InverseJacobiCN

Notations

Traditional name

Inverse of the Jacobi elliptic function cn

Traditional notation

$\text{cn}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiCN[z, m]`

Primary definition

09.38.02.0001.01

$$z = \text{cn}(w \mid m) \;/;\; w = \text{cn}^{-1}(z \mid m)$$

09.38.02.0002.01

$$\text{cn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{m\,t^2 - m + 1}}\,dt \;/;\; -1 < z < 1 \bigwedge m\left(z^2 - 1\right) > -1$$

Specific values

Specialized values

For fixed *z*

09.38.03.0001.01

$$\text{cn}^{-1}(z \mid 0) = \cos^{-1}(z)$$

09.38.03.0002.01

$$\text{cn}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = F\left(\cos^{-1}(z) \,\middle|\, \frac{1}{2}\right)$$

09.38.03.0003.01

$$\text{cn}^{-1}(z \mid 1) = \text{sech}^{-1}(z)$$

For fixed *m*

09.38.03.0004.01

$$\text{cn}^{-1}(-1 \mid m) = 2\,K(m)$$

09.38.03.0005.01

$$\mathrm{cn}^{-1}\left(-\frac{1}{2}\,\middle|\, m\right) = F\left(\frac{2\pi}{3}\,\middle|\, m\right)$$

09.38.03.0006.01

$$\mathrm{cn}^{-1}(0 \mid m) = K(m) \;/;\; m \in \mathbb{R} \land m < 1$$

09.38.03.0007.01

$$\mathrm{cn}^{-1}\left(\frac{1}{2}\,\middle|\, m\right) = F\left(\frac{\pi}{3}\,\middle|\, m\right)$$

09.38.03.0008.01

$$\mathrm{cn}^{-1}(1 \mid m) = 0$$

09.38.03.0009.01

$$\mathrm{cn}^{-1}(i \mid m) = \frac{1}{\sqrt{m-1}}\left(i\left(K\left(\frac{m}{m-1}\right) - F\left(\sin^{-1}(i)\,\middle|\,\frac{m}{m-1}\right)\right)\right)$$

09.38.03.0010.01

$$\mathrm{cn}^{-1}(-i \mid m) = \frac{1}{\sqrt{1-m}}\left(F\left(i\sinh^{-1}(1)\,\middle|\,\frac{m}{m-1}\right) + K\left(\frac{m}{m-1}\right)\right)$$

Values at infinities

09.38.03.0011.01

$$\mathrm{cn}^{-1}(z \mid \infty) = 0$$

09.38.03.0012.01

$$\mathrm{cn}^{-1}(z \mid -\infty) = 0$$

09.38.03.0013.01

$$\mathrm{cn}^{-1}(\infty \mid m) = -\frac{i}{\sqrt{m}}\,K\left(1-\frac{1}{m}\right)$$

09.38.03.0014.01

$$\mathrm{cn}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{1-m}}\,K\left(\frac{m}{m-1}\right) + \frac{i}{\sqrt{m}}\,K\left(\frac{m-1}{m}\right)$$

General characteristics

Domain and analyticity

$\mathrm{cn}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.38.04.0001.01

$$(z * m) \longrightarrow \mathrm{cn}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.38.04.0002.01

$$\mathrm{cn}^{-1}(\bar{z} \mid \bar{m}) = \overline{\mathrm{cn}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.38.04.0003.01

$$\mathrm{cn}^{-1}(-z \mid m) = \frac{2}{\sqrt{1-m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{m}{m-1}\right) + \mathrm{cn}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to *m*

The function $\mathrm{cn}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.38.04.0004.01

$$\mathcal{Sing}_m\left(\mathrm{cn}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\mathrm{cn}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.38.04.0005.01

$$\mathcal{Sing}_z\left(\mathrm{cn}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\mathrm{cn}^{-1}(z \mid m)$ has two branch points: $m = \frac{1}{1-z^2}$, $m = \tilde{\infty}$.

