InverseJacobiDS

Notations

Traditional name

Inverse of the Jacobi elliptic function ds

Traditional notation

$\mathrm{ds}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiDS[z, m]`

Primary definition

09.42.02.0001.01

$$z = \mathrm{ds}(w \mid m) \,/;\, w = \mathrm{ds}^{-1}(z \mid m)$$

09.42.02.0002.01

$$\mathrm{ds}^{-1}(z \mid m) = \int_z^{\infty} \frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt \,/;\, z \in \mathbb{R} \bigwedge z^2+m>1$$

Specific values

Specialized values

For fixed *z*

09.42.03.0001.01

$$\mathrm{ds}^{-1}(z \mid 0) = \csc^{-1}(z)$$

09.42.03.0002.01

$$\mathrm{ds}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = \sqrt{2}\, F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\, z}\right) \,\middle|\, -1\right) \,/;\, z>1$$

09.42.03.0003.01

$$\mathrm{ds}^{-1}(z \mid 1) = \mathrm{csch}^{-1}(z)$$

For fixed *m*

09.42.03.0004.01

$$\mathrm{ds}^{-1}(-1 \mid m) = \frac{1}{\sqrt{m}}\left(K\left(\frac{1}{m}\right) + i\, F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right)\right) \,/;\, m>1$$

09.42.03.0005.01

$$\text{ds}^{-1}\left(-\frac{1}{2} \,\middle|\, m\right) = \frac{2 i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{1}{2\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) + \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(2\sqrt{1-m}\right) \,\middle|\, \frac{m}{m-1}\right) /; m > 1$$

09.42.03.0006.01

$$\text{ds}^{-1}(0 \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m > 1$$

09.42.03.0007.01

$$\text{ds}^{-1}\left(\frac{1}{2} \,\middle|\, m\right) = \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(2\sqrt{1-m}\right) \,\middle|\, \frac{m}{m-1}\right) /; m > 1$$

09.42.03.0008.01

$$\text{ds}^{-1}(1 \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - i\, F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right)\right) /; m > 1$$

Values at infinities

09.42.03.0009.01

$$\text{ds}^{-1}(z \mid \infty) = 0$$

09.42.03.0010.01

$$\text{ds}^{-1}(z \mid -\infty) = 0$$

09.42.03.0011.01

$$\text{ds}^{-1}(\infty \mid m) = 0$$

09.42.03.0012.01

$$\text{ds}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m > 1$$

General characteristics

Domain and analyticity

$\text{ds}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.42.04.0001.01

$$(z * m) \longrightarrow \text{ds}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.42.04.0002.01

$$\text{ds}^{-1}(\bar{z} \mid \bar{m}) = \overline{\text{ds}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.42.04.0003.01

$$\text{ds}^{-1}(-z \mid m) = \frac{2 i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{z}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) + \text{ds}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to *m*

The function $\text{ds}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to *m*.

09.42.04.0004.01

$$\mathcal{S}ing_m\left(\text{ds}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\text{ds}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to *z*.

09.42.04.0005.01

$$\mathcal{S}ing_z\left(\text{ds}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed *z*, the function $\text{ds}^{-1}(z \mid m)$ has three branch points: $m = -z^2$, $m = 1 - z^2$, $m = \tilde{\infty}$.

09.42.04.0006.01

$$\mathcal{BP}_m\left(\text{ds}^{-1}(z \mid m)\right) = \left\{-z^2,\, 1 - z^2,\, \tilde{\infty}\right\}$$

09.42.04.0007.01

$$\mathcal{R}_m\left(\text{ds}^{-1}(z \mid m),\, -z^2\right) = \log$$

09.42.04.0008.01

$$\mathcal{R}_m\left(\text{ds}^{-1}(z \mid m),\, 1 - z^2\right) = \log$$

09.42.04.0009.01

$$\mathcal{R}_m\left(\text{ds}^{-1}(z \mid m),\, \tilde{\infty}\right) = 2$$

With respect to *z*

For fixed *m*, the function $\text{ds}^{-1}(z \mid m)$ has six branch points: $z = 0$, $z = \pm\sqrt{-m}$, $z = \pm\sqrt{1-m}$, $z = \tilde{\infty}$.

