a_{q+1}; \\ ; & 1-b_1, \ldots, 1-b_q; \end{matrix} \; 1-a_k, 1, i; -z\right) /; $$

$$ |z| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

07.32.06.0019.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\pi^q}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\csc(\pi(a_j - a_k))}{\prod_{j=1}^{q}\Gamma(b_j - a_k)} (-z)^{-a_k} $$

$$ {}_{q+1}\tilde{F}_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z}\right) /; $$

$$ z \notin (0, 1) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

Case of poles of order r in the points $a_r + k$ /; $r \in \{2, 3, 4\} \bigwedge k \in \mathbb{N}$

07.32.06.0020.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \; \{z, \infty, \infty\}\right) /; z \notin (0, 1) \bigwedge a_k - a_{k-1} \in \mathbb{N} \bigwedge 2 \leq k \leq r \bigwedge $$

$$ a_k - a_1 \notin \mathbb{Z} \bigwedge r + 1 \leq k \leq q + 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge r+1\leq j\leq q+1\wedge r+1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) \bigwedge r \in \{2, 3, 4\} $$

The major terms for expansions of function ${}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$ at $z == \infty$

07.32.06.0021.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j - a_k)}{\prod_{j=1}^{q}\Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

07.32.06.0022.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto $$

$$ \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left(\sum_{k=r+1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j - a_k)}{\prod_{j=1}^{q}\Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) - \sum_{j=2}^{r+1} \Gamma\text{Res}\left(\begin{matrix} 0; & a_1, \ldots, a_{q+1}; \\ ; & b_1, \ldots, b_q; \end{matrix} \; a_{j-1}, j-1, 0; -z\right)\left(1 + O\left(\frac{1}{z}\right)\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge a_k - a_{k-1} \in \mathbb{N} \bigwedge 2 \leq k \leq r \bigwedge a_k - a_1 \notin \mathbb{Z} \bigwedge r + 1 \leq k \leq q + 1 \bigwedge $$

$$ \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge r+1\leq j\leq q+1\wedge r+1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) \bigwedge r \in \{2, 3, 4\} $$

07.32.06.0023.01

$$ {}_{q+1}\tilde{F}_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) \propto \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)}\left(\frac{\Gamma(a_1)\prod_{k=2}^{q+1}\Gamma(a_k-a_1)}{\prod_{k=1}^{q}\Gamma(b_k-a_1)}(-z)^{-a_1}\left(1+O\left(\frac{1}{z}\right)\right)+\frac{(-1)^{a_2-a_1}\,\Gamma(a_2)\prod_{k=3}^{q+1}\Gamma(a_k-a_2)}{(a_2-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_2)}(-z)^{-a_2}\left(\log(-z)+\psi(a_2-a_1+1)-\psi(a_2)+\sum_{k=3}^{q+1}\psi(a_k-a_2)-\sum_{k=1}^{q}\psi(b_k-a_2)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right)+\sum_{k=3}^{q+1}\frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma\left(a_j-a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-a_k\right)}(-z)^{-a_k}\left(1+O\left(\frac{1}{z}\right)\right)\right) /; $$

$$ (|z|\to\infty)\bigwedge a_2-a_1\in\mathbb{N}\bigwedge a_k-a_1\notin\mathbb{Z}\bigwedge 3\leq k\leq q+1\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\bigwedge j\neq k\bigwedge 3\leq j\leq q+1\bigwedge 3\leq k\leq q+1}\left(a_j-a_k\notin\mathbb{Z}\right) $$

07.32.06.0024.01

$$ {}_{q+1}\tilde{F}_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) \propto \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)}\left(\frac{\Gamma(a_1)\prod_{k=2}^{q+1}\Gamma(a_k-a_1)}{\prod_{k=1}^{q}\Gamma(b_k-a_1)}(-z)^{-a_1}\left(1+O\left(\frac{1}{z}\right)\right)+\frac{(-1)^{a_2-a_1}\,\Gamma(a_2)\prod_{k=3}^{q+1}\Gamma(a_k-a_2)}{(a_2-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_2)}(-z)^{-a_2}\left(\log(-z)+\psi(a_2-a_1+1)-\psi(a_2)+\sum_{k=3}^{q+1}\psi(a_k-a_2)-\sum_{k=1}^{q}\psi(b_k-a_2)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right)+\frac{(-1)^{a_2-a_1}\,\Gamma(a_3)\prod_{k=4}^{q+1}\Gamma(a_k-a_3)}{2\,(a_3-a_2)!\,(a_3-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_3)}(-z)^{-a_3}\left(\left(\log(-z)+\psi(a_3-a_1+1)+\psi(a_3-a_2+1)-\psi(a_3)+\sum_{k=4}^{q+1}\psi(a_k-a_3)-\sum_{k=1}^{q}\psi(b_k-a_3)-\gamma\right)^2+\left(\frac{5\pi^2}{6}-\psi^{(1)}(a_3-a_1+1)-\psi^{(1)}(a_3-a_2+1)+\psi^{(1)}(a_3)+\sum_{k=4}^{q+1}\psi^{(1)}(a_k-a_3)-\sum_{k=1}^{q}\psi^{(1)}(b_k-a_3)\right)\right)\left(O\left(\frac{1}{z}\right)+1\right)+\sum_{k=4}^{q+1}\frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma\left(a_j-a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-a_k\right)}(-z)^{-a_k}\left(1+O\left(\frac{1}{z}\right)\right)\right) /; (|z|\to\infty)\bigwedge a_2-a_1\in\mathbb{N}\bigwedge a_3-a_2\in\mathbb{N}\bigwedge $$

$$ a_k-a_1\notin\mathbb{Z}\bigwedge 4\leq k\leq q+1\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\bigwedge j\neq k\bigwedge 4\leq j\leq q+1\bigwedge 4\leq k\leq q+1}\left(a_j-a_k\notin\mathbb{Z}\right) $$

07.32.06.0025.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto \frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left(\frac{\Gamma(a_1)\prod_{k=2}^{q+1}\Gamma(a_k-a_1)}{\prod_{k=1}^{q}\Gamma(b_k-a_1)} (-z)^{-a_1} \left(1+O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2-a_1}\,\Gamma(a_2)\prod_{k=3}^{q+1}\Gamma(a_k-a_2)}{(a_2-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_2)} (-z)^{-a_2} \left(\log(-z)+\psi(a_2-a_1+1) - \psi(a_2) + \sum_{k=3}^{q+1}\psi(a_k-a_2) - \sum_{k=1}^{q}\psi(b_k-a_2) - \gamma\right) \left(1+O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2-a_1}\,\Gamma(a_3)\prod_{k=4}^{q+1}\Gamma(a_k-a_3)}{2\,(a_3-a_2)!\,(a_3-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_3)} (-z)^{-a_3} \left(\left(\log(-z)+\psi(a_3-a_1+1)+\psi(a_3-a_2+1)-\psi(a_3)+\sum_{k=4}^{q+1}\psi(a_k-a_3)-\sum_{k=1}^{q}\psi(b_k-a_3)-\gamma\right)^2 + \left(\frac{5\pi^2}{6} - \psi^{(1)}(a_3-a_1+1) - \psi^{(1)}(a_3-a_2+1) + \psi^{(1)}(a_3) + \sum_{k=4}^{q+1}\psi^{(1)}(a_k-a_3) - \sum_{k=1}^{q}\psi^{(1)}(b_k-a_3)\right)\right) \left(O\left(\frac{1}{z}\right)+1\right) + \frac{(-1)^{-a_1+a_2-a_3+a_4}\,\Gamma(a_4)\prod_{k=5}^{q+1}\Gamma(a_k-a_4)}{6\,(a_4-a_3)!\,(a_4-a_2)!\,(a_4-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_4)} (-z)^{-a_4} \left(\left(\log(-z)+\psi(a_4-a_1+1)+\psi(a_4-a_2+1)+\psi(a_4-a_3+1)-\psi(a_4)+\sum_{k=5}^{q+1}\psi(a_k-a_4)-\sum_{k=1}^{q}\psi(b_k-a_4)-\gamma\right)^3 + \left(3\left(\psi^{(1)}(a_4)-\psi^{(1)}(a_4-a_1+1)-\psi^{(1)}(a_4-a_2+1)-\psi^{(1)}(a_4-a_3+1)+\sum_{k=5}^{q+1}\psi^{(1)}(a_k-a_4)-\sum_{k=1}^{q}\psi^{(1)}(b_k-a_4)\right) + \frac{7\pi^2}{2}\right) \left(\log(-z)+\psi(a_4-a_1+1)+\psi(a_4-a_2+1) + \psi(a_4-a_3+1)-\psi(a_4)+\sum_{k=5}^{q+1}\psi(a_k-a_4)-\sum_{k=1}^{q}\psi(b_k-a_4)-\gamma\right) + \left(\psi^{(2)}(a_4-a_1+1)+\psi^{(2)}(a_4-a_2+1)+\psi^{(2)}(a_4-a_3+1)-\psi^{(2)}(a_4)+\sum_{k=5}^{q+1}\psi^{(2)}(a_k-a_4)-\sum_{k=1}^{q}\psi^{(2)}(b_k-a_4)-2\,\zeta(3)\right)\right) \left(1+O\left(\frac{1}{z}\right)\right) + \sum_{k=5}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j-a_k)}{\prod_{j=1}^{q}\Gamma(b_j-a_k)} (-z)^{-a_k} \left(1+O\left(\frac{1}{z}\right)\right)\right) /; $$

