InverseJacobiND

Notations

Traditional name

Inverse of the Jacobi elliptic function nd

Traditional notation

$\mathrm{nd}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiND[z, m]`

Primary definition

09.44.02.0001.01

$$z = \mathrm{nd}(w \mid m) \;/;\; w = \mathrm{nd}^{-1}(z \mid m)$$

09.44.02.0002.01

$$\mathrm{nd}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\,z^2 < 1 \bigwedge m > 0$$

Specific values

Specialized values

For fixed *z*

09.44.03.0001.01

$$\mathrm{nd}^{-1}(z \mid 0) = \tilde{\infty}$$

09.44.03.0002.01

$$\mathrm{nd}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = i\sqrt{2}\,F\left(\frac{\pi}{4} \,\middle|\, 2\right) - \sqrt{2}\,i\,F\left(\sin^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, 2\right) \;/;\; -1 < z < 1$$

09.44.03.0003.01

$$\mathrm{nd}^{-1}(z \mid 1) = \cosh^{-1}(z)$$

For fixed *m*

09.44.03.0004.01

$$\mathrm{nd}^{-1}(-1 \mid m) = 2\,i\,K(1-m)$$

09.44.03.0005.01

$$\operatorname{nd}^{-1}\left(-\frac{1}{2} \,\middle|\, m\right) = i \left(F\left(\frac{\pi}{6} \,\middle|\, 1-m\right) + K(1-m)\right)$$

09.44.03.0006.01

$$\operatorname{nd}^{-1}(0 \mid m) = i\, K(1-m)$$

09.44.03.0007.01

$$\operatorname{nd}^{-1}\left(\frac{1}{2} \,\middle|\, m\right) = i \left(K(1-m) - F\left(\frac{\pi}{6} \,\middle|\, 1-m\right)\right)$$

09.44.03.0008.01

$$\operatorname{nd}^{-1}(1 \mid m) = 0$$

09.44.03.0009.01

$$\operatorname{nd}^{-1}(i \mid m) = i\, K(1-m) - i\, F\left(i \sinh^{-1}(1) \,\middle|\, 1-m\right)$$

09.44.03.0010.01

$$\operatorname{nd}^{-1}(-i \mid m) = i\, F\left(i \sinh^{-1}(1) \,\middle|\, 1-m\right) + i\, K(1-m)$$

Values at infinities

09.44.03.0011.01

$$\operatorname{nd}^{-1}(z \mid \infty) = 0$$

09.44.03.0012.01

$$\operatorname{nd}^{-1}(z \mid -\infty) = 0$$

09.44.03.0013.01

$$\operatorname{nd}^{-1}(\infty \mid m) = -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right)$$

09.44.03.0014.01

$$\operatorname{nd}^{-1}(-\infty \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + 2\, i\, K(1-m)$$

General characteristics

Domain and analyticity

$\operatorname{nd}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.44.04.0001.01

$$(z * m) \longrightarrow \operatorname{nd}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.44.04.0002.01

$$\operatorname{nd}^{-1}(\bar{z} \mid \bar{m}) = \overline{\operatorname{nd}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.44.04.0003.01

$$\mathrm{nd}^{-1}(-z \mid m) = 2\, i\, F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) + \mathrm{nd}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to *m*

The function $\mathrm{nd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.44.04.0004.01

$$\mathcal{Sing}_m\left(\mathrm{nd}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\mathrm{nd}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.44.04.0005.01

$$\mathcal{Sing}_z\left(\mathrm{nd}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\mathrm{nd}^{-1}(z \mid m)$ has two branch points: $m = \frac{z^2-1}{z^2}$, $m = \tilde{\infty}$.

09.44.04.0006.01

$$\mathcal{BP}_m\left(\mathrm{nd}^{-1}(z \mid m)\right) = \left\{\frac{z^2-1}{z^2}, \tilde{\infty}\right\}$$

09.44.04.0007.01

$$\mathcal{R}_m\left(\mathrm{nd}^{-1}(z \mid m), \frac{z^2-1}{z^2}\right) = \log$$

09.44.04.0008.01

$$\mathcal{R}_m\left(\mathrm{nd}^{-1}(z \mid m), \tilde{\infty}\right) = 2$$

With respect to *z*

For fixed m, the function $\mathrm{nd}^{-1}(z \mid m)$ has five branch points: $z = \pm 1$, $z = \pm \frac{1}{\sqrt{1-m}}$, $z = \tilde{\infty}$.

