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InverseEllipticNomeQ

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Notations

Traditional name

Inverse nome

Traditional notation

$q^{-1}(z)$

Mathematica StandardForm notation

`InverseEllipticNomeQ[z]`

Primary definition

09.52.02.0001.01

$$q^{-1}(z) = 16\, z \prod_{k=1}^{\infty} \left(\frac{z^{2k} + 1}{z^{2k-1} + 1} \right)^8 \; /; \; |z| < 1$$

Specific values

Values at fixed points

09.52.03.0001.01

$$q^{-1}(0) = 0$$

09.52.03.0002.01

$$q^{-1}(e^{-\pi}) = \frac{1}{2}$$

General characteristics

Domain and analyticity

$q^{-1}(z)$ is an analytical function of z which is defined inside the unite circle of the complex z-plane.

09.52.04.0001.01

$$z \longrightarrow q^{-1}(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.52.04.0002.01

$$q^{-1}(\bar{z}) == \overline{q^{-1}(z)}$$

Periodicity

No periodicity

Poles and essential singularities

On the boundary of analyticity $|q| == 1$ the function $q^{-1}(z)$ has a dense set of poles.

09.52.04.0003.01

$$\mathcal{S}ing_z(q^{-1}(z)) == \{\} /; |q| == 1$$

Branch points

The function $q^{-1}(z)$ does not have branch points.

09.52.04.0004.01

$$\mathcal{BP}_z(q^{-1}(z)) == \{\}$$

Branch cuts

The function $q^{-1}(z)$ does not have branch cuts.

09.52.04.0005.01

$$\mathcal{BC}_z(q^{-1}(z)) == \{\}$$

Natural boundary of analyticity

The unit circle $|q| == 1$ is the natural boundary of the region of analyticity.

09.52.04.0006.01

$$\mathcal{AB}_z(q^{-1}(z)) == \{e^{i(-\pi,\pi)}\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

09.52.06.0003.01

$$q^{-1}(z) \propto q^{-1}(z_0) - \frac{\left(4\,(w-1)\,w\,K(w)^2\right)}{\pi^2\,z_0}\,(z-z_0) + \frac{\left(4\,w\,K(w)^2 - 4\,E(w)\,K(w) + \pi^2\right)\left(2\,(w-1)\,w\,K(w)^2\right)}{\pi^4\,z_0^2}\,(z-z_0)^2 - \frac{1}{3\,\pi^6\,z_0^3}\left(4\,(w-1)\,w\,K(w)^2\right) \left(4\left(2\,w^2+w-1\right)K(w)^4 + 12\,E(w)^2\,K(w)^2 + 6\,\pi^2\,w\,K(w)^2 - 6\,E(w)\left(4\,w\,K(w)^2+\pi^2\right)K(w) + \pi^4\right)(z-z_0)^3 + \frac{1}{3\,\pi^8\,z_0^4}\left(16\left(2\,w^3+3\,w^2-w-2\right)K(w)^6 + 24\,\pi^2\left(2\,w^2+w-1\right)K(w)^4 - 96\,E(w)^3\,K(w)^3 + 22\,\pi^4\,w\,K(w)^2 + 72\,E(w)^2\left(4\,w\,K(w)^2+\pi^2\right)K(w)^2 + 3\,\pi^6 - 2\,E(w)\left(48\left(2\,w^2+w-1\right)K(w)^5 + 72\,\pi^2\,w\,K(w)^3 + 11\,\pi^4\,K(w)\right)\right) \left((w-1)\,w\,K(w)^2\right)(z-z_0)^4 - \frac{4\,(w-1)\,w\,K(w)^2}{15\,\pi^{10}\,z^5}\left(16\left(2\,w^4+6\,w^3+w^2-4\,w-3\right)K(w)^8 + 40\,\pi^2\left(2\,w^3+3\,w^2-w-2\right)K(w)^6 + 240\,E(w)^4\,K(w)^4 + 35\,\pi^4\left(2\,w^2+w-1\right)K(w)^4 - 240\,E(w)^3\left(4\,w\,K(w)^2+\pi^2\right) K(w)^3 + 25\,\pi^6\,w\,K(w)^2 + 3\,\pi^8 + 15\,E(w)^2\left(32\left(2\,w^2+w-1\right)K(w)^6 + 48\,\pi^2\,w\,K(w)^4 + 7\,\pi^4\,K(w)^2\right) - 5\,E(w)\left(32\left(2\,w^3+3\,w^2-w-2\right)K(w)^7 + 48\,\pi^2\left(2\,w^2+w-1\right)K(w)^5 + 42\,\pi^4\,w\,K(w)^3 + 5\,\pi^6\,K(w)\right)\right) (z-z_0)^5 + \ldots\ /;\ (z \to z_0) \bigwedge w = q^{-1}(z_0)$$