09.38.04.0006.01

$$\mathcal{BP}_m\left(\mathrm{cn}^{-1}(z \mid m)\right) = \left\{\frac{1}{1-z^2}, \tilde{\infty}\right\}$$

09.38.04.0007.01

$$\mathcal{R}_m\left(\mathrm{cn}^{-1}(z \mid m), \frac{1}{1-z^2}\right) = \log$$

09.38.04.0008.01

$$\mathcal{R}_m\left(\mathrm{cn}^{-1}(z \mid m), \tilde{\infty}\right) = \log$$

With respect to *z*

For fixed m, the function $\mathrm{cn}^{-1}(z \mid m)$ has six branch points: $z = 0$, $z = \pm 1$, $z = \pm \sqrt{\frac{m-1}{m}}$, $z = \tilde{\infty}$.

09.38.04.0009.01

$$\mathcal{BP}_z\left(\mathrm{cn}^{-1}(z \mid m)\right) = \left\{0, 1, -1, \sqrt{\frac{m-1}{m}}, -\sqrt{\frac{m-1}{m}}, \tilde{\infty}\right\}$$

09.38.04.0010.01

$$\mathcal{R}_z\left(\mathrm{cn}^{-1}(z \mid m), 0\right) = \log$$

09.38.04.0011.01

$$\mathcal{R}_z\left(\mathrm{cn}^{-1}(z \mid m), 1\right) = 2$$

09.38.04.0012.01

$$\mathcal{R}_z\left(\mathrm{cn}^{-1}(z \mid m), -1\right) = 2$$

09.38.04.0013.01

$$\mathcal{R}_z\left(\text{cn}^{-1}(z \mid m), \sqrt{\frac{m-1}{m}}\right) == 2$$

09.38.04.0014.01

$$\mathcal{R}_z\left(\text{cn}^{-1}(z \mid m), -\sqrt{\frac{m-1}{m}}\right) == 2$$

09.38.04.0015.01

$$\mathcal{R}_z\left(\text{cn}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

Branch cuts

Branch cut locations: complicated.

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

09.38.06.0009.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z_0 \mid m) - \frac{z-z_0}{\sqrt{1-z_0^2}\,\sqrt{m z_0^2 - m + 1}} - \frac{\left(2 m z_0^2 - 2 m + 1\right) z_0}{2\left(1-z_0^2\right)^{3/2}\left(m z_0^2 - m + 1\right)^{3/2}} (z-z_0)^2 + \ldots /; (z \to z_0)$$

09.38.06.0010.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z_0 \mid m) - \frac{z-z_0}{\sqrt{1-z_0^2}\,\sqrt{m z_0^2 - m + 1}} - \frac{\left(2 m z_0^2 - 2 m + 1\right) z_0}{2\left(1-z_0^2\right)^{3/2}\left(m z_0^2 - m + 1\right)^{3/2}} (z-z_0)^2 + O\left((z-z_0)^3\right)$$

09.38.06.0011.01

$$\text{cn}^{-1}(z \mid m) = \text{cn}^{-1}(z_0 \mid m) - \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \frac{(1-k)_{2(k-j)-2}}{(k-j-1)!\,(2 z_0)^{k-2j-1}} \sum_{s=0}^{j} (-1)^{j+s} \binom{j}{s} \left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_{j-s} m^{j-s} \left(1-z_0^2\right)^{-s-\frac{1}{2}} \left(m z_0^2 - m + 1\right)^{s-j-\frac{1}{2}} (z-z_0)^k$$

09.38.06.0012.01

$$\text{cn}^{-1}(z \mid m) == \text{cn}^{-1}(z_0 \mid m) - \frac{\pi}{\text{dn}\left(\text{cn}^{-1}(z_0 \mid m) \mid m\right) \text{sn}\left(\text{cn}^{-1}(z_0 \mid m) \mid m\right)} \sum_{k=1}^{\infty} \frac{(2 z_0)^{k-1}}{k} \sum_{j=0}^{k-1} \frac{m^{k-j-1} \left(z_0^2 - 1\right)^{-j} \left(m z_0^2 - m + 1\right)^{j-k+1}}{j!\,(k-j-1)!\,\Gamma\left(\frac{1}{2} - j\right) \Gamma\left(j - k + \frac{3}{2}\right)} {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2} - j; 1 - \frac{1}{z_0^2}\right) {}_2F_1\left(\frac{1}{2}(j-k+2), \frac{1}{2}(j-k+1); j-k+\frac{3}{2}; \frac{1-m}{m z_0^2} + 1\right) (z-z_0)^k$$