09.44.04.0009.01

$$\mathcal{BP}_z\left(\mathrm{nd}^{-1}(z \mid m)\right) = \left\{1, -1, \frac{1}{\sqrt{1-m}}, -\frac{1}{\sqrt{1-m}}, \tilde{\infty}\right\}$$

09.44.04.0010.01

$$\mathcal{R}_z\left(\mathrm{nd}^{-1}(z \mid m), 1\right) = 2$$

09.44.04.0011.01

$$\mathcal{R}_z\left(\mathrm{nd}^{-1}(z \mid m), -1\right) = 2$$

09.44.04.0012.01

$$\mathcal{R}_z\left(\text{nd}^{-1}(z \mid m), \frac{1}{\sqrt{1-m}}\right) = 2$$

09.44.04.0013.01

$$\mathcal{R}_z\left(\text{nd}^{-1}(z \mid m), -\frac{1}{\sqrt{1-m}}\right) = 2$$

09.44.04.0014.01

$$\mathcal{R}_z\left(\text{nd}^{-1}(z \mid m), \tilde{\infty}\right) = \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z = 0$

09.44.06.0001.02

$$\text{nd}^{-1}(z \mid m) \propto i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \left(z + \frac{2-m}{6} z^3 + \frac{8-8m+3m^2}{40} z^5 + \ldots\right) /; (z \to 0)$$

09.44.06.0002.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)\, k!}\, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{1-m}\right) z^{2k+1}$$

09.44.06.0007.01

$$\text{nd}^{-1}(z \mid m) \propto i\, K(1-m)\, (1 + O(z))$$

Expansions at $m = 0$

09.44.06.0003.02

$$\text{nd}^{-1}(z \mid m) \propto \left(-\frac{i}{2}\left(1 + \frac{m}{4} + \frac{9m^2}{64} + \ldots\right) \log(m) + i \log(4) + \frac{i}{4} (\log(4) - 1)\, m + \frac{3i}{128} (6 \log(4) - 7)\, m^2 + \ldots\right) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \left(\tanh^{-1}(z) + \frac{1}{4}\left(\frac{z}{z^2-1} + \tanh^{-1}(z)\right) m + \frac{3}{64}\left(\frac{\left(5z^2-3\right) z}{\left(z^2-1\right)^2} + 3 \tanh^{-1}(z)\right) m^2 + \ldots\right) /; (m \to 0)$$

09.44.06.0004.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)\, k!}\, {}_2F_1\left(k + \frac{1}{2}, k+1; k + \frac{3}{2}; z^2\right) m^k /; |m| < 1$$

09.44.06.0008.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \left(\tanh^{-1}(z) + \frac{1}{2\,z} \sum_{j=1}^{k} \frac{(j-1)!}{\left(\frac{1}{2}\right)_j} \left(\frac{z^2}{z^2-1}\right)^j\right) m^k \;/;\; |m| < 1$$

09.44.06.0005.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, m^k\, z^{2j+2k+1} \left(\frac{1}{2}\right)_k (j+k)!}{(2\,j+2\,k+1)\, j!\, k!^2}$$

09.44.06.0006.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}\; z}{\sqrt{z^2-1}}\, F_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} 1, \frac{1}{2}; \frac{1}{2}; ; \\ \frac{3}{2}; 1; ; \end{matrix} \; -m\, z^2, z^2\right)$$

09.44.06.0009.01

$$\text{nd}^{-1}(z \mid m) \propto \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \tanh^{-1}(z)\,(1+O(m)) - \frac{i}{2} \log\left(\frac{m}{16}\right)(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.44.07.0001.01

$$\text{nd}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\, dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\, z^2 < 1 \bigwedge m > 0$$

09.44.07.0002.01

$$\text{nd}^{-1}(z \mid m) = \frac{\sqrt{(m-1)\, z^2 + 1}\; \text{sc}\left(\text{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\, dt \;/;$$

$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\text{Im}\left(((z-1)\,\tau+1)^2-1\right) = 0 \bigwedge ((z-1)\,\tau+1)^2 - 1 < 0 \bigwedge \text{Im}\left(1-(1-m)\,((z-1)\,\tau+1)^2\right) = 0 \bigwedge 1-(1-m)\,((z-1)\,\tau+1)^2 < 0\right)$$