$$ (|z|\to\infty) \bigwedge a_2-a_1\in\mathbb{N} \bigwedge a_3-a_2\in\mathbb{N} \bigwedge a_4-a_3\in\mathbb{N} \bigwedge a_k-a_1\notin \mathbb{Z} \bigwedge 5\le k\le q+1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\bigwedge j\neq k\bigwedge 5\le j\le q+1\bigwedge 5\le k\le q+1} (a_j-a_k\notin\mathbb{Z}) $$

Expansions at $z == \infty$ for polynomial cases

07.32.06.0026.01

$$ {}_p\tilde{F}_q(-n, a_2, a_3, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{k=2}^{p} (a_k)_n}{\prod_{k=1}^{q} \Gamma(b_k + n)} (-z)^n \, {}_{q+1}F_{p-1}\left(-n, 1-n-b_1, \ldots, 1-n-b_q; 1-n-a_2, 1-n-a_3, \ldots, 1-n-a_p; \frac{(-1)^{p+q-1}}{z}\right) /; n \in \mathbb{N}^+ $$

Asymptotic series expansions

Expansions for $q == p$

07.32.06.0027.01

$$ {}_p\tilde{F}_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z) \propto \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_p; \end{matrix} \{z, \infty, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\exp)}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_p; \end{matrix} \{z, \infty, \infty\}\right) /; (|z| \to \infty) $$

07.32.06.0028.01

$$ {}_p\tilde{F}_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z) \propto \frac{1}{\prod_{j=1}^{p} \Gamma(a_j)} e^z z^\chi \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{1}{\prod_{j=1}^{p} \Gamma(a_j)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p} (a_j - a_k \notin \mathbb{Z}) $$

Expansions for $q == p + 1$

07.32.06.0029.01

$$ {}_p\tilde{F}_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \infty, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \infty, \infty\}\right) /; (|z| \to \infty) $$

07.32.06.0030.01

$$ {}_p\tilde{F}_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto \frac{1}{2\sqrt{\pi} \prod_{k=1}^{p} \Gamma(a_k)} (-z)^\chi \left(e^{i\left(\pi\chi + 2\sqrt{-z}\right)} \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + e^{-i\left(\pi\chi + 2\sqrt{-z}\right)} \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right)\right) + $$

$$ \frac{1}{\prod_{k=1}^{p} \Gamma(a_k)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p+1} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p} (a_j - a_k \notin \mathbb{Z}) $$

07.32.06.0031.01

$$ {}_p\tilde{F}_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto $$

$$ \frac{1}{\sqrt{\pi} \prod_{k=1}^{p} \Gamma(a_k)} (-z)^\chi \left(\cos\left(\pi\chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{c_1}{2\sqrt{-z}} \sin\left(\pi\chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{z}\right)\right)\right) + $$

$$ \frac{1}{\prod_{k=1}^{p} \Gamma(a_k)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p+1} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \chi == \frac{1}{2}\left(A_p - B_{p+1} + \frac{1}{2}\right) \bigwedge c_1 == 2\left(\mathfrak{B} - \mathfrak{A} + \frac{1}{4}(3A_p + B_{p+1} - 2)(A_p - B_{p+1}) - \frac{3}{16}\right) \bigwedge A_p == \sum_{k=1}^{p} a_k \bigwedge $$

$$ B_{p+1} == \sum_{k=1}^{p+1} b_k \bigwedge \mathfrak{A} == \sum_{s=2}^{p} \sum_{j=1}^{s-1} a_s a_j \bigwedge \mathfrak{B} == \sum_{s=2}^{p+1} \sum_{j=1}^{s-1} b_s b_j \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p} (a_j - a_k \notin \mathbb{Z}) $$

Expansions for $q \geq p + 2$

07.32.06.0032.01

$$ {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\propto \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\{z,\infty,\infty\}\right)+\mathcal{A}_{\tilde{F}}^{(\text{exp})}\left(\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\{z,\infty,\infty\}\right)/; $$
$$ q-p\geq 2\land(|z|\to\infty)\land\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p}\,(a_j-a_k\notin\mathbb{Z}) $$

07.32.06.0033.01

$$ {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\propto $$
$$ \frac{(2\pi)^{\frac{1-\beta}{2}}\,1}{\sqrt{\beta}\,\prod_{k=1}^{p}\Gamma(a_k)}z^{\chi}\exp\left(\beta z^{1/\beta}\right)\left(1+O\left(\frac{1}{z^{1/\beta}}\right)\right)+\frac{1}{\prod_{k=1}^{p}\Gamma(a_k)}\sum_{k=1}^{p}\frac{\Gamma(a_k)\prod_{\substack{j=1\\ j\neq k}}^{p}\Gamma(a_j-a_k)}{\prod_{j=1}^{q}\Gamma(b_j-a_k)}(-z)^{-a_k}\left(1+O\left(\frac{1}{z}\right)\right)/; $$
$$ q-p\geq 2\bigwedge(|z|\to\infty)\bigwedge\beta=q-p+1\bigwedge\chi=\frac{1}{\beta}\left(\frac{\beta-1}{2}+\sum_{k=1}^{p}a_k-\sum_{k=1}^{q}b_k\right)\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p}\,(a_j-a_k\notin\mathbb{Z}) $$

Expansions for $_0F_2$

07.32.06.0034.01

$$ {}_0\tilde{F}_2(;b_1,b_2;z)\propto\frac{1}{2\sqrt{3}\,\pi}e^{3\sqrt[3]{z}}\,z^{\frac{1}{3}(1-b_1-b_2)} $$
$$ \left(1+\frac{-3b_1^2+3(b_2+1)b_1-3b_2^2+3b_2-2}{9\sqrt[3]{z}}+\frac{1}{162z^{2/3}}\left(9b_1^4-6(3b_2+2)b_1^3+3\left(9b_2^2-3b_2+1\right)b_1^2-\right.\right. $$
$$ \left.\left.3\left(6b_2^3+3b_2^2-17b_2+4\right)b_1+9b_2^4-12b_2^3+3b_2^2-12b_2+4\right)+\ldots\right)/;(|z|\to\infty) $$

07.32.06.0035.01

$$ {}_0\tilde{F}_2(;b_1,b_2;z)\propto\frac{1}{2\sqrt{3}\,\pi}e^{3\sqrt[3]{z}}\,z^{\frac{1}{3}(1-b_1-b_2)}\left(1+O\left(\frac{1}{\sqrt[3]{z}}\right)\right)/;(|z|\to\infty) $$