09.38.06.0013.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z_0 \mid m) \left(1 + O(z - z_0)\right)$$

Expansions at $z == 0$

09.38.06.0001.02

$$\text{cn}^{-1}(z \mid m) \propto K(m) - \frac{1}{\sqrt{1-m}} \left(z + \frac{(2m-1)z^3}{6(m-1)} + \frac{(3-8m+8m^2)z^5}{40(m-1)^2} + \ldots \right) /; (z \to 0)$$

09.38.06.0002.01

$$\text{cn}^{-1}(z \mid m) == K(m) - \frac{1}{\sqrt{1-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m}{m-1}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} \, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{m-1}{m}\right) z^{2k+1} /; |z| < 1$$

09.38.06.0014.01

$$\text{cn}^{-1}(z \mid m) \propto K(m)(1 + O(z))$$

Expansions at generic point $m == m_0$

For the function itself

09.38.06.0015.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z \mid m_0) + \frac{(1-z^2)^{3/2}}{6} F_1\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1-z^2, (1-z^2)m_0\right)(m-m_0) + \frac{3(1-z^2)^{5/2}}{40} F_1\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; 1-z^2, (1-z^2)m_0\right)(m-m_0)^2 + \ldots /; (m \to m_0)$$

09.38.06.0016.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z \mid m_0) + \frac{(1-z^2)^{3/2}}{6} F_1\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1-z^2, (1-z^2)m_0\right)(m-m_0) + \frac{3(1-z^2)^{5/2}}{40} F_1\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; 1-z^2, (1-z^2)m_0\right)(m-m_0)^2 + O\left((m-m_0)^3\right)$$

09.38.06.0017.01

$$\text{cn}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{\pi} \, (1-z^2)^{k+\frac{1}{2}}}{k! \, (2k+1) \, \Gamma\left(\frac{1}{2}-k\right)} F_1\left(k+\frac{1}{2}; \frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; 1-z^2, m_0(1-z^2)\right)(m-m_0)^k$$

09.38.06.0018.01

$$\text{cn}^{-1}(z \mid m) \propto \text{cn}^{-1}(z \mid m_0)(1 + O(m-m_0))$$

Expansions at $m == 0$

09.38.06.0003.02

$$\text{cn}^{-1}(z \mid m) \propto \cos^{-1}(z) + \frac{1}{4}\left(\cos^{-1}(z) - z\sqrt{1-z^2}\right)m + \frac{3}{64}\left((2z^2-5)\sqrt{1-z^2}\, z + 3\cos^{-1}(z)\right)m^2 + \ldots /; (m \to 0)$$

09.38.06.0004.01

$$\text{cn}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left(\frac{\sqrt{\pi}}{2\,k!} \Gamma\left(k+\frac{1}{2}\right) - z \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-k; \frac{3}{2}; z^2\right)\right) m^k /; |m| < 1$$

09.38.06.0019.01

$$\mathrm{cn}^{-1}(z\mid m)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\cos^{-1}(z)-\frac{z}{2\sqrt{1-z^2}}\sum_{j=1}^{k}\frac{(j-1)!\,\left(1-z^2\right)^j}{\left(\frac{1}{2}\right)_j}\right)m^k\;/;\;|m|<1$$

09.38.06.0005.01

$$\mathrm{cn}^{-1}(z\mid m)=K(m)-\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,m^k\,z^{2j+2k+1}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k^2}{(2j+2k+1)\,(j+k)!\,k!}-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,m^{j+k+1}\,z^{2k+1}\left(\frac{3}{2}\right)_{j+k}^2}{2\,(2k+1)\,k!\,(j+k+1)!\left(\frac{3}{2}\right)_j}$$

