InverseJacobiDN

Notations

Traditional name

Inverse of the Jacobi elliptic function dn

Traditional notation

$\text{dn}^{-1}(z \mid m)$

Mathematica StandardForm notation

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InverseJacobiDN[z, m]
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Primary definition

09.41.02.0001.01

$z = \text{dn}(w \mid m) \;/;\; w = \text{dn}^{-1}(z \mid m)$

09.41.02.0002.01

$$\text{dn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\,dt \;/;\; -1 < z < 1 \bigwedge z^2 + m > 1$$

Specific values

Specialized values

For fixed *z*

09.41.03.0001.01

$\text{dn}^{-1}(z \mid 0) = \tilde{\infty}$

09.41.03.0002.01

$$\text{dn}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = \frac{8\,(1+i)\,\pi^{3/2}}{\Gamma\left(-\frac{1}{4}\right)^2} - i\sqrt{2}\;F\left(\sin^{-1}(z) \,\middle|\, 2\right)$$

09.41.03.0003.01

$\text{dn}^{-1}(z \mid 1) = \text{sech}^{-1}(z)$

For fixed *m*

09.41.03.0004.01

$$\text{dn}^{-1}(-1 \mid m) = \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1$$

09.41.03.0005.01

$$\text{dn}^{-1}\left(-\frac{1}{2} \middle| m\right) = -\frac{1}{\sqrt{m-1}} \left(F\left(\frac{\pi}{6} \middle| \frac{1}{1-m}\right) + K\left(\frac{1}{1-m}\right)\right) /; 0 < m < 1$$

09.41.03.0006.01

$$\text{dn}^{-1}(0 \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1$$

09.41.03.0007.01

$$\text{dn}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - F\left(\frac{\pi}{6} \middle| \frac{1}{1-m}\right)\right) /; m > 1$$

09.41.03.0008.01

$$\text{dn}^{-1}(1 \mid m) = 0$$

09.41.03.0009.01

$$\text{dn}^{-1}\left(\sqrt{1-m} \mid m\right) = K(m) /; 0 < m < 1$$

09.41.03.0010.01

$$\text{dn}^{-1}(i \mid m) = \frac{1}{\sqrt{m-1}} \left(F\left(i \sinh^{-1}(1) \middle| \frac{1}{1-m}\right) - K\left(\frac{1}{1-m}\right)\right) /; 0 < m < 1$$

09.41.03.0011.01

$$\text{dn}^{-1}(-i \mid m) = -\frac{1}{\sqrt{m-1}} \left(F\left(i \sinh^{-1}(1) \middle| \frac{1}{1-m}\right) + K\left(\frac{1}{1-m}\right)\right) /; 0 < m < 1$$

Values at infinities

09.41.03.0012.01

$$\text{dn}^{-1}(z \mid \infty) = 0$$

09.41.03.0013.01

$$\text{dn}^{-1}(z \mid -\infty) = 0$$

09.41.03.0014.01

$$\text{dn}^{-1}(\infty \mid m) = i\, K(1-m)$$

09.41.03.0015.01

$$\text{dn}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - i\, K(1-m) /; m > 1$$

General characteristics

Domain and analyticity

$\text{dn}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.41.04.0001.01

$$(z * m) \longrightarrow \text{dn}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.41.04.0002.01

$$\mathrm{dn}^{-1}(\bar{z} \mid \bar{m}) = \overline{\mathrm{dn}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.41.04.0003.01

$$\mathrm{dn}^{-1}(-z \mid m) = \mathrm{dn}^{-1}(z \mid m) - \frac{2}{\sqrt{m-1}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)$$

Poles and essential singularities

With respect to *m*

The function $\mathrm{dn}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.41.04.0004.01

$$\mathcal{S}ing_m\left(\mathrm{dn}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\mathrm{dn}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.41.04.0005.01

$$\mathcal{S}ing_z\left(\mathrm{dn}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\mathrm{dn}^{-1}(z \mid m)$ has two branch points: $m = 1 - z^2$, $m = \tilde{\infty}$.