Expansions for $_0F_3$

07.32.06.0036.01

$$ {}_0\tilde{F}_3(;b_1,b_2,b_3;z)\propto\frac{1}{4\sqrt{2}\,\pi^{3/2}}e^{4\sqrt[4]{z}}\,z^{\frac{1}{4}\left(\frac{3}{2}-b_1-b_2-b_3\right)}\left(1+\frac{-12b_1^2+8(b_2+b_3+1)b_1-12b_2^2-12b_3^2+8b_3+8b_2(b_3+1)-7}{32\sqrt[4]{z}}+\right. $$
$$ \frac{1}{2048\sqrt{z}}\left(144b_1^4-64(3b_2+3b_3+1)b_1^3+8\left(44b_2^2-8(b_3+3)b_2+44b_3^2-24b_3+1\right)b_1^2-16\right. $$
$$ \left(12b_2^3+4(b_3+3)b_2^2+\left(4b_3^2-40b_3-21\right)b_2+12b_3^3+12b_3^2-21b_3+11\right)b_1+144b_2^4+144b_3^4-64b_3^3+8b_3^2- $$
$$ \left.\left.176b_3-64b_2^3(3b_3+1)+8b_2^2\left(44b_3^2-24b_3+1\right)-16b_2\left(12b_3^3+12b_3^2-21b_3+11\right)+121\right)+\ldots\right)/;(|z|\to\infty) $$

07.32.06.0037.01

$$ {}_0\tilde{F}_3(;b_1,b_2,b_3;z)\propto\frac{1}{4\sqrt{2}\,\pi^{3/2}}e^{4\sqrt[4]{z}}\,z^{\frac{1}{4}\left(\frac{3}{2}-b_1-b_2-b_3\right)}\left(1+O\left(\frac{1}{\sqrt[4]{z}}\right)\right)/;(|z|\to\infty) $$

General formulas of asymptotic series expansions

07.32.06.0038.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \mathcal{A}_{\tilde{F}}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) /; (|z| \to \infty) \bigwedge p \leq q+1 $$

07.32.06.0039.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \left(\theta(q-p) - \delta_{q,p+1}\right) \mathcal{A}_{\tilde{F}}^{(\text{exp})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) + \delta_{q,p+1} \mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) /; (|z| \to \infty) \bigwedge p \leq q+1 $$

Main terms of asymptotic expansions

07.32.06.0040.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \frac{1}{\prod_{j=1}^p \Gamma(a_j)} \left(\sum_{k=1}^p \operatorname{res}_s\left(\frac{\Gamma(s)\prod_{j=1}^p \Gamma(a_j - s)}{\prod_{j=1}^q \Gamma(b_j - s)} (-z)^{-s}\right)(a_k)\left(1 + O\left(\frac{1}{z}\right)\right) + \delta_{q,p+1}\, d_1\, (-z)^{\chi} \cos\left(\pi\,\chi + 2\sqrt{-z}\right)\left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + \left(\theta(q-p) - \delta_{q,p+1}\right) d_2\, z^{\chi}\, e^{\beta z^{1/\beta}}\left(1 + O\left(\frac{1}{z^{1/\beta}}\right)\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \beta == q - p + 1 \bigwedge \chi == \frac{1}{\beta}\left(\frac{\beta - 1}{2} + \sum_{k=1}^p a_k - \sum_{k=1}^q b_k\right) \bigwedge 2\, d_2 == d_1 == \frac{2\,(2\,\pi)^{\frac{1-\beta}{2}}}{\sqrt{\beta}} $$

07.32.06.0041.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \sum_{k=1}^p c_k\, (-z)^{-a_k}\left(1 + O\left(\frac{1}{z}\right)\right) + \delta_{q,p+1}\, e_1\, (-z)^{\chi} \cos\left(\pi\,\chi + 2\sqrt{-z}\right)\left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + \left(\theta(q-p) - \delta_{q,p+1}\right) e_2\, z^{\chi}\, e^{\beta z^{1/\beta}}\left(1 + O\left(\frac{1}{z^{1/\beta}}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \beta == q - p + 1 \bigwedge \chi == \frac{1}{\beta}\left(\frac{\beta - 1}{2} + \sum_{k=1}^p a_k - \sum_{k=1}^q b_k\right) \bigwedge c_k == \frac{\Gamma(a_k)\prod_{\substack{j=1 \\ j \neq k}}^p \Gamma(a_j - a_k)}{\left(\prod_{j=1}^p \Gamma(a_j)\right)\prod_{j=1}^q \Gamma(b_j - a_k)} \bigwedge $$

$$ 2\, e_2 == e_1 == \frac{2\,(2\,\pi)^{\frac{1-\beta}{2}}}{\sqrt{\beta}\, \prod_{k=1}^p \Gamma(a_k)} \bigwedge \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq p \land 1 \leq k \leq p} (a_j \neq a_k) $$

07.32.06.0042.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto c\,(1 + O(z-1)) + d\, (1-z)^{\psi_q}\, (1 + O(z-1)) /; $$

$$ (z \to 1) \bigwedge \psi_q == \sum_{j=1}^q b_j - \sum_{j=1}^{q+1} a_j \bigwedge c == {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1) \bigwedge d == \frac{\Gamma(-\psi_q)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \bigwedge \psi_q \notin \mathbb{Z} $$

Residue representations

07.32.06.0043.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \frac{1}{\prod_{k=1}^p \Gamma(a_k)} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s)\prod_{k=1}^p \Gamma(a_k - s)}{\prod_{k=1}^q \Gamma(b_k - s)} (-z)^{-s}\right)(-j) /; p < q + 1 \bigvee p == q + 1 \bigwedge |z| < 1 $$

07.32.06.0044.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = -\frac{1}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1}\sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s)\prod_{k=1}^{q+1}\Gamma(a_k-s)}{\prod_{k=1}^{q}\Gamma(b_k-s)}(-z)^{-s}\right)(a_k+j) /; |z|>1 $$

Continued fraction representations

07.32.10.0001.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = $$

$$ \frac{1}{\prod_{k=1}^{q}\Gamma(b_k)}\left(1+\left(z\prod_{k=1}^{p}a_k\Bigg/\left(\prod_{k=1}^{q}b_k\right)\right)\Bigg/\left(1+-\frac{z\prod_{j=1}^{p}(1+a_j)}{2\prod_{j=1}^{q}(1+b_j)}\Bigg/\left(1+\frac{z\prod_{j=1}^{p}(1+a_j)}{2\prod_{j=1}^{q}(1+b_j)}+\frac{-\frac{z\prod_{j=1}^{p}(2+a_j)}{3\prod_{j=1}^{q}(2+b_j)}}{1+\frac{z\prod_{j=1}^{p}(2+a_j)}{3\prod_{j=1}^{q}(2+b_j)}+\ldots}\right)\right)\right) $$

07.32.10.0002.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = $$

$$ \frac{1}{\prod_{k=1}^{q}\Gamma(b_k)}\left(1+\left(z\prod_{k=1}^{p}a_k\right)\Bigg/\left(\left(\prod_{k=1}^{q}b_k\right)\left(1+\mathrm{K}_k\left(-\frac{z\prod_{j=1}^{p}(k+a_j)}{(k+1)\prod_{j=1}^{q}(k+b_j)}, \frac{z\prod_{j=1}^{p}(k+a_j)}{(k+1)\prod_{j=1}^{q}(k+b_j)}+1\right)_1^{\infty}\right)\right)\right) $$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

The differential equation for the function ${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ has the order $\max(p, q+1)$. It has two ($z = 0$, $z = \tilde{\infty}$ for $p \leq q$) or three ($z = 0$, $z = 1$, $z = \tilde{\infty}$, for $p = q+1$) singular points. If $p \leq q$, then the point $z = 0$ is a regular singular point, while $z = \tilde{\infty}$ is a nonregular (essential) singular point; if $p = q+1$, then all three singular points are regular.