09.38.06.0006.01

$$\mathrm{cn}^{-1}(z\mid m)=K(m)-z\,F_{2\times0\times0}^{1\times2\times2}\left(\begin{matrix}\frac{1}{2};\;\frac{1}{2},\;\frac{1}{2};\;1,\;\frac{1}{2};\\ 1,\;\frac{3}{2};;;\end{matrix}\;-m\,z^2,\,z^2\right)-\frac{m\,z}{2}\,F_{1\times1\times1}^{2\times1\times1}\left(\begin{matrix}\frac{3}{2},\;\frac{3}{2};\;\frac{1}{2};\;1;\\ 2;\;\frac{3}{2};\;\frac{3}{2};\end{matrix}\;-m\,z^2,\,m\right)$$

09.38.06.0007.01

$$\mathrm{cn}^{-1}(z\mid m)=K(m)-\frac{1}{\sqrt{1-m}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^{2j+2k+1}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k}{(2j+2k+1)\,j!\,k!}\left(\frac{m}{m-1}\right)^k$$

09.38.06.0008.01

$$\mathrm{cn}^{-1}(z\mid m)=K(m)-\frac{1}{\sqrt{1-m}}\,z\,F_{1\times0\times0}^{1\times1\times1}\left(\begin{matrix}\frac{1}{2};\;\frac{1}{2};\;\frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\;\frac{m\,z^2}{m-1},\,z^2\right)$$

09.38.06.0020.01

$$\mathrm{cn}^{-1}(z\mid m)\propto\cos^{-1}(z)\,(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.38.07.0001.02

$$\mathrm{cn}^{-1}(z\mid m)=\int_z^1\frac{1}{\sqrt{1-t^2}\,\sqrt{m\,t^2-m+1}}\,dt\;/;\;\left(-1<z<1\bigwedge m\left(z^2-1\right)>-1\right)\bigvee\left(|z|<1\land|m|<1\right)$$

09.38.07.0002.01

$$\mathrm{cn}^{-1}(z\mid m)=\frac{\sqrt{1-z^2}\;\mathrm{ds}\left(\mathrm{cn}^{-1}(z\mid m)\mid m\right)}{\sqrt{m\left(z^2-1\right)+1}}\int_z^1\frac{1}{\sqrt{1-t^2}\,\sqrt{m\,t^2-m+1}}\,dt\;/;$$

$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left(1-((z-1)\,\tau+1)^2\right)=0\bigwedge 1-((z-1)\,\tau+1)^2<0\bigwedge \mathrm{Im}\left(m\,((z-1)\,\tau+1)^2-m+1\right)=0\bigwedge m\,((z-1)\,\tau+1)^2-m+1<0\right)$$

09.38.07.0003.01

$$\text{cn}^{-1}(z\mid m)=\text{cn}^{-1}(z_0\mid m)-\frac{\sqrt{1-z^2}\ \text{ds}\left(\text{cn}^{-1}(z\mid m)\mid m\right)}{\sqrt{m\left(z^2-1\right)+1}}\int_{z_0}^{z}\frac{1}{\sqrt{1-t^2}\ \sqrt{m\,t^2-m+1}}\,dt\ /;$$
$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\text{Im}\left(1-(\tau\,(z-z_0)+z_0)^2\right)=0\bigwedge 1-(\tau\,(z-z_0)+z_0)^2<0\bigwedge\right.$$
$$\left.\text{Im}\left(m\,(\tau\,(z-z_0)+z_0)^2-m+1\right)=0\bigwedge m\,(\tau\,(z-z_0)+z_0)^2-m+1<0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.38.13.0001.01

$$w''(z)-\left(2\,m\,z^2-2\,m+1\right)z\,w'(z)^3=0\ /;\ w(z)=\text{cn}^{-1}(z\mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.38.16.0001.01

$$\text{cn}^{-1}(-z\mid m)=\frac{2}{\sqrt{1-m}}\,F\!\left(\sin^{-1}(z)\,\middle|\,\frac{m}{m-1}\right)+\text{cn}^{-1}(z\mid m)$$

Products, sums, and powers of the direct function

Sums of the direct function

09.38.16.0002.01

$$\text{cn}^{-1}(z_1\mid m)+\text{cn}^{-1}(z_2\mid m)=\text{cn}^{-1}\!\left(\frac{z_1\,z_2-\sqrt{\left(1-z_1^2\right)\left(m\,z_1^2+(1-m)\right)\left(1-z_2^2\right)\left(m\,z_2^2+(1-m)\right)}}{1-m\left(1-z_1^2\right)\left(1-z_2^2\right)}\,\middle|\,m\right)$$