09.52.06.0004.01

$$q^{-1}(z) \propto q^{-1}(z_0)\,(1 + O(z-z_0))$$

Expansions at $z == 0$

09.52.06.0001.01

$$q^{-1}(z) \propto 16\,z - 128\,z^2 + 704\,z^3 - 3072\,z^4 + 11\,488\,z^5 - 38\,400\,z^6 + 117\,632\,z^7 - 335\,872\,z^8 + 904\,784\,z^9 - 2\,320\,128\,z^{10} + 5\,702\,208\,z^{11} - 13\,504\,512\,z^{12} + 30\,952\,544\,z^{13} - 68\,901\,888\,z^{14} + 149\,403\,264\,z^{15} - 316\,342\,272\,z^{16} + 655\,445\,792\,z^{17} - 1\,331\,327\,616\,z^{18} + 2\,655\,115\,712\,z^{19} - 5\,206\,288\,384\,z^{20} + O\left(z^{21}\right)\ /;\ (z \to 0)$$

09.52.06.0005.01

$$q^{-1}(z) = 1 - \frac{\left(1 + 2\sum_{k=1}^{\infty}(-1)^k\,z^{k^2}\right)^4}{\left(1 + 2\sum_{k=1}^{\infty}z^{k^2}\right)^4}$$

09.52.06.0002.01

$$q^{-1}(z) \propto 16\,z\,(1 + O(z))\ /;\ (z \to 0)$$

Expansions at $z == 1$

09.52.06.0006.01

$$q^{-1}(z) \propto 1 - 16\,e^{\frac{\pi^2}{\log(z)}} + 128\,e^{\frac{2\pi^2}{\log(z)}} - 704\,e^{\frac{3\pi^2}{\log(z)}} + 3072\,e^{\frac{4\pi^2}{\log(z)}} - 11\,488\,e^{\frac{5\pi^2}{\log(z)}} + 38\,400\,e^{\frac{6\pi^2}{\log(z)}} - 117\,632\,e^{\frac{7\pi^2}{\log(z)}} + 335\,872\,e^{\frac{8\pi^2}{\log(z)}} - 904\,784\,e^{\frac{9\pi^2}{\log(z)}} + 2\,320\,128\,e^{\frac{10\pi^2}{\log(z)}} - 5\,702\,208\,e^{\frac{11\pi^2}{\log(z)}} + 13\,504\,512\,e^{\frac{12\pi^2}{\log(z)}} - 30\,952\,544\,e^{\frac{13\pi^2}{\log(z)}} + 68\,901\,888\,e^{\frac{14\pi^2}{\log(z)}} - 149\,403\,264\,e^{\frac{15\pi^2}{\log(z)}} + 316\,342\,272\,e^{\frac{16\pi^2}{\log(z)}} - 655\,445\,792\,e^{\frac{17\pi^2}{\log(z)}} + 1\,331\,327\,616\,e^{\frac{18\pi^2}{\log(z)}} - 2\,655\,115\,712\,e^{\frac{19\pi^2}{\log(z)}} + 5\,206\,288\,384\,e^{\frac{20\pi^2}{\log(z)}} + O\left(e^{\frac{21\pi^2}{\log(z)}}\right)\ /;\ |z| < 1 \land (z \to 1)$$

09.52.06.0007.01

$$q^{-1}(z) = \frac{\left(1 + 2\sum_{k=1}^{\infty} (-1)^k e^{\frac{k^2\pi^2}{\log(z)}}\right)^4}{\left(1 + 2\sum_{k=1}^{\infty} e^{\frac{k^2\pi^2}{\log(z)}}\right)^4}$$