Representation of fundamental system solutions near point $z = 0$ for $p \leq q+1$ in the general case

07.32.13.0002.01

$$z^q w^{(q+1)}(z) + z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) - z^p w^{(p)}(z) - z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{l=1}^{p} a_l\right) w^{(p-1)}(z) + \left(\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz} + b_k - 1\right)\right) w(z) - \prod_{l=1}^{p}\left(z\frac{d}{dz} + a_l\right) w(z) - z^q w^{(q+1)}(z) - z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) + z^p w^{(p)}(z) + z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{k=1}^{p} a_k\right) w^{(p-1)}(z) + w(z)\prod_{l=1}^{p} a_l\right) - w(z)\prod_{l=1}^{p} a_l == 0 \;/;$$

$$\left(w(z) == c_1\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + c_2 \sum_{k=1}^{q} G_{p,q+1}^{2,p}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_k, 1-b_1, \ldots, 1-b_{k-1}, 1-b_{k+1}, \ldots, 1-b_q \end{matrix}\right) + \ldots + c_q \sum_{k=1}^{q} G_{p,q+1}^{q,p}\left((-1)^q z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_{k-1}, 1-b_{k+1}, \ldots, 1-b_q, 1-b_k \end{matrix}\right) + c_{q+1}\, G_{p,q+1}^{q+1,p}\left((-1)^{q+1} z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)\right)$$

07.32.13.0003.01

$$z^q w^{(q+1)}(z) + z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) - z^p w^{(p)}(z) - z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{l=1}^{p} a_l\right) w^{(p-1)}(z) + \left(\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz} + b_k - 1\right)\right) w(z) - \prod_{l=1}^{p}\left(z\frac{d}{dz} + a_l\right) w(z) - z^q w^{(q+1)}(z) - z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) + z^p w^{(p)}(z) + z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{k=1}^{p} a_k\right) w^{(p-1)}(z) + w(z)\prod_{l=1}^{p} a_l\right) - w(z)\prod_{l=1}^{p} a_l == 0 \;/; \left(w(z) == c_1\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + \sum_{k=1}^{q} c_{k+1} z^{1-b_k}\, {}_p\tilde{F}_q(a_1 - b_k + 1, \ldots, a_p - b_k + 1; 2 - b_k, b_1 - b_k + 1, \ldots, b_{k-1} - b_k + 1, b_{k+1} - b_k + 1, \ldots, b_q - b_k + 1; z) \;/; \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq q\land 1\leq k\leq q} (b_j - b_k \notin \mathbb{Z}) \land b_k \notin \mathbb{Z}\right)$$

07.32.13.0001.01

$$\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz} + b_k - 1\right) - \prod_{l=1}^{p}\left(z\frac{d}{dz} + a_l\right)\right) w(z) == 0 \;/; \left(w(z) == c_1\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + \sum_{k=1}^{q} c_{k+1} z^{1-b_k}\, {}_p\tilde{F}_q(a_1 - b_k + 1, \ldots, a_p - b_k + 1; 2 - b_k, b_1 - b_k + 1, \ldots, b_{k-1} - b_k + 1, b_{k+1} - b_k + 1, \ldots, b_q - b_k + 1; z) \;/; \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq q\land 1\leq k\leq q} (b_j - b_k \notin \mathbb{Z}) \land b_k \notin \mathbb{Z}\right)$$

07.32.13.0004.01

$$W_z\Big({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z), z^{1-b_1}\, {}_p\tilde{F}_q(a_1 - b_1 + 1, \ldots, a_p - b_1 + 1; 2 - b_1, 1 - b_1 + b_2, \ldots, 1 - b_1 + b_q; z), \ldots, z^{1-b_k}\, {}_p\tilde{F}_q(a_1 - b_k + 1, \ldots, a_p - b_k + 1; 2 - b_k, b_1 - b_k + 1, \ldots, b_{k-1} - b_k + 1, b_{k+1} - b_k + 1, \ldots, b_q - b_k + 1; z), \ldots, z^{1-b_q}\, {}_p\tilde{F}_q(a_1 - b_q + 1, \ldots, a_p - b_q + 1; 2 - b_q, b_1 - b_q + 1, \ldots, b_{q-1} - b_q + 1; z)\Big) \equiv \pi^{-\frac{q(1+q)}{2}} \left(\prod_{k=1}^{q} \sin(\pi\, b_k)\right) \prod_{k=1}^{q} \prod_{j=1}^{k-1} \sin\big(\pi\,(b_j - b_k)\big)\, z^{-\frac{1}{2}(q-1)\, q - \sum_{k=1}^{q} b_k} \left(\delta_{p,q+1}\, (1-z)^{-q-\sum_{l=1}^{q+1} a_l + \sum_{k=1}^{q} b_k} + e^z\, \delta_{p,q} + \theta(-p + q - 1)\right)$$

Representation of fundamental system solutions near point $z \equiv 1$ for $p \equiv q + 1$ in the general case

Below representation includes functions of two kinds. The function $G_{q+1,q+1}^{q+1,0}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)$ is the piecewise analytical function with a discontinuity on the unite circle $|z| \equiv 1$. It has singularity near point $z \equiv 1$of the form const $(1-z)^{\psi_q}\, (1 + O(z-1))$, when $|z| < 1$. The functions $G_{q+3,q+3}^{2,q+3}\left(z \,\middle|\, \begin{matrix} 0, b_k, 1-a_1, \ldots, 1-a_{q+1} \\ 0, b_k, 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)$ are the analytical functions and are bounded near point $z \equiv 1$.

The function $G_{q+1,q+1}^{q+1,0}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)$ inside of $|z| < 1$can be reprezented through hypergeometric functions defined for all complex z.

07.32.13.0005.01

$$\left(\frac{d}{dz} \prod_{k=1}^{q} \left(z \frac{d}{dz} + b_k - 1\right) - \prod_{l=1}^{q+1} \left(z \frac{d}{dz} + a_l\right)\right) w(z) \equiv 0 \,/;$$

$$\left(w(z) \equiv c_1\, G_{q+1,q+1}^{q+1,0}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right) + \sum_{k=1}^{q} c_{k+1}\, G_{q+3,q+3}^{2,q+3}\left(z \,\middle|\, \begin{matrix} 0, b_k, 1-a_1, \ldots, 1-a_{q+1} \\ 0, b_k, 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right) \bigwedge |z| < 1 \bigwedge \psi_q \equiv \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q \notin \mathbb{Z} \bigwedge \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \le j \le q \land 1 \le k \le q} \left(b_j - b_k \notin \mathbb{Z}\right) \bigwedge b_k \notin \mathbb{Z}\right)$$

Representation of fundamental system solutions near point $z = \tilde{\infty}$ for $p \geq q + 1$ in the general case

07.32.13.0006.01

$$\left(\frac{d}{dz} \prod_{k=1}^{q} \left(z \frac{d}{dz} + b_k - 1\right) - \prod_{l=1}^{p} \left(z \frac{d}{dz} + a_l\right)\right) w(z) \equiv 0 \,/;$$

$$\left(w(z) \equiv \sum_{k=1}^{p} c_k\, z^{-a_k}\, {}_{q+1}\tilde{F}_{p-1}\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_p + a_k; \frac{(-1)^{1-p+q}}{z}\right) \bigwedge \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \le j \le p \land 1 \le k \le p} \left(a_j - a_k \notin \mathbb{Z}\right)\right)$$

Transformations

Products, sums, and powers of the direct function

Products of the direct function

07.32.16.0001.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; c\,z)\,{}_r\tilde{F}_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; d\,z) = \sum_{k=0}^{\infty} c_k\, z^k \,/;$$

$$ c_k = \frac{(-1)^{k\,t}\, d^k \prod_{j=1}^{r} \Gamma(1-\alpha_j)}{k! \prod_{j=1}^{s} \Gamma(k+\beta_j)}\, {}_{p+s+1}\tilde{F}_{q+r}\Bigg(-k, 1-\beta_1-k, \ldots, 1-\beta_s-k, a_1, \ldots, a_p; 1-\alpha_1-k, \ldots, 1-\alpha_r-k, b_1, \ldots, b_q; \frac{(-1)^{r+s-1}\, c}{d}\Bigg) \bigvee c_k = \frac{(-1)^{k\,p}\, c^k \prod_{j=1}^{p} \Gamma(1-a_j)}{k! \prod_{j=1}^{q} \Gamma(k+b_j)}$$

$$ {}_{q+r+1}\tilde{F}_{p+s}\Bigg(-k, 1-b_1-k, \ldots, 1-b_q-k, \alpha_1, \ldots, \alpha_r; 1-a_1-k, \ldots, 1-a_r-k, \beta_1, \ldots, \beta_s; \frac{(-1)^{p+q-1}\, d}{c}\Bigg)$$