09.44.07.0003.01

$$\text{nd}^{-1}(z \mid m) = \text{nd}^{-1}(z_0 \mid m) + \frac{\sqrt{(m-1)\, z^2 + 1}\; \text{sc}\left(\text{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \int_{z_0}^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\, dt \;/;$$

$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\text{Im}\left((\tau\,(z-z_0)+z_0)^2-1\right) = 0 \bigwedge (\tau\,(z-z_0)+z_0)^2 - 1 < 0 \bigwedge \text{Im}\left(1-(1-m)\,(\tau\,(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-(1-m)\,(\tau\,(z-z_0)+z_0)^2 < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.44.13.0001.01

$$w''(z) - \left(2\,(1-m)\,z^2 + m - 2\right) z\, w'(z)^3 == 0 \;/;\; w(z) == \mathrm{nd}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.44.16.0001.01

$$\mathrm{nd}^{-1}(-z \mid m) == 2\, i\, F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) + \mathrm{nd}^{-1}(z \mid m)$$

Identities

Functional identities

09.44.17.0001.01

$$\left((m-1)\left(z_2^2-1\right) z_1^2 - (m-1)\, z_2^2 - 1\right) \mathrm{nd}(w(z_1)+w(z_2) \mid m)^2 + 2\, m\, z_1\, z_2\, \mathrm{nd}(w(z_1)+w(z_2) \mid m) - z_2^2 - z_1^2\left((m-1)\, z_2^2 + 1\right) + 1 == 0 \;/;\; w(z) == \mathrm{nd}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.44.20.0001.02

$$\frac{\partial \mathrm{nd}^{-1}(z \mid m)}{\partial z} == \frac{\mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2 - 1}$$

09.44.20.0002.01

$$\frac{\partial \mathrm{nd}^{-1}(z \mid m)}{\partial z} == \frac{1}{\sqrt{z^2-1}\,\sqrt{1-(1-m)\, z^2}} \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\, z^2 < 1 \bigwedge m > 0$$

09.44.20.0003.02

$$\frac{\partial^2 \mathrm{nd}^{-1}(z \mid m)}{\partial z^2} == -\frac{z\left(-2\, z^2 + m\left(2\, z^2 - 1\right) + 2\right) \mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\left(z^2-1\right)^2 \left((m-1)\, z^2 + 1\right)}$$

09.44.20.0011.01

$$\frac{\partial^2 \mathrm{nd}^{-1}(z \mid m)}{\partial z^2} = \frac{\sqrt{1-(1-m)\, z^2}\; \mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \frac{\partial \frac{1}{\sqrt{z^2-1}\,\sqrt{1-(1-m)\, z^2}}}{\partial z}$$

With respect to *m*

09.44.20.0004.01

$$\frac{\partial \mathrm{nd}^{-1}(z \mid m)}{\partial m} == \frac{1}{2\,(m-1)\, m} \left(\frac{\mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{z} - E\left(\mathrm{am}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right) + (1-m)\, \mathrm{nd}^{-1}(z \mid m)\right)$$

09.44.20.0005.01

$$\frac{\partial \operatorname{nd}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)} \left(\frac{(m-1)\sqrt{z^2-1}\, z}{m\sqrt{(m-1)z^2+1}} + \frac{i\, E(1-m)}{m} + i\, F\left(\sin^{-1}(z) \mid 1-m\right) - i\, K(1-m) - \frac{i\, E\left(\sin^{-1}(z) \mid 1-m\right)}{m} \right) /;$$

$$z \in \mathbb{R} \land z > 1 \land m < 0$$

09.44.20.0006.02

$$\frac{\partial^2 \operatorname{nd}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4(m-1)^2 m^2 z} \Bigg(3\, z \operatorname{nd}^{-1}(z \mid m)\,(m-1)^2 + z\, F\left(\operatorname{am}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \mid m\right)(m-1) +$$

$$2(2m-1)\, z\, E\left(\operatorname{am}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \mid m\right) + \left(-\sqrt{\frac{1}{z^2}}\, z + m\left(\sqrt{\frac{1}{z^2}}\, z + \frac{1}{(m-1)z^2+1} - 6 \right) + 3 \right) \operatorname{sc}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \Bigg)$$