09.41.04.0006.01

$$\mathcal{BP}_m\left(\mathrm{dn}^{-1}(z \mid m)\right) = \left\{1 - z^2, \tilde{\infty}\right\}$$

09.41.04.0007.01

$$\mathcal{R}_m\left(\mathrm{dn}^{-1}(z \mid m), 1 - z^2\right) = \log$$

09.41.04.0008.01

$$\mathcal{R}_m\left(\mathrm{dn}^{-1}(z \mid m), \tilde{\infty}\right) = 2$$

With respect to *z*

For fixed m, the function $\mathrm{dn}^{-1}(z \mid m)$ has six branch points: $z = 0$, $z = \pm 1$, $z = \pm \sqrt{1-m}$, $z = \tilde{\infty}$.

09.41.04.0009.01

$$\mathcal{BP}_z\left(\mathrm{dn}^{-1}(z \mid m)\right) = \left\{0, 1, -1, \sqrt{1-m}, -\sqrt{1-m}, \tilde{\infty}\right\}$$

09.41.04.0010.01

$$\mathcal{R}_z\left(\mathrm{dn}^{-1}(z \mid m), 0\right) = \log$$

09.41.04.0011.01

$$\mathcal{R}_z\left(\operatorname{dn}^{-1}(z \mid m), 1\right) = 2$$

09.41.04.0012.01

$$\mathcal{R}_z\left(\operatorname{dn}^{-1}(z \mid m), -1\right) = 2$$

09.41.04.0013.01

$$\mathcal{R}_z\left(\operatorname{dn}^{-1}(z \mid m), \sqrt{1-m}\right) = 2$$

09.41.04.0014.01

$$\mathcal{R}_z\left(\operatorname{dn}^{-1}(z \mid m), -\sqrt{1-m}\right) = 2$$

09.41.04.0015.01

$$\mathcal{R}_z\left(\operatorname{dn}^{-1}(z \mid m), \tilde{\infty}\right) = \log$$

Branch Cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z = 0$

09.41.06.0001.02

$$\operatorname{dn}^{-1}(z \mid m) \propto \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - z - \frac{(2-m) z^3}{6(1-m)} - \frac{\left(8 - 8m + 3m^2\right) z^5}{40(-1+m)^2} - \ldots \right) /; (z \to 0)$$

09.41.06.0002.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - \sum_{k=0}^{\infty} \frac{(1-m)^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)\, k!} \, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1} \right)$$

09.41.06.0007.01

$$\operatorname{dn}^{-1}(z \mid m) \propto \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right)(1 + O(z))$$

Expansions at $m = 0$

09.41.06.0003.02

$$\operatorname{dn}^{-1}(z \mid m) \propto \frac{\sqrt{1-m}}{\sqrt{m-1}} \left(-\frac{1}{2}\left(1 + \frac{m}{4} + \frac{9m^2}{64} + \ldots\right) \log(-m) + \log(4) + \frac{\log(4) - 1}{4} m + \frac{3(6\log(4) - 7)}{128} m^2 + \ldots \right) -$$
$$\frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} \left(\frac{\tanh^{-1}(z)}{z} + \frac{1}{4}\left(\frac{\tanh^{-1}(z)}{z} + \frac{1}{1-z^2}\right) m + \frac{3\left(-3z^3 + 5z + 3\left(z^2-1\right)^2 \tanh^{-1}(z)\right)}{64 z \left(z^2-1\right)^2} m^2 + \ldots \right) /; (m \to 0)$$

09.41.06.0004.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \, {}_2F_1\left(\frac{1}{2}, k+1; \frac{3}{2}; z^2\right) m^k /; |m| < 1$$

09.41.06.0005.01

$$\mathrm{dn}^{-1}(z \mid m) \propto \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{m^k z^{2j} (j+k)! \left(\frac{1}{2}\right)_k}{(2j+1)\, j!\, k!} \;/;\; |m| < 1$$

09.41.06.0008.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \left(\tanh^{-1}(z) + \frac{z}{2} \sum_{j=1}^{k} \frac{\left(\frac{1}{1-z^2}\right)^j (j-1)!}{\left(\frac{1}{2}\right)_j}\right) m^k \;/;\; |m| < 1$$

09.41.06.0006.01

$$\mathrm{dn}^{-1}(z \mid m) \propto \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} F_{0\times 1\times 0}^{1\times 1\times 1}\left(\begin{matrix} 1;\ \frac{1}{2};\ \frac{1}{2}; \\ ;\ \frac{3}{2};\ ; \end{matrix}\ z^2, m\right)$$