09.52.06.0008.01

$$q^{-1}(z) \propto 1 - 16\, e^{\frac{\pi^2}{\log(z)}} + O\left(e^{\frac{2\pi^2}{\log(z)}}\right) /;\ |z| < 1 \land (z \to 1)$$

Product representations

09.52.08.0001.01

$$q^{-1}(z) = 16\, z \prod_{k=1}^{\infty} \left(\frac{z^{2k} + 1}{z^{2k-1} + 1}\right)^8 /;\ |z| < 1$$

Differential equations

Ordinary nonlinear differential equations

09.52.13.0001.01

$$w'(z) = \frac{2\, K(w(z))^2\, (w(z) - 1)\, w(z)}{\pi\, z\, (K(1 - w(z))\, (K(w(z)) - E(w(z))) - E(1 - w(z))\, K(w(z)))} /;\ w(z) = q^{-1}(z)$$

09.52.13.0002.01

$$3 z^2 w''(z)^2 w(z)^4 - 2 w'(z)^2 w(z)^3 - 6 z^2 w''(z)^2 w(z)^3 - z^2 w'(z)^4 w(z)^2 + w'(z)^2 w(z)^2 + 3 z^2 w''(z)^2 w(z)^2 - 2 z^2 (w(z) - 1)^2 w'(z)\, w^{(3)}(z)\, w(z)^2 + z^2 w'(z)^4 w(z) - z^2 w'(z)^4 = 0 /;\ w(z) = q^{-1}(z)$$

Identities

Functional identities

09.52.17.0001.01

$$2 w(z^2)\, w(z)^2 - w(z^2)^2 w(z)^2 - w(z)^2 - 16\, w(z^2)\, w(z) + 16\, w(z^2) = 0 /;\ w(z) = q^{-1}(z)$$

09.52.17.0002.01

$$\sqrt[8]{(1 - w(z))\,(1 - w(z^7))} + \sqrt[8]{w(z)\, w(z^7)} = 1 /;\ w(z) = q^{-1}(z) \bigwedge 0 \leq z \leq 1$$

09.52.17.0003.01

$$v^4 - u^4 + 2 v^3 u^3 - 2 v u = 0 /;\ u = \sqrt[8]{q^{-1}(z^3)} \bigwedge v = \sqrt[8]{q^{-1}(z)} \bigwedge \operatorname{Re}(z) > 0$$

09.52.17.0004.01

$$v^6 - u^6 + 5 (v^2 - u^2)\, v^2 u^2 - 4 v\, (1 - u^4 v^4)\, u = 0 /;\ u = \sqrt[8]{q^{-1}(z^5)} \bigwedge v = \sqrt[8]{q^{-1}(z)} \bigwedge 0 \leq z \leq 1$$

09.52.17.0005.01

$$u^8 - 8 v^7 u^7 + 28 v^6 u^6 - 56 v^5 u^5 + 70 v^4 u^4 - 56 v^3 u^3 + 28 v^2 u^2 - 8 v u + v^8 = 0 /;\ u = \sqrt[8]{q^{-1}(z^7)} \bigwedge v = \sqrt[8]{q^{-1}(z)} \bigwedge 0 \leq z \leq 1$$

Differentiation

Low-order differentiation

09.52.20.0001.02

$$\frac{\partial q^{-1}(z)}{\partial z} = \frac{\left(4\left(1-q^{-1}(z)\right)q^{-1}(z)\right)K\left(q^{-1}(z)\right)^2}{z\,\pi^2}$$

09.52.20.0002.02

$$\frac{\partial^2 q^{-1}(z)}{\partial z^2} = \frac{4\,K\left(q^{-1}(z)\right)^2\left(q^{-1}(z)-1\right)q^{-1}(z)\left(4\,q^{-1}(z)\,K\left(q^{-1}(z)\right)^2-4\,E\left(q^{-1}(z)\right)K\left(q^{-1}(z)\right)+\pi^2\right)}{\pi^4\,z^2}$$