09.44.20.0012.01

$$\frac{\partial^2 \operatorname{nd}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{8(m-1)^3 m^3}$$

$$\Bigg((-23(m-1)m - 8)\, E\left(\operatorname{am}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \mid m\right) - (m-1)(11m-7)\, F\left(\operatorname{am}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{z\left((m-1)z^2+1\right)^2}$$

$$\Bigg(\Bigg(m^4 \left(45 - 11\sqrt{\frac{1}{z^2}}\, z \right) z^4 + m^3 \left(40\sqrt{\frac{1}{z^2}}\, z^3 - 135 z^2 - 21\sqrt{\frac{1}{z^2}}\, z + 75 \right) z^2 - \left(7\sqrt{\frac{1}{z^2}}\, z - 15 \right)\left(z^2-1\right)^2 +$$

$$m\left(z^2-1\right)\left(32\sqrt{\frac{1}{z^2}}\, z^3 - 75 z^2 - 17\sqrt{\frac{1}{z^2}}\, z + 40 \right) + m^2 \Bigg(-54\sqrt{\frac{1}{z^2}}\, z^5 + 150 z^4 + 56\sqrt{\frac{1}{z^2}}\, z^3 -$$

$$160 z^2 - 10\sqrt{\frac{1}{z^2}}\, z + 33 \Bigg) \Bigg) \operatorname{sc}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) - 15(m-1)^3 z\left((m-1)z^2+1\right)^2 \operatorname{nd}^{-1}(z \mid m) \Bigg) \Bigg)$$

Symbolic differentiation

With respect to z

09.44.20.0013.01

$$\frac{\partial^n \operatorname{nd}^{-1}(z \mid m)}{\partial z^n} = \operatorname{nd}^{-1}(z \mid m)\, \delta_n +$$

$$\frac{\operatorname{sc}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right)}{z^2-1} \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-m)^{j-k} \left(z^2-1\right)^{-k} \left(1-(1-m)z^2\right)^{k-j} /;\ n \in \mathbb{N}$$

09.44.20.0014.01

$$\frac{\partial^n \mathrm{nd}^{-1}(z \mid m)}{\partial z^n} = \mathrm{nd}^{-1}(z \mid m)\,\delta_n + \frac{\mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2-1} \sum_{j=0}^{n-1} \frac{2^{2j-n+1}\,(1-m)^j\, z^{2j-n+1}\left(1-(1-m)\,z^2\right)^{-j}\left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!}\; {}_2F_1\!\left(\frac{1}{2}, -j; \frac{1}{2}-j; -\frac{m\,z^2-z^2+1}{(1-m)\left(z^2-1\right)}\right) /; n \in \mathbb{N}$$

09.44.20.0015.01

$$\frac{\partial^n \mathrm{nd}^{-1}(z \mid m)}{\partial z^n} = \mathrm{nd}^{-1}(z \mid m)\,\delta_n + \frac{\sqrt{1-(1-m)\,z^2}\;\mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \frac{\partial^{n-1} \frac{1}{\sqrt{z^2-1}\,\sqrt{1-(1-m)\,z^2}}}{\partial z^{n-1}} /; n \in \mathbb{N}^+$$

09.44.20.0007.01

$$\frac{\partial^n \mathrm{nd}^{-1}(z \mid m)}{\partial z^n} = \frac{2^{n-1}\,\pi\, z^{n-1}\,(n-1)!\;\mathrm{sc}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2-1} \sum_{j=0}^{n-1} \frac{(m-1)^{n-j-1}\left(z^2-1\right)^{-j}\left((m-1)\,z^2+1\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\!\left(\frac{1}{2}-j\right)\Gamma\!\left(j-n+\frac{3}{2}\right)} \, {}_2F_1\!\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; 1-\frac{1}{z^2}\right) {}_2F_1\!\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; 1+\frac{1}{(m-1)\,z^2}\right) /; n \in \mathbb{N}^+$$

With respect to *m*

09.44.20.0008.02

$$\frac{\partial^n \mathrm{nd}^{-1}(z \mid m)}{\partial m^n} = \frac{\pi\, i}{2}\,(m-1)^{-n}\, {}_2\tilde{F}_1\!\left(\frac{1}{2}, \frac{1}{2}; 1-n; 1-m\right) + \frac{i\,\sqrt{\pi}\; z^{2n+1}}{(2n+1)\,\Gamma\!\left(\frac{1}{2}-n\right)} F_1\!\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, (1-m)\,z^2\right) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.44.20.0009.01

$$\frac{\partial^\alpha \mathrm{nd}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{i\,K(1-m)\,z^{-\alpha}}{\Gamma(1-\alpha)} - i\,z^{1-\alpha}\,\sqrt{\pi}\;\tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\!\left(\begin{matrix}\frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2};\\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;;\end{matrix}\; z^2, (1-m)\,z^2\right) /; z \in \mathbb{R} \land m > 1$$