07.32.16.0002.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; c\,z)\,{}_r\tilde{F}_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; d\,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \frac{\left(\prod_{j=1}^{p} (a_j)_m\right)\left(\prod_{j=1}^{r} (\alpha_j)_{k-m}\right) d^{k-m}\, c^m\, z^k}{\left(\prod_{j=1}^{q} \Gamma(b_j+m)\right)\left(\prod_{j=1}^{s} \Gamma(\beta_j+k-m)\right) m!\,(k-m)!}$$

07.32.16.0003.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; c\,z)\,{}_r\tilde{F}_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; d\,z) = \tilde{F}_{0:q;s}^{0:p;r}\left(\begin{matrix} : a_1, \ldots, a_p; \alpha_1, \ldots, \alpha_r; \\ : b_1, \ldots, b_q; \beta_1, \ldots, \beta_s; \end{matrix}\; c\,z, d\,z\right)$$

Identities

Recurrence identities

Distant neighbors with respect to *q*

07.32.17.0001.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{1}{\prod_{j=3}^{q+1} \Gamma(a_j)} \sum_{k=0}^{\infty} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})\, {}_2\tilde{F}_1(a_1, a_2; a_1+a_2+\psi_q+k; z) \,/;\, \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j$$

Functional identities

Relations between contiguous functions

07.32.17.0002.01

$$ b\, {}_p\tilde{F}_q(a, b+1, a_3, \ldots, a_p; b_1, \ldots, b_q; z) - a\, {}_p\tilde{F}_q(a+1, b, a_3, \ldots, a_p; b_1, \ldots, b_q; z) + (a-b)\, {}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; b_1, \ldots, b_q; z) = 0$$

07.32.17.0003.01

$$ {}_p\tilde{F}_q(a, a_2, \ldots, a_p; c, b_2, \ldots, b_q; z) - a\, {}_p\tilde{F}_q(a+1, a_2, \ldots, a_p; c+1, b_2, \ldots, b_q; z) + (a-c)\, {}_p\tilde{F}_q(a, a_2, \ldots, a_p; c+1, b_2, \ldots, b_q; z) = 0$$

07.32.17.0004.01

$$d\,{}_p\tilde{F}_q(a_1, \ldots, a_p; c+1, d, b_3, \ldots, b_q; z) - {}_p\tilde{F}_q(a_1, \ldots, a_p; c, d+1, b_3, \ldots, b_q; z) + (c-d)\,{}_p\tilde{F}_q(a_1, \ldots, a_p; c+1, d+1, b_3, \ldots, b_q; z) == 0$$

07.32.17.0005.01

$$(a-b)\,{}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; c, b_2, \ldots, b_q; z) - a(c-b)\,{}_p\tilde{F}_q(a+1, b, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) + b(c-a)\,{}_p\tilde{F}_q(a, b+1, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) == 0$$

07.32.17.0006.01

$$(d-a)\,{}_p\tilde{F}_q(a, a_2, \ldots, a_p; c, d+1, b_3, \ldots, b_q; z) - d(c-a)\,{}_p\tilde{F}_q(a, a_2, \ldots, a_p; c+1, d, b_3, \ldots, b_q; z) + a(c-d)\,{}_p\tilde{F}_q(a+1, a_2, \ldots, a_p; c+1, d+1, b_3, \ldots, b_q; z) == 0$$

07.32.17.0007.01

$${}_p\tilde{F}_q(a, a_2, \ldots, a_p; b_1, \ldots, b_q; z) - {}_p\tilde{F}_q(a+1, a_2, \ldots, a_p; b_1, \ldots, b_q; z) + z\left(\prod_{j=2}^{p} a_j\right) {}_p\tilde{F}_q(a+1, a_2+1, \ldots, a_p+1; b_1+1, \ldots, b_q+1; z) == 0$$

07.32.17.0008.01

$${}_p\tilde{F}_q(a_1, \ldots, a_p; c, b_2, \ldots, b_q; z) - c\,{}_p\tilde{F}_q(a_1, \ldots, a_p; c+1, b_2, \ldots, b_q; z) - z\left(\prod_{j=1}^{p} a_j\right) {}_p\tilde{F}_q(a_1+1, \ldots, a_p+1; c+2, b_2+1, \ldots, b_q+1; z) == 0$$

07.32.17.0009.01

$${}_p\tilde{F}_q(a, b+1, a_3, \ldots, a_p; b_1, \ldots, b_q; z) - {}_p\tilde{F}_q(a+1, b, a_3, \ldots, a_p; b_1, \ldots, b_q; z) - z\,(b-a)\left(\prod_{j=3}^{p} a_j\right) {}_p\tilde{F}_q(a+1, b+1, a_3+1, \ldots, a_p+1; b_1+1, \ldots, b_q+1; z) == 0$$

07.32.17.0010.01

$${}_p\tilde{F}_q(a, a_2, \ldots, a_p; c, b_2, \ldots, b_q; z) - c\,{}_p\tilde{F}_q(a+1, a_2, \ldots, a_p; c+1, b_2, \ldots, b_q; z) - z\,(c-a)\left(\prod_{j=2}^{p} a_j\right) {}_p\tilde{F}_q(a+1, a_2+1, \ldots, a_p+1; c+2, b_2+1, \ldots, b_q+1; z) == 0$$

07.32.17.0011.01

$${}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; c, b_2, \ldots, b_q; z) - a\,{}_p\tilde{F}_q(a+1, b+1, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) - (c-a)\,{}_p\tilde{F}_q(a, b+1, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) + z\,a\left(\prod_{j=3}^{p} a_j\right) {}_p\tilde{F}_q(a+1, b+1, a_3+1, \ldots, a_p+1; c+1, b_2+1, \ldots, b_q+1; z) == 0$$

07.32.17.0012.01

$${}_p\tilde{F}_q(a+1, b+1, a_3, \ldots, a_p; c+1, d+1, e+1, b_4, \ldots, b_q; z) - \frac{(a-e)\,(b-e)}{a\,b\,(c-e)\,(d-e)}\,{}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; c, d, e+1, b_4, \ldots, b_q; z) - \frac{(a-d)\,(b-d)}{a\,b\,(c-d)\,(e-d)}\,{}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; c, d+1, e, b_4, \ldots, b_q; z) - \frac{(a-c)\,(b-c)}{a\,b\,(d-c)\,(e-c)}\,{}_p\tilde{F}_q(a, b, a_3, \ldots, a_p; c+1, d, e, b_4, \ldots, b_q; z) == 0$$

07.32.17.0013.01

$$ {}_p\tilde{F}_q(a, b, c, a_4, \ldots, a_p; d, e, b_3, \ldots, b_q; z) - \frac{a\, b\, (d-c)\, (e-c)}{(a-c)\, (b-c)}\, {}_p\tilde{F}_q(a+1, b+1, c, a_4, \ldots, a_p; d+1, e+1, b_3, \ldots, b_q; z) - \frac{a\, c\, (d-b)\, (e-b)}{(a-b)\, (c-b)}\, {}_p\tilde{F}_q(a+1, b, c+1, a_4, \ldots, a_p; d+1, e+1, b_3, \ldots, b_q; z) - \frac{b\, c\, (d-a)\, (e-a)}{(b-a)\, (c-a)}\, {}_p\tilde{F}_q(a, b+1, c+1, a_4, \ldots, a_p; d+1, e+1, b_3, \ldots, b_q; z) = 0 $$