Identities

Functional identities

09.38.17.0001.01

$$\left(\left(z_2^2-1\right)m\,z_1^2-m\,z_2^2+m-1\right)\text{cn}(w(z_1)+w(z_2)\mid m)^2+2\,z_1\,z_2\,\text{cn}(w(z_1)+w(z_2)\mid m)+\left(-m\,z_2^2+m-1\right)z_1^2+(m-1)\left(z_2^2-1\right)=0\ /;$$
$$w(z)=\text{cn}^{-1}(z\mid m)$$

Differentiation

Low-order differentiation

With respect to z

09.38.20.0001.02

$$\frac{\partial \operatorname{cn}^{-1}(z \mid m)}{\partial z} = -\frac{\operatorname{ds}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{m z^2 - m + 1}$$

09.38.20.0002.01

$$\frac{\partial \operatorname{cn}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\sqrt{1-z^2} \sqrt{m z^2 - m + 1}} \;/;\; -1 < z < 1 \bigwedge m\left(z^2-1\right) > -1$$

09.38.20.0003.02

$$\frac{\partial^2 \operatorname{cn}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(2 m\left(z^2-1\right)+1\right) \operatorname{ds}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{\left(z^2-1\right)\left(m\left(z^2-1\right)+1\right)^2}$$

09.38.20.0011.01

$$\frac{\partial^2 \operatorname{cn}^{-1}(z \mid m)}{\partial z^2} = \frac{\sqrt{1-z^2}\, \operatorname{ds}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m\left(z^2-1\right)+1}} \frac{\partial \frac{1}{\sqrt{1-z^2} \sqrt{m z^2 - m + 1}}}{\partial z}$$

With respect to *m*

09.38.20.0004.02

$$\frac{\partial \operatorname{cn}^{-1}(z \mid m)}{\partial m} = -\frac{E\left(\operatorname{am}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1) \operatorname{cn}^{-1}(z \mid m) - m z \operatorname{sd}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{2(m-1) m}$$

09.38.20.0005.01

$$\frac{\partial \operatorname{cn}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)}\left(\frac{\sqrt{1-z^2}\, z}{\sqrt{\left(z^2-1\right) m + 1}} + \frac{i}{\sqrt{m}}\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) - E\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) - F\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right)\right)\right) /;\; -1 < z < 1 \land m < 1$$

09.38.20.0006.02

$$\frac{\partial^2 \operatorname{cn}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4(m-1)^2 m^2}\left(3 \operatorname{cn}^{-1}(z \mid m)(m-1)^2 + F\left(\operatorname{am}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) \mid m\right)(m-1) + (4m-2) E\left(\operatorname{am}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) \mid m\right) - \frac{m z\left(m\left(-2 z^2 + 4 m\left(z^2-1\right)+5\right)-1\right) \operatorname{sd}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{m\left(z^2-1\right)+1}\right)$$

09.38.20.0012.01

$$\frac{\partial^3 \operatorname{cn}^{-1}(z \mid m)}{\partial m^3} = -\frac{1}{8(m-1)^3 m^3}\left((m-1)(m(15m-19)+8) \operatorname{cn}^{-1}(z \mid m) + \frac{1}{\left(m\left(z^2-1\right)+1\right)^3}\left((23(m-1)m+8) E\left(\operatorname{am}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) \mid m\right)\left(m\left(z^2-1\right)+1\right)^3 + m z\left(z^2-1\right)\left(m^2(23(m-1)m+8) z^4 - (m-1) m(m(46m-35)+9) z^2 + (m-1)^2(m(23m-12)+4)\right) \operatorname{ds}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)\right)\right)$$