With respect to *m*

09.44.20.0010.01

$$\frac{\partial^\alpha \mathrm{nd}^{-1}(z \mid m)}{\partial m^\alpha} = -\frac{i\,z\,m^{-\alpha}\,\sqrt{\pi}}{2\,\sqrt{1-z^2}}\,\tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\!\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1;\\ \frac{3}{2};; 1-\alpha;\end{matrix}\; \frac{z^2}{z^2-1}, \frac{m\,z^2}{z^2-1}\right) + i\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \left(\psi(k+1)-\psi\!\left(k+\frac{1}{2}\right)\right) m^k}{k!\,\Gamma(k-\alpha+1)} - \frac{i\,m^{-\alpha}}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \mathcal{FC}_{\log}^{(\alpha)}(m, k)\, m^k}{k!^2} /; -1 < z < 1 \land -1 < m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.44.21.0001.01

$$\int \mathrm{nd}^{-1}(z \mid m)\, dz = \mathrm{nd}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{m-1}} \log\left(\frac{\mathrm{cd}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{m-1}} + \mathrm{sd}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right) \right)$$

Involving only one direct function with respect to *m*

09.44.21.0002.01

$$\int \mathrm{nd}^{-1}(z \mid m)\, dm = \frac{2}{z} \left(-i \sqrt{1-z^2} \sqrt{(m-1) z^2 + 1} - \sqrt{m-1}\, z\, E\left(i \sinh^{-1}\left(\sqrt{m-1}\right) \,\middle|\, \frac{1}{1-m} \right) + \sqrt{m-1}\, z\, E\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \,\middle|\, \frac{1}{1-m} \right) \right) /; z > 1 \land m > 1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.44.26.0001.01

$$\mathrm{nd}^{-1}(z \mid m) = i\, K(1-m) + \frac{\sqrt{1-z^2}\, z}{\sqrt{z^2-1}} F_{1\times 1\times 0}^{2\times 1\times 0}\left(\begin{matrix} 1, \frac{1}{2}; \frac{1}{2}; ; \\ \frac{3}{2}; 1; ; \end{matrix} -m z^2, z^2 \right)$$

Through other functions

Involving some hypergeometric-type functions

09.44.26.0002.01

$$\mathrm{nd}^{-1}(z \mid m) = i \left(K(1-m) - z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, (1-m) z^2 \right) \right) /; -1 < z < 1 \land -1 < m < 1$$

09.44.26.0003.01

$$\mathrm{nd}^{-1}(z \mid m) = i \left(K(1-m) - \frac{z}{\sqrt{1-z^2}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{z^2}{z^2-1}, \frac{m z^2}{z^2-1} \right) \right) /; -1 < z < 1 \land -1 < m < 1$$

09.44.26.0004.01

$$\mathrm{nd}^{-1}(z \mid m) = i\, K(1-m) - i\, z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, (1-m) z^2 \right) /; -1 < z < 1 \land -1 < m < 1$$

09.44.26.0005.01

$$\mathrm{nd}^{-1}(z \mid m) = i\, K(1-m) - \frac{i\, z}{\sqrt{1-z^2}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{z^2}{z^2-1}, \frac{m z^2}{z^2-1} \right) /; -1 < z < 1 \land -1 < m < 1$$

Representations through equivalent functions

With inverse function

09.44.27.0001.01

$$\mathrm{nd}\left(\mathrm{nd}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving cd^{-1}

09.44.27.0002.01

$$\text{nd}^{-1}(z \mid m) = i\, \text{cd}^{-1}(z \mid 1-m)$$

Involving cn^{-1}

09.44.27.0003.01

$$\text{nd}^{-1}(z \mid m) = -i\left(\text{cn}^{-1}\left(\sqrt{1-z^2} \;\middle|\; 1-m\right) - K(1-m)\right) /;\ 0 < z < 1 \land m > 0$$