09.42.04.0010.01

$$\mathcal{BP}_z\left(\text{ds}^{-1}(z \mid m)\right) = \left\{0,\, \sqrt{-m},\, -\sqrt{-m},\, \sqrt{1-m},\, -\sqrt{1-m},\, \tilde{\infty}\right\}$$

09.42.04.0011.01

$$\mathcal{R}_z\left(\text{ds}^{-1}(z \mid m),\, 0\right) = \log$$

09.42.04.0012.01

$$\mathcal{R}_z\left(\text{ds}^{-1}(z \mid m),\, \sqrt{-m}\right) = 2$$

09.42.04.0013.01

$$\mathcal{R}_z\left(\text{ds}^{-1}(z \mid m),\, -\sqrt{-m}\right) = 2$$

09.42.04.0014.01

$$\mathcal{R}_z\left(\text{ds}^{-1}(z \mid m),\, \sqrt{1-m}\right) = 2$$

09.42.04.0015.01

$$\mathcal{R}_z\left(\mathrm{ds}^{-1}(z \mid m), -\sqrt{1-m}\right) == 2$$

09.42.04.0016.01

$$\mathcal{R}_z\left(\mathrm{ds}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z == 0$

09.42.06.0001.02

$$\mathrm{ds}^{-1}(z \mid m) \propto \sqrt{\frac{1}{m}}\, K\left(\frac{1}{m}\right) - \frac{z}{\sqrt{m-1}\,\sqrt{m}} + \frac{(2\,m-1)\,z^3}{6\,(m-1)^{3/2}\,m^{3/2}} - \ldots\ /;\ (z \to 0)$$

09.42.06.0002.01

$$\mathrm{ds}^{-1}(z \mid m) == \sqrt{\frac{1}{m}}\, K\left(\frac{1}{m}\right) - \frac{1}{\sqrt{m-1}\,\sqrt{m}} \sum_{k=0}^{\infty} \frac{(-1)^k\, m^{-k} \left(\frac{1}{2}\right)_k}{(2\,k+1)\,k!}\, {}_2F_1\left(\frac{1}{2}, -k;\ \frac{1}{2}-k;\ \frac{m}{m-1}\right) z^{2\,k+1}$$

09.42.06.0007.01

$$\mathrm{ds}^{-1}(z \mid m) \propto \sqrt{\frac{1}{m}}\, K\left(\frac{1}{m}\right)(1 + O(z))$$

Expansions at $m == 0$

09.42.06.0003.02

$$\mathrm{ds}^{-1}(z \mid m) \propto \csc^{-1}(z) + \frac{1}{4}\left(\csc^{-1}(z) + \frac{z^2+1}{z\left(1-z^2\right)} \sqrt{1-\frac{1}{z^2}}\right) m + \frac{1}{64}\left(9\csc^{-1}(z) - \frac{\left(9\,z^6 - 12\,z^4 - 11\,z^2 + 6\right)}{z^3\left(z^2-1\right)^2} \sqrt{1-\frac{1}{z^2}}\right) m^2 + \ldots\ /;$$

$$(m \to 0)$$

09.42.06.0008.01

$$\mathrm{ds}^{-1}(z \mid m) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^j \left(\frac{1}{2}\right)_{j-k} \left(\frac{1}{2}\right)_k z^{-2\,j-1}}{(2\,j+1)\,(j-k)!\,k!}\, {}_2F_1\left(j+\frac{1}{2}, k+\frac{1}{2};\ j+\frac{3}{2};\ \frac{1}{z^2}\right) m^j$$

09.42.06.0004.01

$$\mathrm{ds}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{-2\,j-2\,k-1}\,(-1)^{j+k} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k}{(2\,j+2\,k+1)\,j!\,k!}\, {}_2F_1\left(j+\frac{1}{2}, j+k+\frac{1}{2};\ j+k+\frac{3}{2};\ \frac{1}{z^2}\right) m^{j+k}$$