07.32.17.0014.01

$$ \left(a + z \sum_{j=1}^{q} (a_{j+1} - b_j)\right) {}_{q+1}\tilde{F}_q(a, a_2, \ldots, a_{q+1}; b_1, \ldots, b_q; z) + z \sum_{j=1}^{q} \frac{(b_j - a) \prod_{k=1}^{q} (b_j - a_{k+1})}{\prod_{k=1, k \neq j}^{q} (b_j - b_k)}\, {}_{q+1}\tilde{F}_q(a, a_2, \ldots, a_{q+1}; b_1, \ldots, b_{j-1}, b_j + 1, b_{j+1}, \ldots, b_q; z) = a(1-z)\, {}_{q+1}\tilde{F}_q(a+1, a_2, \ldots, a_{q+1}; b_1, \ldots, b_q; z) $$

Relations of special kind

07.32.17.0015.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; -n, b_2, \ldots, b_q; z) = z^{n+1} \left(\prod_{j=1}^{p} (a_j)_{n+1}\right) {}_p\tilde{F}_q(a_1 + n + 1, \ldots, a_p + n + 1; n + 2, b_2 + n + 1, \ldots, b_q + n + 1; z) \;/;\; n \in \mathbb{N} $$

07.32.17.0016.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; c, 1-c, b_3, \ldots, b_q; z) - {}_p\tilde{F}_q(a_1, \ldots, a_p; -c, 1+c, b_3, \ldots, b_q; z) = 2\, c\, {}_p\tilde{F}_q(a_1, \ldots, a_p; 1+c, 1-c, b_3, \ldots, b_q; z) $$

07.32.17.0017.01

$$ {}_p\tilde{F}_q(a, a_2, \ldots, a_p; -a, 1+a, b_3, \ldots, b_q; z) + 2\, a\, {}_p\tilde{F}_q(a, a_2, \ldots, a_p; 1-a, 1+a, b_3, \ldots, b_q; z) = \frac{1}{\Gamma(a)}\, {}_{p-1}\tilde{F}_{q-1}(a_2, \ldots, a_p; 1-a, b_3, \ldots, b_q; z) $$

07.32.17.0018.01

$$ \frac{1}{\Gamma(1-a)}\, {}_p\tilde{F}_q(a, a_2, \ldots, a_p; 1+a, b_2, \ldots, b_q; z) + \frac{1}{\Gamma(1+a)}\, {}_p\tilde{F}_q(-a, a_2, \ldots, a_p; 1-a, b_2, \ldots, b_q; z) = 2\, {}_{p+1}\tilde{F}_{q+1}(a, -a, a_2, \ldots, a_p; 1+a, 1-a, b_2, \ldots, b_q; z) $$

07.32.17.0019.01

$$ {}_p\tilde{F}_q(a, 1-a, a_3, \ldots, a_p; b_1, \ldots, b_q; z) + {}_p\tilde{F}_q(-a, 1+a, a_3, \ldots, a_p; b_1, \ldots, b_q; z) = 2\, {}_p\tilde{F}_q(a, -a, a_3, \ldots, a_p; b_1, \ldots, b_q; z) $$

Division on even and odd parts and generalization

07.32.17.0020.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = A^+(z) + A^-(z) \;/;\; A^+(z) = \frac{1}{2}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -z)\right) \bigwedge A^-(z) = \frac{1}{2}\left({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) - {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -z)\right) $$

07.32.17.0021.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = A^+(z) + A^-(z) /; $$

$$ A^+(z) = 2^{q-\eta} \pi^{\frac{q+1}{2}} {}_{2p}\tilde{F}_{2q+1}\left(\frac{a_1}{2}, \ldots, \frac{a_p}{2}, \frac{a_1+1}{2}, \ldots, \frac{a_p+1}{2}; \frac{1}{2}, \frac{b_1}{2}, \ldots, \frac{b_q}{2}, \frac{b_1+1}{2}, \ldots, \frac{b_q+1}{2}; 4^{p-q-1} z^2\right) \bigwedge $$

$$ A^-(z) = 2^{-\eta-1} \pi^{\frac{q+1}{2}} \left(\prod_{j=1}^{p} a_j\right) z $$

$$ {}_{2p}\tilde{F}_{2q+1}\left(\frac{a_1+1}{2}, \ldots, \frac{a_p+1}{2}, \frac{a_1+2}{2}, \ldots, \frac{a_p+2}{2}; \frac{3}{2}, \frac{b_1+1}{2}, \ldots, \frac{b_q+1}{2}, \frac{b_1+2}{2}, \ldots, \frac{b_q+2}{2}; 4^{p-q-1} z^2\right) \bigwedge \eta = \sum_{j=1}^{q} b_j $$

07.32.17.0022.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = $$

$$ (2\pi)^{\frac{n-1}{2}(q+1)} \sum_{k=0}^{n-1} n^{-(q+1)k-\eta} z^k \left(\prod_{j=1}^{p} (a_j)_k\right) {}_{np+1}\tilde{F}_{nq+n}\left(1, \frac{a_1+k}{n}, \ldots, \frac{a_1+k+n-1}{n}, \ldots, \frac{a_p+k}{n}, \ldots, \frac{a_p+k+n-1}{n}; \frac{k+1}{n}, \right. $$

$$ \left. \ldots, \frac{k+n}{n}, \frac{b_1+k}{n}, \ldots, \frac{b_1+k+n-1}{n}, \ldots, \frac{b_q+k}{n}, \ldots, \frac{b_q+k+n-1}{n}; n^{n(p-q-1)} z^n\right) /; \eta = \sum_{j=1}^{q} b_j + \frac{q-1}{2} $$

Case ${}_{q+1}F_q$

07.32.17.0023.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\pi^q}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{q+1} \csc(\pi(a_j - a_k))}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{-a_k} $$

$$ {}_{q+1}\tilde{F}_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z}\right) /; $$

$$ z \notin (0, 1) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

07.32.17.0024.01

$$ \sum_{k=1}^{m} \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + b_k)}{\prod_{j=1, j\neq k}^{m} \sin(\pi(b_j - b_k)) \prod_{j=n+1}^{q+1} \Gamma(a_j - b_k)} z^{b_k} $$

$$ {}_{q+1}\tilde{F}_q(1 - a_1 + b_k, \ldots, 1 - a_{q+1} + b_k; 1 - b_1 + b_k, \ldots, 1 - b_{k-1} + b_k, 1 - b_{k+1} + b_k, \ldots, 1 - b_{q+1} + b_k; (-1)^{q-m-n+1} z) = $$

$$ \pi^{n-m} \sum_{k=1}^{n} \frac{\prod_{j=1}^{m} \Gamma(1 - a_k + b_j)}{\prod_{j=1, j\neq k}^{n} \sin(\pi(a_k - a_j)) \prod_{j=m+1}^{q+1} \Gamma(a_k - b_j)} z^{a_k - 1} $$

$$ {}_{q+1}\tilde{F}_q\left(1 - a_k + b_1, \ldots, 1 - a_k + b_{q+1}; 1 + a_1 - a_k, \ldots, 1 + a_{k-1} - a_k, 1 + a_{k+1} - a_k, \ldots, 1 + a_{q+1} - a_k; \frac{(-1)^{q-m-n+1}}{z}\right) /; $$

$$ m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge q \in \mathbb{N} \bigwedge m \leq q+1 \bigwedge n \leq q+1 \bigwedge (m + n - q > 2 \bigvee (m + n - q = 2 \bigwedge z \notin (-1, 0))) $$

07.32.17.0025.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; w z) = (1 - z)^{-a_1} \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} {}_{q+1}\tilde{F}_q(-k, a_2, \ldots, a_{q+1}; b_1, \ldots, b_q; w) \left(\frac{z}{z-1}\right)^k $$

Differentiation

Low-order differentiation

With respect to a_1

07.32.20.0001.01

$$ {}_p\tilde{F}_q^{(\{1,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^{\infty} \frac{\psi(k+a_1)\left(\prod_{j=1}^{p}(a_j)_k\right) z^k}{k!\prod_{j=1}^{q}\Gamma(k+b_j)} - \psi(a_1)\,{}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) \,/;\, q = p-1 \land |z|<1 \lor q \geq p $$