09.52.20.0003.01

$$\frac{\partial^3 q^{-1}(z)}{\partial z^3} = -\frac{8\,K\left(q^{-1}(z)\right)^2\left(q^{-1}(z)-1\right)q^{-1}(z)}{\pi^6\,z^3}\left(4\left(q^{-1}(z)+1\right)\left(2\,q^{-1}(z)-1\right)K\left(q^{-1}(z)\right)^4-\right.$$
$$\left.24\,E\left(q^{-1}(z)\right)q^{-1}(z)\,K\left(q^{-1}(z)\right)^3+6\left(2\,E\left(q^{-1}(z)\right)^2+\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^2-6\,\pi^2\,E\left(q^{-1}(z)\right)K\left(q^{-1}(z)\right)+\pi^4\right)$$

09.52.20.0004.01

$$\frac{\partial^4 q^{-1}(z)}{\partial z^4} = \frac{8\,K\left(q^{-1}(z)\right)^2\left(q^{-1}(z)-1\right)q^{-1}(z)}{\pi^8\,z^4}$$
$$\left(16\left(q^{-1}(z)+1\right)\left(2\,q^{-1}(z)^2+q^{-1}(z)-2\right)K\left(q^{-1}(z)\right)^6-96\,E\left(q^{-1}(z)\right)\left(q^{-1}(z)+1\right)\left(2\,q^{-1}(z)-1\right)K\left(q^{-1}(z)\right)^5+\right.$$
$$24\left(12\,q^{-1}(z)\,E\left(q^{-1}(z)\right)^2-\pi^2+2\,\pi^2\,q^{-1}(z)^2+\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^4-48\,E\left(q^{-1}(z)\right)\left(2\,E\left(q^{-1}(z)\right)^2+3\,\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^3+$$
$$\left.2\,\pi^2\left(36\,E\left(q^{-1}(z)\right)^2+11\,\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^2-22\,\pi^4\,E\left(q^{-1}(z)\right)K\left(q^{-1}(z)\right)+3\,\pi^6\right)$$

09.52.20.0005.01

$$\frac{\partial^5 q^{-1}(z)}{\partial z^5} = -\frac{32\,K\left(q^{-1}(z)\right)^2\left(q^{-1}(z)-1\right)q^{-1}(z)}{\pi^{10}\,z^5}$$
$$\left(16\left(2\,q^{-1}(z)^4+6\,q^{-1}(z)^3+q^{-1}(z)^2-4\,q^{-1}(z)-3\right)K\left(q^{-1}(z)\right)^8-160\,E\left(q^{-1}(z)\right)\left(q^{-1}(z)+1\right)\left(2\,q^{-1}(z)^2+q^{-1}(z)-2\right)K\left(q^{-1}(z)\right)^7+\right.$$
$$40\left(q^{-1}(z)+1\right)\left(24\,q^{-1}(z)\,E\left(q^{-1}(z)\right)^2-12\,E\left(q^{-1}(z)\right)^2-2\,\pi^2+2\,\pi^2\,q^{-1}(z)^2+\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^6-$$
$$240\,E\left(q^{-1}(z)\right)\left(4\,q^{-1}(z)\,E\left(q^{-1}(z)\right)^2-\pi^2+2\,\pi^2\,q^{-1}(z)^2+\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^5+$$
$$5\left(48\,E\left(q^{-1}(z)\right)^4+144\,\pi^2\,q^{-1}(z)\,E\left(q^{-1}(z)\right)^2-7\,\pi^4+14\,\pi^4\,q^{-1}(z)^2+7\,\pi^4\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^4-30\,\pi^2\,E\left(q^{-1}(z)\right)$$
$$\left.\left(8\,E\left(q^{-1}(z)\right)^2+7\,\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^3+5\,\pi^4\left(21\,E\left(q^{-1}(z)\right)^2+5\,\pi^2\,q^{-1}(z)\right)K\left(q^{-1}(z)\right)^2-25\,\pi^6\,E\left(q^{-1}(z)\right)K\left(q^{-1}(z)\right)+3\,\pi^8\right)$$

Operations

Limit operation

09.52.25.0002.01

$$\lim_{z\to 1^-} q^{-1}(z) = 1$$

09.52.25.0001.01

$$\lim_{z\to 0}\frac{q^{-1}(z)-\left(-3072\,z^4+704\,z^3-128\,z^2+16\,z\right)}{z^4}==0$$