09.42.06.0005.01

$$\mathrm{ds}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+k}\, m^{j+k}\, z^{-2\,j-2\,k-2\,l-1} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j+l}}{(2\,j+2\,k+2\,l+1)\,j!^2\,k!^2}$$

09.42.06.0006.01

$$\mathrm{ds}^{-1}(z\mid m) \equiv \frac{1}{z}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2 z^{-2k}}{\left(\frac{3}{2}\right)_k} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} k+\frac{1}{2}; k+\frac{1}{2}; \frac{1}{2}; \\ k+\frac{3}{2}; 1; 1; \end{matrix} -\frac{m}{z^2}, -\frac{m}{z^2}\right)$$

09.42.06.0009.01

$$\mathrm{ds}^{-1}(z\mid m) \propto \csc^{-1}(z)\,(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.42.07.0001.01

$$\mathrm{ds}^{-1}(z\mid m) \equiv \int_z^{\infty}\frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt\,/;\, z\in\mathbb{R}\bigwedge z^2+m>1$$

09.42.07.0002.01

$$\mathrm{ds}^{-1}(z\mid m) = \mathrm{ds}^{-1}(z_0\mid m) - \frac{\sqrt{z^2+m-1}\,\mathrm{nc}\left(\mathrm{ds}^{-1}(z\mid m)\,\middle|\,m\right)}{\sqrt{z^2+m}}\int_{z_0}^{z}\frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt\,/;$$

$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2+m\right)=0\bigwedge(\tau\,(z-z_0)+z_0)^2+m<0\bigwedge \mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2+m-1\right)=0\bigwedge(\tau\,(z-z_0)+z_0)^2+m-1<0\right)$$

09.42.07.0003.01

$$\mathrm{ds}^{-1}(z\mid m) = \frac{\sqrt{z^2+m-1}\,\mathrm{nc}\left(\mathrm{ds}^{-1}(z\mid m)\,\middle|\,m\right)}{\sqrt{z^2+m}}\int_{z}^{\infty}\frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt\,/;$$

$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+m\right)=0\bigwedge\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+m<0\bigwedge \mathrm{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+m-1\right)=0\bigwedge\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+m-1<0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.42.13.0001.01

$$w''(z)+\left(2\,z^2+2\,m-1\right)z\,w'(z)^3 \equiv 0\,/;\, w(z)\equiv \mathrm{ds}^{-1}(z\mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.42.16.0001.01

$$\operatorname{ds}^{-1}(-z \mid m) = \frac{2 i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + \operatorname{ds}^{-1}(z \mid m)$$

Identities

Functional identities

09.42.17.0001.01

$$\left(z_1^2 - z_2^2\right)^2 \operatorname{ds}(w(z_1) + w(z_2) \mid m)^4 - 2\left(z_2^2 z_1^4 + \left(z_2^4 + (4m-2) z_2^2 + (m-1) m\right) z_1^2 + (m-1) m z_2^2\right) \operatorname{ds}(w(z_1) + w(z_2) \mid m)^2 + \left((m-1) m - z_1^2 z_2^2\right)^2 = 0 \;/;\; w(z) = \operatorname{ds}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.42.20.0001.02

$$\frac{\partial \operatorname{ds}^{-1}(z \mid m)}{\partial z} = -\frac{\operatorname{nc}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)}{z^2 + m}$$

09.42.20.0002.01

$$\frac{\partial \operatorname{ds}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\sqrt{z^2 + m} \sqrt{z^2 + m - 1}} \;/;\; z \in \mathbb{R} \bigwedge z^2 + m > 1$$

09.42.20.0003.02

$$\frac{\partial^2 \operatorname{ds}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(2z^2 + 2m - 1\right) \operatorname{nc}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)}{\left(z^2 + m - 1\right)\left(z^2 + m\right)^2}$$