07.32.20.0002.01

$$ {}_p\tilde{F}_q^{(\{1,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = z\,\Gamma(a_1)\left(\prod_{j=1}^{p} a_j\right)\tilde{F}_{q+1\times0\times1}^{p\,1\times2}\left(\begin{matrix} a_1+1,\ldots,a_p+1;1;1,a_1; \\ 2,b_1+1,\ldots,b_q+1;;a_1+1; \end{matrix}\, z, z\right) $$

With respect to b_1

07.32.20.0003.01

$$ {}_p\tilde{F}_q^{(\{0,\ldots,0\},\{1,0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = -\sum_{k=0}^{\infty} \frac{\psi(k+b_1)\left(\prod_{j=1}^{p}(a_j)_k\right) z^k}{k!\prod_{j=1}^{q}\Gamma(k+b_j)} \,/;\, q = p-1 \land |z|<1 \lor q \geq p $$

07.32.20.0004.01

$$ {}_p\tilde{F}_q^{(\{0,\ldots,0\},\{1,0,\ldots,0\},0)}(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = -z\,\Gamma(b_1)\left(\prod_{j=1}^{p} a_j\right)\tilde{F}_{q+1\times0\times1}^{p\,1\times2}\left(\begin{matrix} a_1+1,\ldots,a_p+1;1;1,b_1; \\ 2,b_1+1,\ldots,b_q+1;;b_1+1; \end{matrix}\, z, z\right) - \psi(b_1)\,{}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) $$

With respect to element of parameters ||| With respect to element of parameters

07.32.20.0019.01

$$ \frac{\partial^n\,{}_p\tilde{F}_q(a,a_2,\ldots,a_p;a+1,b_2,\ldots,b_q;z)}{\partial a^n} = n!\,z\left(\prod_{j=2}^{p} a_j\right)\Big(\Gamma(a+1)\,{}_{p+1}\tilde{F}_{q+1}(a+1,a+1,a_2+1,\ldots,a_p+1;a+2,a+2,b_2+1,\ldots,b_q+1;z) + {}_p\tilde{F}_q(a+1,a_2+1,\ldots,a_p+1;a+2,b_2+1,\ldots,b_q+1;z)\,\psi(a+1)\Big) - \frac{\psi(a+1)}{\Gamma(a+1)}\,{}_{p-1}\tilde{F}_{q-1}(a_2,\ldots,a_p;b_2,\ldots,b_q;z) $$

07.32.20.0020.01

$$ \frac{\partial\,{}_p\tilde{F}_q(a+1,a_2,\ldots,a_p;a,b_2,\ldots,b_q;z)}{\partial a} = -\frac{\psi(a)}{\Gamma(a)}\,{}_{p-1}\tilde{F}_{q-1}(a_2,\ldots,a_p;b_2,\ldots,b_q;z) - \frac{z\left(\prod_{j=2}^{p} a_j\right)\psi(a+1)}{\Gamma(a+1)}\,{}_{p-1}\tilde{F}_{q-1}(a_2+1,\ldots,a_p+1;b_2+1,\ldots,b_q+1;z) $$

With respect to z

07.32.20.0005.01

$$\frac{\partial\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z} == \left(\prod_{j=1}^{p} a_j\right) {}_p\tilde{F}_q(a_1+1, \ldots, a_p+1; b_1+1, \ldots, b_q+1; z)$$

07.32.20.0006.01

$$\frac{\partial^2\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^2} == \left(\prod_{j=1}^{p} a_j\,(a_j+1)\right) {}_p\tilde{F}_q(a_1+2, \ldots, a_p+2; b_1+2, \ldots, b_q+2; z)$$

Symbolic differentiation

With respect to a_1

07.32.20.0007.01

$${}_p\tilde{F}_q^{(\{n,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \sum_{k=0}^{\infty} \frac{\prod_{j=2}^{p} (a_j)_k}{k!\,\prod_{j=1}^{q} \Gamma(k+b_j)}\, \frac{\partial^n (a_1)_k}{\partial a_1^n}\, z^k \,/;\, n \in \mathbb{N}^+ \bigwedge q == p-1 \bigwedge |z| < 1 \bigvee q \geq p$$

With respect to b_1

07.32.20.0008.01

$${}_p\tilde{F}_q^{(\{0,\ldots,0\},\{n,0,\ldots,0\},0)}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) == \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{k!\,\prod_{j=2}^{q} \Gamma(k+b_j)}\, \frac{\partial^n \frac{1}{\Gamma(k+b_1)}}{\partial b_1^n}\, z^k \,/;$$

$$|z| < 1 \bigwedge n \in \mathbb{N}^+ \bigwedge q == p-1 \bigwedge |z| < 1 \bigvee q \geq p$$

With respect to element of parameters ||| With respect to element of parameters

07.32.20.0021.01

$$\frac{\partial^n\, {}_p\tilde{F}_q(a, a_2, \ldots, a_p; a+1, b_2, \ldots, b_q; z)}{\partial a^n} ==$$

$$\frac{\partial^n \frac{1}{\Gamma(a+1)}}{\partial a^n}\, {}_{p-1}\tilde{F}_{q-1}(a_2, \ldots, a_p; b_2, \ldots, b_q; z) - n!\, z \left(\prod_{j=2}^{p} a_j\right) \sum_{k=0}^{n} \frac{(-1)^k\, \Gamma(a+1)^{k+1}}{(n-k)!}\, \frac{\partial^{n-k} \frac{1}{\Gamma(a+1)}}{\partial a^{n-k}}$$

$${}_{p+k}\tilde{F}_{q+k}(a+1, \ldots, a+1, a_2+1, \ldots, a_p+1; a+2, \ldots, a+2, b_2+1, \ldots, b_q+1; z) \,/;\, n \in \mathbb{N}^+$$

07.32.20.0022.01

$$\frac{\partial^n\, {}_p\tilde{F}_q(a+1, a_2, \ldots, a_p; a, b_2, \ldots, b_q; z)}{\partial a^n} ==$$

$$\frac{\partial^n \frac{1}{\Gamma(a)}}{\partial a^n}\, {}_{p-1}\tilde{F}_{q-1}(a_2, \ldots, a_p; b_2, \ldots, b_q; z) + z\, \frac{\partial^n \frac{1}{\Gamma(a+1)}}{\partial a^n} \left(\prod_{j=2}^{p} a_j\right) {}_{p-1}\tilde{F}_{q-1}(a_2+1, \ldots, a_p+1; b_2+1, \ldots, b_q+1; z) \,/;\, n \in \mathbb{N}^+$$

With respect to z

07.32.20.0009.01

$$\frac{\partial^n\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^n} == \left(\prod_{j=1}^{p} (a_j)_n\right) {}_p\tilde{F}_q(a_1+n, \ldots, a_p+n; b_1+n, \ldots, b_q+n; z) \,/;\, n \in \mathbb{N}^+$$

07.32.20.0010.01

$$\frac{\partial^n {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^n} = z^{-n} \; {}_{p+1}\tilde{F}_{q+1}(1, a_1, \ldots, a_p; 1-n, b_1, \ldots, b_q; z) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0011.01

$$\frac{\partial^n \left(z^\alpha \, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = z^{\alpha-n} \, \Gamma(\alpha+1) \, {}_{p+1}\tilde{F}_{q+1}(\alpha+1, a_1, \ldots, a_p; \alpha-n+1, b_1, \ldots, b_q; z) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0012.01

$$\frac{\partial^n \left(z^{a+n-1} \, {}_p\tilde{F}_q(a, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = (a)_n \, z^{a-1} \, {}_p\tilde{F}_q(a+n, a_2, \ldots, a_p; b_1, \ldots, b_q; z) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0013.01

$$\frac{\partial^n \left(z^{c-1} \, {}_p\tilde{F}_q(a_1, \ldots, a_p; c, b_2, \ldots, b_q; z)\right)}{\partial z^n} = z^{c-n-1} \, {}_p\tilde{F}_q(a_1, \ldots, a_p; c-n, b_2, \ldots, b_q; z) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0014.01

$$\frac{\partial^n \left(z^n \, {}_p\tilde{F}_q\left(-n, a_2, \ldots, a_p; \frac{1}{2}, b_2, \ldots, b_q; z\right)\right)}{\partial z^n} = n! \, {}_{p+1}\tilde{F}_{q+1}\left(-n, n+1, a_2, \ldots, a_p; \frac{1}{2}, 1, b_2, \ldots, b_q; z\right) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0015.01

$$\frac{\partial^n \left(z^\alpha \, {}_p\tilde{F}_q(-n, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = \Gamma(\alpha+1) \, z^{\alpha-n} \, {}_{p+1}\tilde{F}_{q+1}(-n, \alpha+1, a_2, \ldots, a_p; \alpha-n+1, b_1, \ldots, b_q; z) \;/;\; n \in \mathbb{N}^+$$