09.41.06.0009.01

$$\mathrm{dn}^{-1}(z \mid m) \propto -\frac{\sqrt{1-m}}{2\sqrt{m-1}} \log\left(-\frac{m}{4}\right)(1+O(m)) - \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \tanh^{-1}(z)\,(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.41.07.0001.01

$$\mathrm{dn}^{-1}(z \mid m) \propto \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\, dt \;/;\; -1 < z < 1 \bigwedge z^2 + m > 1$$

09.41.07.0002.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{\sqrt{1-z^2}\ \mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+m-1}} \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\, dt \;/;$$

$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left(1-((z-1)\tau+1)^2\right) = 0 \bigwedge 1-((z-1)\tau+1)^2 < 0 \bigwedge \mathrm{Im}\left(((z-1)\tau+1)^2+m-1\right) = 0 \bigwedge ((z-1)\tau+1)^2+m-1 < 0\right)$$

09.41.07.0003.01

$$\mathrm{dn}^{-1}(z \mid m) = \mathrm{dn}^{-1}(z_0 \mid m) - \frac{\sqrt{1-z^2}\ \mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+m-1}} \int_{z_0}^{z} \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\, dt \;/;$$

$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left(1-(\tau(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-(\tau(z-z_0)+z_0)^2 < 0 \bigwedge \mathrm{Im}\left((\tau(z-z_0)+z_0)^2+m-1\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2+m-1 < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.41.13.0001.01

$$w''(z) - \left(2 z^2 + m - 2\right) z\, w'(z)^3 == 0 \;/;\; w(z) == \mathrm{dn}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.41.16.0001.01

$$\mathrm{dn}^{-1}(-z \mid m) == \mathrm{dn}^{-1}(z \mid m) - \frac{2}{\sqrt{m-1}} F\!\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)$$

Products, sums, and powers of the direct function

09.41.16.0002.01

$$\mathrm{dn}^{-1}(z_1 \mid m) + \mathrm{dn}^{-1}(z_2 \mid m) == \mathrm{dn}^{-1}\!\left(\frac{m\, z_1\, z_2 + \sqrt{\left(1-z_1^2\right)\left(z_1^2+m-1\right)\left(1-z_2^2\right)\left(z_2^2+m-1\right)}}{m - \left(1-z_1^2\right)\left(1-z_2^2\right)} \,\middle|\, m\right)$$

Identities

Functional identities

09.41.17.0001.01

$$\left(z_2^2 + m + \left(z_2^2 - 1\right)\left(-z_1^2\right) - 1\right) \mathrm{dn}(w(z_1) + w(z_2) \mid m)^2 - 2\, m\, z_1\, z_2\, \mathrm{dn}(w(z_1) + w(z_2) \mid m) + \left(z_2^2 + m - 1\right) z_1^2 + (m-1)\left(z_2^2 - 1\right) == 0 \;/;\; w(z) == \mathrm{dn}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to z

09.41.20.0001.02

$$\frac{\partial \mathrm{dn}^{-1}(z \mid m)}{\partial z} == -\frac{\mathrm{cs}\!\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2 + m - 1}$$

09.41.20.0002.01

$$\frac{\partial \mathrm{dn}^{-1}(z \mid m)}{\partial z} == -\frac{1}{\sqrt{1-z^2}\,\sqrt{z^2+m-1}} \;/;\; -1 < z < 1 \bigwedge z^2 + m > 1$$

09.41.20.0003.02

$$\frac{\partial^2 \mathrm{dn}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(2 z^2 + m - 2\right) \mathrm{cs}\!\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{\left(z^2 - 1\right)\left(z^2 + m - 1\right)^2}$$

09.41.20.0011.01

$$\frac{\partial^2 \mathrm{dn}^{-1}(z \mid m)}{\partial z^2} = -\frac{\sqrt{1-z^2}\ \mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+m-1}} \frac{\partial \frac{1}{\sqrt{1-z^2}\ \sqrt{z^2+m-1}}}{\partial z}$$

With respect to *m*

09.41.20.0004.01

$$\frac{\partial \mathrm{dn}^{-1}(z \mid m)}{\partial m} = \frac{E\left(\mathrm{am}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\,\mathrm{dn}^{-1}(z \mid m) - z\,\mathrm{sc}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)}{2\,(1-m)\,m}$$