09.42.20.0011.01

$$\frac{\partial^2 \operatorname{ds}^{-1}(z \mid m)}{\partial z^2} = -\frac{\sqrt{z^2 + m - 1} \operatorname{nc}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2 + m}} \frac{\partial \frac{1}{\sqrt{z^2+m} \sqrt{z^2+m-1}}}{\partial z}$$

With respect to *m*

09.42.20.0004.02

$$\frac{\partial \operatorname{ds}^{-1}(z \mid m)}{\partial m} = -\frac{E\left(\operatorname{am}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1) \operatorname{ds}^{-1}(z \mid m) - \frac{m z \operatorname{nc}\left(\operatorname{ds}^{-1}(z|m)|m\right)}{z^2+m}}{2(m-1) m}$$

09.42.20.0005.01

$$\frac{\partial \operatorname{ds}^{-1}(z \mid m)}{\partial m} = \frac{1}{2}\left(\frac{z\left(z^2+2m-1\right)}{(m-1)\, m \sqrt{z^2+m-1}\, \sqrt{z^2+m}} + \frac{1}{(m-1)\sqrt{m}}\left((m-1)\, K\!\left(\frac{1}{m}\right) - m\, E\!\left(\frac{1}{m}\right)\right) + \frac{i}{\sqrt{m}}\, E\!\left(\sin^{-1}\!\left(\frac{z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + \frac{i}{\sqrt{m-1}}\, E\!\left(i \sinh^{-1}\!\left(\frac{z}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) + \frac{1}{\sqrt{m-1}\; m}\left((m-1)\, E\!\left(\frac{1}{1-m}\right) - m\, K\!\left(\frac{1}{1-m}\right)\right)\right) /;\, z \in \mathbb{R} \land m > 1$$

09.42.20.0006.01

$$\frac{\partial^2 \operatorname{ds}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4\,(m-1)^2\, m^2}\Bigg((4m-2)\, E\!\left(\operatorname{am}\!\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\, F\!\left(\operatorname{am}\!\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left(z^2+m\right)\left(z^2+m-1\right)^2}\Bigg(3\left(z^2+m\right) \operatorname{ds}^{-1}(z \mid m)\left(m^2+\left(z^2-2\right) m - z^2+1\right)^2 + m \operatorname{cs}\!\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)\Bigg(\sqrt{\frac{z^2}{z^2+m}}\left(m^2+\left(z^2-2\right) m - z^2+1\right) - \left(-z^4+z^2+5 m^3+m^2\left(8 z^2-7\right)+m\left(3 z^4-7 z^2+2\right)\right) \operatorname{dn}\!\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)\Bigg)\Bigg)\Bigg)$$

09.42.20.0012.01

$$\frac{\partial^3 \mathrm{ds}^{-1}(z \mid m)}{\partial m^3} = \frac{1}{8\,(m-1)^3\, m^3} \left(\frac{1}{z\left(z^2+m-1\right)^3 \left(z^2+m\right)} \left(m\, z\, \mathrm{cn}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) \left(-(m-1)\, m\, z \left((3\,m-1)\, z^4 + (m\,(8\,m-7)+1)\, z^2 + (m-1)\, m\,(5\,m-2)\right) - (m-1)\left(z^2+m-1\right) \sqrt{\frac{z^2}{z^2+m}} \left(-z^6 + (5\,m-3)\, z^4 + (m\,(18\,m-19)+4)\, z^2 + (m-1)\, m\,(12\,m-7)\right) \mathrm{sn}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) + \mathrm{dn}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) \left(\frac{m\, z\left(z^2+m-1\right)(m-1)^2}{\sqrt{\frac{z^2}{z^2+m}}} + \left((3\,m\,(5\,m-4)+5)\, z^8 + (m\,(m\,(73\,m-94)+47)-10)\, z^6 + (m\,(m\,(m\,(139\,m-248)+158)-46)+5)\, z^4 + (m-1)\, m\,(m\,(m\,(119\,m-152)+66)-11)\, z^2 + (m-1)^2\, m^2\,(m\,(38\,m-29)+9)\right) \mathrm{sn}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) \right) \right) - z\left(z^2+m-1\right)^3 \left(z^2+m\right) \left((23\,(m-1)\, m+8)\, E\left(\mathrm{am}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\,(11\,m-7)\, F\left(\mathrm{am}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) \right) \right) - 15\,(m-1)^3\, \mathrm{ds}^{-1}(z \mid m) \right)$$