Involving cs^{-1}

09.44.27.0004.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \text{cs}^{-1}\left(\frac{i}{z} \;\middle|\; m\right) /;\ -1 < z < 1 \land m > 0$$

Involving dc^{-1}

09.44.27.0005.01

$$\text{nd}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}}\, \text{dc}^{-1}\left(z \;\middle|\; \frac{1}{1-m}\right) /;\ -1 < z < 1 \land 0 < m < 1$$

Involving dn^{-1}

09.44.27.0006.01

$$\text{nd}^{-1}(z \mid m) = \text{dn}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /;\ z < -1 \land m < 0 \lor z > 1 \land m > 1$$

Involving ds^{-1}

09.44.27.0007.01

$$\text{nd}^{-1}(z \mid m) = i\left(K(1-m) - \frac{1}{\sqrt{m}}\, \text{ds}^{-1}\left(\frac{1}{\sqrt{m}\, z} \;\middle|\; \frac{m-1}{m}\right)\right) /;\ 0 < z < 1 \land m \in \mathbb{R}$$

Involving nc^{-1}

09.44.27.0008.01

$$\text{nd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{nc}^{-1}\left(z \;\middle|\; \frac{1}{m}\right) /;\ -1 < z < 1 \land m > 1$$

Involving ns^{-1}

09.44.27.0009.01

$$\text{nd}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}}\left(K\left(\frac{1}{1-m}\right) - \text{ns}^{-1}\left(z \;\middle|\; \frac{1}{1-m}\right)\right) /;\ -1 < z < 1$$

Involving sc^{-1}

09.44.27.0010.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \text{sc}^{-1}(-i\, z \mid m) /;\ -1 < z < 1 \land m \in \mathbb{R}$$

Involving sd^{-1}

09.44.27.0011.01

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) - \frac{1}{\sqrt{m}}\, \text{sd}^{-1}\left(i\, z\, \sqrt{m} \;\middle|\; \frac{1}{m}\right)$$

Involving sn^{-1}

09.44.27.0012.01

$$\text{nd}^{-1}(z \mid m) == -i\left(\text{sn}^{-1}(z \mid 1-m) - K(1-m)\right) /; z > 1 \land m > 1$$

Involving elliptic integrals

09.44.27.0013.01

$$\text{nd}^{-1}(z \mid m) == i\left(F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) - K(1-m)\right) /; z > 1 \land m > 1$$

09.44.27.0015.01

$$\text{nd}^{-1}(z \mid m) = i\,K(1-m) - i\,F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) /; |z| < 1$$

09.44.27.0016.01

$$\text{nd}^{-1}(z \mid m) = \frac{\sqrt{(m-1)z^2+1}\ \text{sc}\left(\text{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \left(\frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) + i\,K(1-m)\right) /;$$
$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\text{Im}\left(((z-1)\tau+1)^2-1\right) = 0 \bigwedge ((z-1)\tau+1)^2 - 1 < 0 \bigwedge \text{Im}\left(1-(1-m)((z-1)\tau+1)^2\right) = 0 \bigwedge 1-(1-m)((z-1)\tau+1)^2 < 0\right)$$

09.44.27.0017.01

$$\text{nd}^{-1}(z \mid m) = \text{nd}^{-1}(z_0 \mid m) + \frac{\sqrt{(m-1)z^2+1}\ \text{sc}\left(\text{nd}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \left(\frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) - \frac{\sqrt{1-z_0^2}}{\sqrt{z_0^2-1}} F\left(\sin^{-1}(z_0) \,\middle|\, 1-m\right)\right) /;$$
$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\text{Im}\left((\tau(z-z_0)+z_0)^2-1\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2 - 1 < 0 \bigwedge \text{Im}\left(1-(1-m)(\tau(z-z_0)+z_0)^2\right) = 0 \bigwedge 1-(1-m)(\tau(z-z_0)+z_0)^2 < 0\right)$$

Involving other related functions

09.44.27.0014.01

$$\text{nd}^{-1}(z \mid m) == -\frac{i\sqrt{z_2^2}}{z_2}\left(K(1-m) + \text{elog}(z_1, z_2; a, b)\right) /;$$
$$\{a, b, z_1\} == \left\{m-2, 1-m, \frac{1}{z^2}\right\} \bigwedge z_1^3 + a\,z_1^2 + b\,z_1 - z_2^2 == 0 \bigwedge z > 1 \bigwedge m > 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

Copyright