09.41.20.0005.01

$$\frac{\partial \mathrm{dn}^{-1}(z \mid m)}{\partial m} = \frac{1}{2\,m}\left(\frac{\sqrt{1-z^2}\ z}{(m-1)\sqrt{z^2+m-1}} + \frac{1}{\sqrt{m-1}}\left(E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) - E\left(\frac{1}{1-m}\right)\right)\right) /;\ 0<z<1 \land m>1$$

09.41.20.0006.02

$$\frac{\partial^2 \mathrm{dn}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4\,(m-1)^2\,m^2}\Bigg(3\,\mathrm{dn}^{-1}(z \mid m)\,(m-1)^2 + F\left(\mathrm{am}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) \mid m\right)(m-1) + (4\,m-2)\,E\left(\mathrm{am}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{\left((3-5\,m)\,z^3 + \left(-6\,m^2+9\,m-3\right)z + (m-1)\sqrt{z^2}\ \left(z^2+m-1\right)\right)\mathrm{sc}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)}{z^2+m-1}\Bigg)$$

09.41.20.0012.01

$$\frac{\partial^2 \mathrm{dn}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{8\,(m-1)^3\,m^3}\Bigg((-23\,(m-1)\,m-8)\,E\left(\mathrm{am}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) \mid m\right) - (m-1)\,(11\,m-7)\,F\left(\mathrm{am}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left(z^2+m-1\right)^3}\Big(\left(z^2+m-1\right)\Big((m\,(33\,m-40)+15)\,z^5 - (m-1)\,(10\,m-7)\sqrt{z^2}\ z^4 + 5\,(m-1)\,(m\,(15\,m-17)+6)\,z^3 + 15\,(m-1)^2\,(3\,(m-1)\,m+1)\,z - (m-1)^2\sqrt{z^2}\ \left(11\,m^2-18\,m+7\,(3\,m-2)\,z^2+7\right)\Big)\,\mathrm{sc}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) - 15\,(m-1)^3\left(z^2+m-1\right)^3\mathrm{dn}^{-1}(z \mid m)\Big)\Bigg)$$

Symbolic differentiation

With respect to *z*

09.41.20.0013.01

$$\frac{\partial^n \mathrm{dn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \, \mathrm{dn}^{-1}(z \mid m) - \frac{\mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2+m-1} \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(1-z^2\right)^{-k} \left(z^2+m-1\right)^{k-j} /;\, n \in \mathbb{N}$$

09.41.20.0014.01

$$\frac{\partial^n \mathrm{dn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \, \mathrm{dn}^{-1}(z \mid m) - \frac{\mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2+m-1} \sum_{j=0}^{n-1} \frac{(-1)^j\, 2^{2j-n+1}\, z^{2j-n+1} \left(z^2+m-1\right)^{-j} \left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(-j+n-1)!} \, {}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2}-j; -\frac{z^2+m-1}{1-z^2}\right) /;\, n \in \mathbb{N}$$

09.41.20.0015.01

$$\frac{\partial^n \mathrm{dn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \, \mathrm{dn}^{-1}(z \mid m) - \frac{\sqrt{1-z^2}\; \mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+m-1}} \frac{\partial^{n-1} \frac{1}{\sqrt{1-z^2}\,\sqrt{z^2+m-1}}}{\partial z^{n-1}} /;\, n \in \mathbb{N}^+$$

09.41.20.0007.01

$$\frac{\partial^n \mathrm{dn}^{-1}(z \mid m)}{\partial z^n} = -\frac{2^{n-1}\, \pi\, z^{n-1}\, (n-1)!\, \mathrm{cs}\left(\mathrm{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2+m-1} \sum_{j=0}^{n-1} \frac{\left(z^2-1\right)^{-j} \left(z^2+m-1\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right) \Gamma\left(j-n+\frac{3}{2}\right)} \, {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; 1-\frac{1}{z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; \frac{m-1}{z^2}+1\right) /;\, n \in \mathbb{N}^+$$

With respect to m

09.41.20.0008.02

$$\frac{\partial^n \mathrm{dn}^{-1}(z \mid m)}{\partial m^n} = \frac{\sqrt{\pi}\, (m-1)^{-n-\frac{1}{2}}}{2\,\Gamma\left(\frac{1}{2}-n\right)} \left(\pi \, {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; 1; \frac{1}{1-m}\right) - 2\, z\, F_1\left(\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{1-m}\right)\right) /;\, n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