09.38.20.0013.01

$$\frac{\partial^3 \mathrm{cn}^{-1}(z \mid m)}{\partial m^3} = -\frac{1}{8 (m-1)^3 m^3} \Bigg(15\, \mathrm{cn}^{-1}(z \mid m) (m-1)^3 + (11 m - 7) F\left(\mathrm{am}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right) \mid m\right) (m-1) + (23 (m-1) m + 8) E\left(\mathrm{am}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right) \mid m\right) - \frac{1}{\left(m\left(z^2-1\right)+1\right)^{7/2}} \Big(m \left(z^2-1\right) z \Big((m-1) \,\mathrm{ns}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right) \left(m\left(z^2-1\right)+1\right)^3 + \left(m^2 (m (23 m - 24) + 9) z^4 - (m-1) m (m (46 m - 37) + 11) z^2 + (m-1)^2 (m (23 m - 13) + 5)\right) \mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right) \sqrt{m\left(z^2-1\right)+1}\Big)\Big)\Bigg)$$

Symbolic differentiation

With respect to *z*

09.38.20.0014.01

$$\frac{\partial^n \mathrm{cn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \,\mathrm{cn}^{-1}(z \mid m) - \frac{\mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{m\left(z^2-1\right)+1} \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} \left(1-z^2\right)^{-k} \left(m z^2 - m + 1\right)^{k-j} /;\, n \in \mathbb{N}$$

09.38.20.0015.01

$$\frac{\partial^n \mathrm{cn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \,\mathrm{cn}^{-1}(z \mid m) - \frac{\mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{m\left(z^2-1\right)+1} \sum_{j=0}^{n-1} \frac{(-1)^j\, 2^{2j-n+1} m^j z^{2j-n+1} \left(m z^2 - m + 1\right)^{-j} \left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!} \, {}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2} - j; -\frac{m z^2 - m + 1}{m\left(1-z^2\right)}\right) /;\, n \in \mathbb{N}$$

09.38.20.0016.01

$$\frac{\partial^n \mathrm{cn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \,\mathrm{cn}^{-1}(z \mid m) - \frac{\sqrt{1-z^2}\; \mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m\left(z^2-1\right)+1}} \frac{\partial^{n-1} \frac{1}{\sqrt{1-z^2}\, \sqrt{m z^2 - m + 1}}}{\partial z^{n-1}} /;\, n \in \mathbb{N}^+$$

09.38.20.0007.01

$$\frac{\partial^n \mathrm{cn}^{-1}(z \mid m)}{\partial z^n} = -\frac{2^{n-1} \pi z^{n-1} (n-1)!\, \mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{m\left(z^2-1\right)+1} \sum_{j=0}^{n-1} \frac{m^{n-j-1} \left(z^2-1\right)^{-j} \left(m z^2 - m + 1\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right) \Gamma\left(j-n+\frac{3}{2}\right)} \, {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2} - j; 1 - \frac{1}{z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; \frac{1-m}{m z^2} + 1\right) /;\, n \in \mathbb{N}^+$$

With respect to *m*

09.38.20.0008.02

$$\frac{\partial^n \mathrm{cn}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} \left(1-z^2\right)^{n+\frac{1}{2}}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; 1-z^2, m\left(1-z^2\right)\right) /;\, n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.38.20.0009.01

$$\frac{\partial^\alpha \operatorname{cn}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{z^{-\alpha}}{\sqrt{1-m}\,\Gamma(1-\alpha)} K\left(\frac{m}{m-1}\right) - \frac{z^{1-\alpha}\sqrt{\pi}}{\sqrt{1-m}} \tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\left(\begin{matrix}\frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;;\end{matrix} z^2, \frac{m z^2}{m-1}\right) /; -1 < z < 1 \land -1 < m < 1$$

With respect to *m*

09.38.20.0010.01

$$\frac{\partial^\alpha \operatorname{cn}^{-1}(z \mid m)}{\partial m^\alpha} = \frac{m^{-\alpha}\sqrt{\pi}}{2} \sqrt{1-z^2}\, \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha;\end{matrix} 1-z^2, m\left(1-z^2\right)\right) /; -1 < z < 1 \land -1 < m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.38.21.0001.01

$$\int \operatorname{cn}^{-1}(z \mid m)\, dz = z \operatorname{cn}^{-1}(z \mid m) + \frac{i}{\sqrt{m}} \log\left(\frac{i \operatorname{dn}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m}} - \operatorname{sn}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)\right)$$