Symbolic differentiation

With respect to *z*

09.42.20.0013.01

$$\frac{\partial^n \mathrm{ds}^{-1}(z \mid m)}{\partial z^n} = \delta_n\, \mathrm{ds}^{-1}(z \mid m) - \frac{\mathrm{nc}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right)}{z^2+m} \sum_{j=0}^{n-1} \frac{(1-n)_{2\,(n-j)-2}}{(n-j-1)!\,(2\,z)^{-2\,j+n-1}} \sum_{k=0}^{j} (-1)^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2+m-1\right)^{-k} \left(z^2+m\right)^{k-j} \;/;\; n \in \mathbb{N}$$

09.42.20.0014.01

$$\frac{\partial^n \mathrm{ds}^{-1}(z \mid m)}{\partial z^n} = \delta_n\, \mathrm{ds}^{-1}(z \mid m) - \frac{\mathrm{nc}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right)}{z^2+m} \sum_{j=0}^{n-1} \frac{(-1)^j\, 2^{2\,j-n+1}\, z^{2\,j-n+1} \left(z^2+m\right)^{-j} \left(\frac{1}{2}\right)_j (1-n)_{2\,(n-j)-2}}{(-j+n-j-1)!}\; {}_2F_1\!\left(\frac{1}{2}, -j;\; \frac{1}{2}-j;\; \frac{z^2+m}{z^2+m-1}\right) \;/;\; n \in \mathbb{N}$$

09.42.20.0015.01

$$\frac{\partial^n \operatorname{ds}^{-1}(z \mid m)}{\partial z^n} = \delta_n \operatorname{ds}^{-1}(z \mid m) - \frac{\sqrt{z^2+m-1}\ \operatorname{nc}\left(\operatorname{ds}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+m}} \frac{\partial^{n-1} \frac{1}{\sqrt{z^2+m}\ \sqrt{z^2+m-1}}}{\partial z^{n-1}} /; n \in \mathbb{N}^+$$

09.42.20.0007.01

$$\frac{\partial^n \operatorname{ds}^{-1}(z \mid m)}{\partial z^n} = -\frac{2^{n-1} \pi z^{n-1} (n-1)! \operatorname{nc}\left(\operatorname{ds}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2+m} \sum_{j=0}^{n-1} \frac{\left(z^2+m-1\right)^{j-n+1} \left(z^2+m\right)^{-j}}{j! (n-j-1)! \Gamma\left(\frac{1}{2}-j\right) \Gamma\left(j-n+\frac{3}{2}\right)}$$
$$\, _2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; \frac{m}{z^2}+1\right) \, _2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; \frac{m-1}{z^2}+1\right) /; n \in \mathbb{N}^+$$

With respect to m

09.42.20.0008.02

$$\frac{\partial^n \operatorname{ds}^{-1}(z \mid m)}{\partial m^n} = \frac{z^{-2n-1}}{2n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}-k\right)_k \left(k-n+\frac{1}{2}\right)_{n-k} F_1\left(n+\frac{1}{2}; -k+n+\frac{1}{2}, k+\frac{1}{2}; n+\frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right) /;$$
$$|z| > 1 \land |m| > 1 \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

09.42.20.0009.01

$$\frac{\partial^\alpha \operatorname{ds}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{z^{-\alpha}}{\sqrt{m}\ \Gamma(1-\alpha)} K\left(\frac{1}{m}\right) - \frac{z^{1-\alpha} \sqrt{\pi}}{\sqrt{m-1}\ \sqrt{m}} \tilde{F}_{2\times0\times0}^{2\times1\times1}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;; \end{matrix} \frac{z^2}{1-m}, -\frac{z^2}{m}\right) /; z \in \mathbb{R} \land m > 1$$