09.41.20.0009.01

$$\frac{\partial^\alpha \mathrm{dn}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{z^{-\alpha}}{\sqrt{m-1}\; \Gamma(1-\alpha)} K\left(\frac{1}{1-m}\right) - \frac{z^{1-\alpha}\sqrt{\pi}}{\sqrt{m-1}} \tilde{F}^{2\times 1\times 1}_{2\times 0\times 0}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;; \end{matrix} \; z^2, \frac{z^2}{1-m}\right) /; \; -1<z<1 \bigwedge z^2+m>1 \bigwedge m>0$$

With respect to m

09.41.20.0010.01

$$\frac{\partial^{\alpha} \operatorname{dn}^{-1}(z \mid m)}{\partial m^{\alpha}} = -\frac{m^{-\alpha} \sqrt{\pi}}{2\sqrt{1-z^2}} \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha; \end{matrix} \frac{1}{1-z^2}, \frac{m}{1-z^2}\right) + i \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \left(\psi(k+1) - \psi\left(k+\frac{1}{2}\right)\right) m^k}{k!\, \Gamma(k-\alpha+1)} - \frac{i\, m^{-\alpha}}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \mathcal{FC}_{\log}^{(\alpha)}(m, k)\, m^k}{k!^2} /; z > 1 \land -1 < m < 0$$

Integration

Indefinite integration

For the direct function with respect to *z*

09.41.21.0001.01

$$\int \operatorname{dn}^{-1}(z \mid m)\, dz = \operatorname{dn}^{-1}(z \mid m)\, z - i \log\left(i \operatorname{cn}\left(\operatorname{dn}^{-1}(z \mid m) \,\middle|\, m\right) + \operatorname{sn}\left(\operatorname{dn}^{-1}(z \mid m) \,\middle|\, m\right)\right)$$

For the direct function with respect to *m*

09.41.21.0002.01

$$\int \operatorname{dn}^{-1}(z \mid m)\, dm = 2\sqrt{m-1} \left(E\left(\frac{1}{1-m}\right) - E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)\right) /; z < 1 \land m > 1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.41.26.0001.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} F_{0\times 1\times 0}^{1\times 1\times 1}\left(\begin{matrix} 1; \frac{1}{2}; \frac{1}{2}; \\ ; \frac{3}{2}; ; \end{matrix} z^2, m\right)$$

Through other functions

Involving some hypergeometric-type functions

09.41.26.0002.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{1-m}\right)\right) /; -1 < z < 1 \bigwedge z^2 + m > 1 \bigwedge m > 0$$

Representations through equivalent functions

With inverse function

09.41.27.0001.01

$$\operatorname{dn}\left(\operatorname{dn}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving cd^{-1}

09.41.27.0002.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \mathrm{cd}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /; -1 < z < 1 \land m > 1$$

Involving cn^{-1}

09.41.27.0003.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \mathrm{cn}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m > 1$$

09.41.27.0004.01

$$\mathrm{dn}^{-1}(z \mid m) = \mathrm{cn}^{-1}\left(\sqrt{\frac{z^2+m-1}{m}} \,\middle|\, m\right) /; 0 < z < 1 \land m > 1$$

Involving cs^{-1}

09.41.27.0005.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(i\, K\left(\frac{1}{1-m}\right) + \mathrm{cs}^{-1}\left(\frac{i}{z} \,\middle|\, \frac{m}{m-1}\right)\right) /; -1 < z < 1 \land m > 1$$

Involving dc^{-1}

09.41.27.0006.01

$$\mathrm{dn}^{-1}(z \mid m) = 2\, K(m) + i\, \mathrm{dc}^{-1}(z \mid 1-m) /; -1 < z < 1 \land m < 0 \lor z \in \mathbb{R} \land m > 1$$

Involving ds^{-1}

09.41.27.0007.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{1}{\sqrt{m}} \mathrm{ds}^{-1}\left(\frac{\sqrt{m-1}}{\sqrt{m}\, z} \,\middle|\, \frac{1}{m}\right) /; 0 < z < 1 \land m > 1$$

Involving nc^{-1}

09.41.27.0008.01

$$\mathrm{dn}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} \mathrm{nc}^{-1}\left(z \,\middle|\, 1 - \frac{1}{m}\right) /; z < 1 \land m > 1$$