Involving only one direct function with respect to *m*

09.38.21.0002.01

$$\int \operatorname{cn}^{-1}(z \mid m)\, dm = 2\left(\frac{z\sqrt{m\left(z^2-1\right)+1} - z}{\sqrt{1-z^2}} + i\sqrt{m}\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) - E\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) - F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + F\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right)\right)\right) /; z < 1 \land 0 < m < 1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.38.26.0001.01

$$\operatorname{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}} z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;;\end{matrix} \frac{m z^2}{m-1}, z^2\right)$$

09.38.26.0002.01

$$\operatorname{cn}^{-1}(z \mid m) = K(m) - z\, F_{2\times 0\times 0}^{1\times 2\times 2}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; 1, \frac{1}{2}; \\ 1, \frac{3}{2};;;\end{matrix} -m z^2, z^2\right) - \frac{m z}{2} F_{1\times 1\times 1}^{2\times 1\times 1}\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}; \frac{1}{2}; 1; \\ 2; \frac{3}{2}; \frac{3}{2};\end{matrix} -m z^2, m\right)$$

Through other functions

Involving some hypergeometric-type functions

09.38.26.0003.01

$$\operatorname{cn}^{-1}(z \mid m) = \sqrt{1-z^2}\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1-z^2, m\left(1-z^2\right)\right) /; -1 < z < 1 \land -1 < m < 1$$

09.38.26.0004.01

$$\operatorname{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) - \frac{z}{\sqrt{1-m}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \frac{m z^2}{m-1}\right) /; -1 < z < 1 \land -1 < m < 1$$

Representations through equivalent functions

With inverse function

09.38.27.0001.01

$$\operatorname{cn}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) = z$$

With related functions

Involving cd^{-1}

09.38.27.0002.01

$$\operatorname{cn}^{-1}(z \mid m) = K(m) - \operatorname{cd}^{-1}\left(\sqrt{1-z^2} \mid m\right) /; 0 < z < 1 \land m < 1$$

Involving cs^{-1}

09.38.27.0003.01

$$\operatorname{cn}^{-1}(z \mid m) = i \operatorname{cs}^{-1}\left(\frac{1}{\sqrt{z^2-1}} \,\middle|\, 1-m\right) /; z > 1 \land m > 1$$

Involving dc^{-1}

09.38.27.0004.01

$$\operatorname{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}} \operatorname{dc}^{-1}\left(\sqrt{1-z^2} \,\middle|\, \frac{1}{m}\right) /; 0 < z < 1 \land 0 < m < 1$$

Involving dn^{-1}

09.38.27.0005.01

$$\operatorname{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m < 1$$

09.38.27.0006.01

$$\operatorname{cn}^{-1}(z \mid m) = \operatorname{dn}^{-1}\left(\sqrt{m z^2 - m + 1} \mid m\right) /; 0 < z < 1 \land m > 1 \lor -1 < z < 0 \land m < 0$$

Involving ds^{-1}

09.38.27.0007.01

$$\operatorname{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(\frac{1}{\sqrt{\left(z^2-1\right)(m-1)}} \,\middle|\, \frac{m}{m-1}\right) /; z > 1 \land m > 1$$

Involving nc^{-1}

09.38.27.0008.01

$$\text{cn}^{-1}(z \mid m) = -i\, \text{nc}^{-1}(z \mid 1-m)\ /;\ -1 < z < 1 \land m \in \mathbb{R}$$

09.38.27.0009.01

$$\text{cn}^{-1}(z \mid m) = \text{nc}^{-1}\left(\frac{1}{z} \,\middle|\, m\right)\ /;\ z < 1 \land m \in \mathbb{R}$$

Involving nd^{-1}

09.38.27.0010.01

$$\text{cn}^{-1}(z \mid m) = K(m) + i\, \text{nd}^{-1}\left(\sqrt{1-z^2} \,\middle|\, 1-m\right)\ /;\ z > 0 \land m \in \mathbb{R}$$

Involving ns^{-1}

09.38.27.0011.01

$$\text{cn}^{-1}(z \mid m) = \text{ns}^{-1}\left(\frac{1}{\sqrt{1-z^2}} \,\middle|\, m\right)\ /;\ 0 < z < 1 \land m \in \mathbb{R}$$