Representations through equivalent functions

With inverse function

09.52.27.0001.01

$$q^{-1}(q(m))==m$$

With related functions

Involving theta functions

09.52.27.0002.01

$$q^{-1}(z)==\frac{\vartheta_2(0,\,z)^4}{\vartheta_3(0,\,z)^4}$$

09.52.27.0003.01

$$\sqrt{q^{-1}(z)}==\frac{\vartheta_2(0,\,z)^2\,\vartheta_3(0,\,z)^2+\vartheta_1(0,\,z)^2\,\vartheta_4(0,\,z)^2}{\vartheta_2(0,\,z)^4+\vartheta_4(0,\,z)^4}$$

Involving Weierstrass functions

09.52.27.0004.01

$$q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_3}{\omega_1}\right)\right)==\exp(2\,(\omega_3\,\zeta\,(\omega_1;\,g_2,\,g_3)+(\omega_1+\omega_3)\,\zeta\,(\omega_3;\,g_2,\,g_3)))\left(\frac{\sigma\,(\omega_1;\,g_2,\,g_3)}{\sigma\,(\omega_1+\omega_3;\,g_2,\,g_3)}\right)^4\,/;$$
$$\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}$$

09.52.27.0005.01

$$q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_3}{\omega_1}\right)\right)==\frac{e_2-e_3}{e_1-e_3}\,/;$$
$$\{\omega_1,\,\omega_2,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,-\omega_1(g_2,\,g_3)-\omega_3(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}\bigwedge q==\exp\left(\pi\,i\,\frac{\omega_3}{\omega_1}\right)\bigwedge e_n==\wp(\omega_n;\,g_2,\,g_3)\bigwedge n\in\{1,\,2,\,3\}$$

09.52.27.0006.01

$$g_2^3\left(2\,m^3-3\,m^2-3\,m+2\right)^2-108\,g_3^2\left(m^2-m+1\right)^3==0\,/;\,m==q^{-1}\left(\exp\left(\pi\,i\,\frac{\omega_3}{\omega_1}\right)\right)\bigwedge\{\omega_1,\,\omega_3\}=\{\omega_1(g_2,\,g_3),\,\omega_3(g_2,\,g_3)\}$$

Involving other related functions

09.52.27.0007.01

$$\lambda(z)==q^{-1}\left(e^{i\,\pi\,z}\right)\,/;\,\mathrm{Im}(z)>0$$

Zeros

09.52.30.0001.01

$$q^{-1}(z)==0\,/;\,z==0$$

Theorems

The solutions of the quintic $t^5 - t + \rho == 0$

The solutions of the quintic $t^5 - t + \rho == 0$ can be expressed in terms of q^{-1} (C. Hermite). For one of the roots we have ($\rho \in \mathbb{R}$):

$$\frac{1}{2\,5^{3/4}\sqrt{k}\,\sqrt{1-k^2}}\left(\left(\sqrt[8]{k^2}\,\operatorname{sgn}(\rho)\right)\left((-1)^{3/4}\left(\sqrt[8]{q^{-1}\left(\sqrt[5]{q}\,e^{-\frac{1}{5}2\,i\,\pi}\right)}+i\,\sqrt[8]{q^{-1}\left(e^{\frac{2\,i\,\pi}{5}}\sqrt[5]{q}\right)}\right)\left(\sqrt[8]{q^{-1}\left(\sqrt[5]{q}\,e^{-\frac{1}{5}4\,i\,\pi}\right)}+\sqrt[8]{q^{-1}\left(e^{\frac{4\,i\,\pi}{5}}\sqrt[5]{q}\right)}\right)\left(q^{5/8}\,\sqrt[8]{q^{-1}\left(q^5\right)}\Big/\sqrt[8]{q^5}+\sqrt[8]{q^{-1}\left(\sqrt[5]{q}\right)}\right)\right)\right)\,/;$$

$$k=\tan\left(\frac{1}{4}\sin^{-1}\left(\frac{16}{25\sqrt{5}\,\rho^2}\right)\right)\bigwedge q=q\left(k^2\right).$$

History

– C. Hermite (1858)

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