Involving nd^{-1}

09.41.27.0009.01

$$\mathrm{dn}^{-1}(z \mid m) = \mathrm{nd}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; -1 < z < 0 \land m < 0 \lor z < 1 \land m > 1$$

Involving ns^{-1}

09.41.27.0010.01

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - \mathrm{ns}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{1-m}\right)\right) /; -1 < z < 1 \land m > 1$$

Involving sc^{-1}

09.41.27.0011.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - i \operatorname{sc}^{-1}\left(-i z \,\middle|\, \frac{m}{m-1}\right)\right) /; -1 < z < 1 \land m > 1$$

Involving sd^{-1}

09.41.27.0012.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + \frac{i}{\sqrt{m}} \operatorname{sd}^{-1}\left(\frac{i \sqrt{m}\, z}{\sqrt{m-1}} \,\middle|\, 1 - \frac{1}{m}\right) /; -1 < z < 1 \land m > 1$$

Involving sn^{-1}

09.41.27.0013.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(\operatorname{sn}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) - K\left(\frac{1}{1-m}\right)\right) /; z \in \mathbb{R} \bigwedge z^2 + m < 1 \bigwedge m > 0$$

09.41.27.0014.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - \operatorname{sn}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right)\right) /; -1 < z < 1 \land m > 1$$

09.41.27.0015.01

$$\operatorname{dn}^{-1}(z \mid m) = \operatorname{sn}^{-1}\left(\sqrt{\frac{1-z^2}{m}} \,\middle|\, m\right) /; 0 < z < 1 \land m > 1$$

Involving elliptic integrals

09.41.27.0016.02

$$\operatorname{dn}^{-1}(z \mid m) = -\frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)\right) /; (z < 1 \land m > 1) \lor (|z| < 1 \land 0 < m < 1)$$

09.41.27.0018.01

$$\operatorname{dn}^{-1}(z \mid m) = \frac{\sqrt{1-z^2}\, \operatorname{cs}\left(\operatorname{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+m-1}} \left(\frac{1}{\sqrt{m}} \sqrt{\frac{m}{m-1}}\, K\left(\frac{1}{1-m}\right) - \frac{1}{\sqrt{z^2+m-1}} \sqrt{\frac{z^2+m-1}{m-1}}\, F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\operatorname{Im}\left(1 - ((z-1)\tau+1)^2\right) = 0 \bigwedge 1 - ((z-1)\tau+1)^2 < 0 \bigwedge \operatorname{Im}\left(((z-1)\tau+1)^2 + m - 1\right) = 0 \bigwedge ((z-1)\tau+1)^2 + m - 1 < 0\right)$$

09.41.27.0019.01

$$\operatorname{dn}^{-1}(z \mid m) = \operatorname{dn}^{-1}(z_0 \mid m) - \frac{\sqrt{1-z^2}\, \operatorname{cs}\left(\operatorname{dn}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+m-1}}$$

$$\left(\frac{1}{\sqrt{z^2+m-1}} \sqrt{\frac{z^2+m-1}{m-1}}\, F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) - \frac{1}{\sqrt{z_0^2+m-1}} \sqrt{\frac{z_0^2+m-1}{m-1}}\, F\left(\sin^{-1}(z_0) \,\middle|\, \frac{1}{1-m}\right)\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\operatorname{Im}\left(1 - (\tau(z-z_0)+z_0)^2\right) = 0 \bigwedge 1 - (\tau(z-z_0)+z_0)^2 < 0 \bigwedge \operatorname{Im}\left((\tau(z-z_0)+z_0)^2 + m - 1\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2 + m - 1 < 0\right)$$

Involving other related functions

09.41.27.0017.01

$$\mathrm{dn}^{-1}(z \mid m) == -\frac{i\sqrt{z_2^2}}{z_2}\,(K(1-m) + \mathrm{elog}(z_1,\, z_2;\, a,\, b))\ /;$$
$$\{a,\, b,\, z_1\} == \{m-2,\, 1-m,\, z^2\} \bigwedge z_1^3 + a\, z_1^2 + b\, z_1 - z_2^2 == 0 \bigwedge 0 < z < 1 \bigwedge m > 1$$

History

– N. H. Abel (1826)
– A.G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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