07.32.20.0016.01

$$\frac{\partial^n \left(z^\alpha \, {}_p\tilde{F}_q\left(-\frac{n}{r}, \frac{-n+1}{r}, \ldots, \frac{-n+r-1}{r}, a_{r+1}, \ldots, a_p; b_1, \ldots, b_q; z^m\right)\right)}{\partial z^n} =$$

$$m^{n-\alpha-\frac{1}{2}} (2\pi)^{\frac{m-1}{2}} \, \Gamma(\alpha+1) \, z^{\alpha-n} \, {}_{p+m}\tilde{F}_{q+m}\left(-\frac{n}{r}, \frac{-n+1}{r}, \ldots, \frac{-n+r-1}{r}, \frac{\alpha+1}{m}, \frac{\alpha+2}{m}, \ldots, \frac{\alpha+m}{m}, \right.$$

$$\left. a_{r+1}, \ldots, a_p; \frac{\alpha-n+1}{m}, \frac{\alpha-n+2}{m}, \ldots, \frac{\alpha-n+m}{m}, b_1, \ldots, b_q; z^m\right) \;/;\; r \in \mathbb{N}^+ \land m \in \mathbb{N}^+ \land n \in \mathbb{N}^+$$

07.32.20.0017.01

$$\frac{\partial^n \left(e^{-z} \, {}_p\tilde{F}_q(-n, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} =$$

$$(-1)^n \, e^{-z} \sum_{k=0}^{n} \frac{(-n)_k \, z^k}{k!} \, {}_{p+1}\tilde{F}_q(-n, k-n, a_2+k, \ldots, a_p+k; b_1+k, \ldots, b_q+k; z) \;/;\; n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

07.32.20.0018.01

$$\frac{\partial^\alpha {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^\alpha} = z^{-\alpha} \; {}_{p+1}\tilde{F}_{q+1}(1, a_1, \ldots, a_p; 1-\alpha, b_1, \ldots, b_q; z)$$

Integration

Indefinite integration

Involving only one direct function

07.32.21.0001.01

$$\int {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\, dz = \frac{1}{\prod_{j=1}^{p}(a_j - 1)}\, {}_p\tilde{F}_q(a_1 - 1, \ldots, a_p - 1; b_1 - 1, \ldots, b_q - 1; z)$$

Involving one direct function and elementary functions

Involving power function

07.32.21.0002.01

$$\int z^{\alpha-1}\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\, dz = \Gamma(\alpha)\, z^{\alpha}\, {}_{p+1}\tilde{F}_{q+1}(\alpha, a_1, \ldots, a_p; \alpha + 1, b_1, \ldots, b_q; z)$$

Definite integration

For the direct function itself

07.32.21.0003.01

$$\int_0^{\infty} t^{\alpha-1}\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -t)\, dt = \frac{\Gamma(\alpha) \prod_{k=1}^{p} \Gamma(a_k - \alpha)}{\left(\prod_{k=1}^{p} \Gamma(a_k)\right) \prod_{k=1}^{q} \Gamma(b_k - \alpha)} \;/;$$

$$0 < \operatorname{Re}(\alpha) < \min(\operatorname{Re}(a_1), \ldots, \operatorname{Re}(a_p)) \land p - 1 \leq q \leq p \bigvee$$

$$0 < \operatorname{Re}(\alpha) < \min\left(\operatorname{Re}(a_1), \ldots, \operatorname{Re}(a_p), \frac{1}{4} - \frac{1}{2} \operatorname{Re}\left(\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} b_k\right)\right) \bigwedge q = p + 1$$

Summation

Infinite summation

07.32.23.0001.01

$$\sum_{k=0}^{\infty} \frac{(a_1)_k}{k!}\, {}_{q+1}\tilde{F}_q(-k, a_2, \ldots, a_{q+1}; b_1, \ldots, b_q; w)\, z^k = \left(\frac{1}{1-z}\right)^{a_1} {}_{q+1}\tilde{F}_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; \frac{z\, w}{z - 1}\right)$$

Operations

Limit operation

07.32.25.0001.01

$$\lim_{z \to 1} (1 - z)^{-\psi_q}\, {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\Gamma(-\psi_q)}{\prod_{j=1}^{q+1} \Gamma(a_j)} \;/;\; \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \operatorname{Re}(\psi_q) < 0$$

07.32.25.0002.01

$$\lim_{a \to \infty} {}_p\tilde{F}_q\left(a, a_2, \ldots, a_p; b_1, \ldots, b_q; \frac{z}{a}\right) = {}_{p-1}\tilde{F}_q(a_2, \ldots, a_p; b_1, \ldots, b_q; z)$$

07.32.25.0003.01

$$\lim_{a \to \infty} \left(\Gamma(b)\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b, b_2, \ldots, b_q; b\, z)\right) = {}_p\tilde{F}_{q-1}(a_1, \ldots, a_p; b_2, \ldots, b_q; z)$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_pF_q$

07.32.26.0001.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q} \Gamma(b_k)} \, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \,/;\, \neg\left(\{b_1, \ldots, b_q\} \in \mathbb{Z} \bigwedge (b_1 \leq 0 \bigvee \ldots \bigvee b_q \leq 0)\right) $$

Through hypergeometric functions of two variables

07.32.26.0002.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{q} \Gamma(b_k)} \, F_{0\,q\,0}^{0\,p\,0}\left(\begin{matrix} ;\, a_1, \ldots, a_p;; \\ ;\, b_1, \ldots, b_q;; \end{matrix} \, z, 0 \right) $$

07.32.26.0003.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \tilde{F}_{0\,q\,0}^{0\,p\,0}\left(\begin{matrix} ;\, a_1, \ldots, a_p;; \\ ;\, b_1, \ldots, b_q;; \end{matrix} \, z, 0 \right) $$

Through Meijer *G*

Classical cases for the direct function itself

07.32.26.0004.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{p} \Gamma(a_k)} \, G_{p,q+1}^{1,p}\left(-z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix} \right) $$

07.32.26.0005.01

$$ {}_{q+1}\tilde{F}_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = $$

$$ \frac{1}{\pi \sin(\psi_q \pi) \prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{j=1}^{q} \frac{\prod_{k=1}^{q+1} \sin(\pi(b_j - a_k))}{\prod_{k=1,\, k\neq j}^{q} \sin(\pi(b_j - b_k))} \, G_{q+1,q+1}^{2,q+1}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_j, 1-b_1, \ldots, 1-b_{j-1}, 1-b_{j+1}, \ldots, 1-b_q \end{matrix} \right) - $$

$$ \frac{\pi}{\sin(\psi_q \pi) \prod_{k=1}^{q+1} \Gamma(a_k)} \left((1-z)^{\psi_q} (z-1)^{-\psi_q} G_{q+1,q+1}^{0,q+1}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix} \right) + G_{q+1,q+1}^{q+1,0}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix} \right) \right) /; $$

$$ \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge z \notin (-1, 0) \bigwedge \psi_q \notin \mathbb{Z} $$

07.32.26.0006.01

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{\prod_{k=1}^{p} \Gamma(a_k)} \, G_{4,3}^{3,1}\left(-\frac{1}{z} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix} \right) /;\, z \notin (0, \infty) $$

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