With respect to m

09.42.20.0010.01

$$\frac{\partial^\alpha \operatorname{ds}^{-1}(z \mid m)}{\partial m^\alpha} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_j (j+k)! (-1)^{j+k} z^{-2j-2k-1} m^{j+k-\alpha}}{(2j+2k+1) \Gamma(j+k-\alpha+1) k! j!} \, _2F_1\left(\frac{1}{2}+j+k, j+\frac{1}{2}; \frac{3}{2}+j+k; \frac{1}{z^2}\right) /; z > 1 \land -1 < m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.42.21.0001.01

$$\int \operatorname{ds}^{-1}(z \mid m)\, dz = z \operatorname{ds}^{-1}(z \mid m) + \log\left(\operatorname{cs}\left(\operatorname{ds}^{-1}(z \mid m) \,\middle|\, m\right) + \operatorname{ns}\left(\operatorname{ds}^{-1}(z \mid m) \,\middle|\, m\right)\right)$$

Involving only one direct function with respect to m

09.42.21.0002.01

$$\int \operatorname{ds}^{-1}(z \mid m)\, dm = 2\left(\frac{\sqrt{z^2+m-1}\,\sqrt{z^2+m}}{z} - z\left(\log\left(\frac{\sqrt{z^2+m-1}+\sqrt{z^2+m}}{2z}\right)+1\right)+\sqrt{m-1}\, i\left(E\left(i\sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right) \,\middle|\, \frac{m}{m-1}\right) - F\left(i\sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right) \,\middle|\, \frac{m}{m-1}\right)\right)\right) /; z>0 \land m>0$$

Representations through more general functions

Through hypergeometric functions of two variables

09.42.26.0001.01

$$\operatorname{ds}^{-1}(z \mid m) = \frac{1}{z}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 z^{-2k}}{\left(\frac{3}{2}\right)_k} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} k+\frac{1}{2};\, k+\frac{1}{2};\, \frac{1}{2}; \\ k+\frac{3}{2};\, 1;\, 1; \end{matrix}\; -\frac{m}{z^2}, -\frac{m}{z^2}\right)$$

Through other functions

Involving some hypergeometric-type functions

09.42.26.0002.01

$$\operatorname{ds}^{-1}(z \mid m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right) /; z^2+m>1$$

09.42.26.0003.01

$$\operatorname{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - \frac{z}{\sqrt{m-1}\,\sqrt{m}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{z^2}{1-m}, -\frac{z^2}{m}\right) /; z \in \mathbb{R} \land m>1$$

Representations through equivalent functions

With inverse function

09.42.27.0001.01

$$\operatorname{ds}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right) = z$$

With related functions

Involving cd^{-1}

09.42.27.0002.01

$$\operatorname{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\left(K\left(\frac{m}{m-1}\right) - \operatorname{cd}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{m}{m-1}\right)\right) /; z>0 \land m>0$$

Involving cn^{-1}

09.42.27.0003.01

$$\operatorname{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \operatorname{cn}^{-1}\left(\frac{\sqrt{z^2+m-1}}{z} \,\middle|\, \frac{m}{m-1}\right) /; z>1 \land m>1$$

Involving cs^{-1}

09.42.27.0004.01

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}} \text{cs}^{-1}\left(\frac{i z}{\sqrt{1-m}} \,\middle|\, \frac{1}{1-m}\right) /; z \in \mathbb{R} \land m > 1$$

Involving dc^{-1}

09.42.27.0005.01

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - \text{dc}^{-1}\left(\frac{i z}{\sqrt{m}} \,\middle|\, \frac{m-1}{m}\right)\right) /; m > 1$$

Involving dn^{-1}

09.42.27.0006.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{1}{\sqrt{m}} \text{dn}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{1}{m}\right) /; z > 0 \land m > 1$$