Involving sc^{-1}

09.38.27.0012.01

$$\text{cn}^{-1}(z \mid m) = -i\, \text{sc}^{-1}\left(i \sqrt{1-z^2} \,\middle|\, 1-m\right)\ /;\ 0 < z < 1 \land m \in \mathbb{R}$$

Involving sd^{-1}

09.38.27.0013.01

$$\text{cn}^{-1}(z \mid m) = -\frac{i}{\sqrt{1-m}}\, \text{sd}^{-1}\left(\sqrt{(m-1)\left(1-z^2\right)} \,\middle|\, \frac{1}{1-m}\right)\ /;\ 0 < z < 1 \land 0 < m < 1$$

Involving sn^{-1}

09.38.27.0014.01

$$\text{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - \text{sn}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right)\right)\ /;\ -1 < z < 1 \land m < 1$$

09.38.27.0015.01

$$\text{cn}^{-1}(z \mid m) = \text{sn}^{-1}\left(\sqrt{1-z^2} \,\middle|\, m\right)\ /;\ 0 < z < 1 \land m < 1$$

Involving elliptic integrals

09.38.27.0016.01

$$\text{cn}^{-1}(z \mid m) = F\left(\cos^{-1}(z) \,\middle|\, m\right)\ /;\ -1 < z < 1 \land m \in \mathbb{R}$$

09.38.27.0017.01

$$\text{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}}\, F\left(\sin^{-1}(z) \,\middle|\, \frac{m}{m-1}\right)\ /;\ (-1 < z < 1 \land m < 1) \lor |z| < 1$$

09.38.27.0019.01

$$\mathrm{cn}^{-1}(z \mid m) = \frac{\sqrt{1-z^2}\ \mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m\left(z^2-1\right)+1}} \left(\sqrt{\frac{1}{1-m}}\ K\left(\frac{m}{m-1}\right) - \frac{1}{\sqrt{m z^2 - m + 1}} \sqrt{\frac{-m z^2 + m - 1}{m-1}}\ F\left(\sin^{-1}(z) \,\middle|\, \frac{m}{m-1}\right)\right) /;$$

$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left(1-((z-1)\,\tau+1)^2\right) = 0 \bigwedge 1-((z-1)\,\tau+1)^2 < 0 \bigwedge \mathrm{Im}\left(m\,((z-1)\,\tau+1)^2 - m + 1\right) = 0 \bigwedge m\,((z-1)\,\tau+1)^2 - m + 1 < 0\right)$$

09.38.27.0020.01

$$\mathrm{cn}^{-1}(z \mid m) = \mathrm{cn}^{-1}(z_0 \mid m) - \frac{\sqrt{1-z^2}\ \mathrm{ds}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m\left(z^2-1\right)+1}} \left(\frac{\sqrt{\frac{-m z^2+m-1}{m-1}}\ F\left(\sin^{-1}(z) \,\middle|\, \frac{m}{m-1}\right)}{\sqrt{m z^2 - m + 1}} - \frac{\sqrt{\frac{-m z_0^2+m-1}{m-1}}\ F\left(\sin^{-1}(z_0) \,\middle|\, \frac{m}{m-1}\right)}{\sqrt{m z_0^2 - m + 1}}\right) /;$$

$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left(1-(\tau\,(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-(\tau\,(z-z_0)+z_0)^2 < 0 \bigwedge \mathrm{Im}\left(m\,(\tau\,(z-z_0)+z_0)^2 - m + 1\right) = 0 \bigwedge m\,(\tau\,(z-z_0)+z_0)^2 - m + 1 < 0\right)$$

Involving other related functions

09.38.27.0018.01

$$\mathrm{cn}^{-1}(z \mid m) = -\frac{i\sqrt{z_2^2}}{z_2\sqrt{m}} \left(K\left(1-\frac{1}{m}\right) + \mathrm{elog}(z_1, z_2; a, b)\right) /;$$

$$\{a, b, z_1\} = \left\{\frac{1}{m} - 2,\ 1 - \frac{1}{m},\ z^2\right\} \bigwedge z_1^3 + a\, z_1^2 + b\, z_1 - z_2^2 = 0 \bigwedge 0 < z < 1 \bigwedge 0 < m < 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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