Involving nc^{-1}

09.42.27.0007.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + i\, \text{nc}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, 1-m\right) /; z > 0 \land m < 1$$

Involving nd^{-1}

09.42.27.0008.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) + i\, \text{nd}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{1}{1-m}\right)\right) /; z > 0 \land m > 0$$

Involving ns^{-1}

09.42.27.0009.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{ns}^{-1}\left(\frac{z}{\sqrt{1-m}} \,\middle|\, \frac{m}{m-1}\right) /; z > 0 \land m > 0$$

Involving sc^{-1}

09.42.27.0010.01

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}} \text{sc}^{-1}\left(-\frac{i \sqrt{1-m}}{z} \,\middle|\, \frac{1}{1-m}\right) /; z > 0 \land m > 0$$

Involving sd^{-1}

09.42.27.0011.01

$$\text{ds}^{-1}(z \mid m) = \text{sd}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; z > 0 \land m > 1$$

Involving sn^{-1}

09.42.27.0012.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{sn}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{m}{m-1}\right) /; z > 0 \land m > 0$$

Involving elliptic integrals

09.42.27.0013.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(\frac{\sqrt{1-m}}{z}\right) \,\middle|\, \frac{m}{m-1}\right) /; z > 1 \land m > 1$$

09.42.27.0014.01

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - \frac{i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{z}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) /; z \in \mathbb{R} \land m > 1$$

09.42.27.0016.01

$$\text{ds}^{-1}(z \mid m) = \text{ds}^{-1}(z_0 \mid m) - \frac{\sqrt{z^2+m-1}\; \text{nc}\left(\text{ds}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+m}} \left(\frac{\sqrt{\frac{z^2+m-1}{m-1}} \sqrt{\frac{z^2+m}{m}}}{\sqrt{\frac{1}{1-m}} \sqrt{z^2+m-1} \sqrt{z^2+m}} F\left(\sin^{-1}\left(\sqrt{\frac{1}{1-m}}\, z\right) \,\middle|\, \frac{m-1}{m}\right) - \frac{\sqrt{\frac{z_0^2+m-1}{m-1}} \sqrt{\frac{z_0^2+m}{m}}}{\sqrt{\frac{1}{1-m}} \sqrt{z_0^2+m-1} \sqrt{z_0^2+m}} F\left(\sin^{-1}\left(\sqrt{\frac{1}{1-m}}\, z_0\right) \,\middle|\, \frac{m-1}{m}\right) \right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left((\tau (z-z_0)+z_0)^2+m\right) = 0 \bigwedge (\tau (z-z_0)+z_0)^2+m < 0 \bigwedge \text{Im}\left((\tau (z-z_0)+z_0)^2+m-1\right) = 0 \bigwedge (\tau (z-z_0)+z_0)^2+m-1 < 0 \right)$$

09.42.27.0017.01

$$\text{ds}^{-1}(z \mid m) = -\frac{i\, z^2\, \text{nc}\left(\text{ds}^{-1}(z \mid m) \mid m\right)}{\sqrt{m}\, (z^2+m)} \sqrt{\frac{z^2+m}{z^2}} \sqrt{\frac{z^2+m-1}{z^2}}\; F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{z}\right) \,\middle|\, 1-\frac{1}{m}\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left(\left(z+\tan\left(\frac{\pi \tau}{2}\right)\right)^2+m\right) = 0 \bigwedge \left(z+\tan\left(\frac{\pi \tau}{2}\right)\right)^2+m < 0 \bigwedge \text{Im}\left(\left(z+\tan\left(\frac{\pi \tau}{2}\right)\right)^2+m-1\right) = 0 \bigwedge \left(z+\tan\left(\frac{\pi \tau}{2}\right)\right)^2+m-1 < 0 \right)$$

Involving other related functions

09.42.27.0015.01

$$\text{ds}^{-1}(z \mid m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{2m-1, m(m-1), z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m > 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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