InverseJacobiSC

Notations

Traditional name

Inverse of the Jacobi elliptic function sc

Traditional notation

$\mathrm{sc}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiSC[z, m]`

Primary definition

09.46.02.0001.01

$z = \mathrm{sc}(w \mid m) \;/;\; w = \mathrm{sc}^{-1}(z \mid m)$

09.46.02.0002.01

$$\mathrm{sc}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{t^2+1}\,\sqrt{(1-m)\,t^2+1}}\,dt \;/;\; z \in \mathbb{R} \bigwedge (1-m)\,z^2 > -1$$

Specific values

Specialized values

For fixed *z*

09.46.03.0001.01

$\mathrm{sc}^{-1}(z \mid 0) = \tan^{-1}(z)$

09.46.03.0002.01

$$\mathrm{sc}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = -i\sqrt{2}\, F\left(i \sinh^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, 2\right)$$

09.46.03.0003.01

$\mathrm{sc}^{-1}(z \mid 1) = \sinh^{-1}(z)$

For fixed *m*

09.46.03.0004.01

$\mathrm{sc}^{-1}(-1 \mid m) = i\, F\left(i \sinh^{-1}(1) \,\middle|\, 1-m\right)$

09.46.03.0005.01

$$\text{sc}^{-1}\left(-\frac{1}{2} \,\middle|\, m\right) = i\, F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \,\middle|\, 1-m\right)$$

09.46.03.0006.01

$$\text{sc}^{-1}(0 \mid m) = 0$$

09.46.03.0007.01

$$\text{sc}^{-1}\left(\frac{1}{2} \,\middle|\, m\right) = -i\, F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \,\middle|\, 1-m\right)$$

09.46.03.0008.01

$$\text{sc}^{-1}(1 \mid m) = -i\, F\left(i \sinh^{-1}(1) \,\middle|\, 1-m\right)$$

09.46.03.0009.01

$$\text{sc}^{-1}(i \mid m) = i\, K(1-m)$$

09.46.03.0010.01

$$\text{sc}^{-1}(-i \mid m) = -i\, K(1-m)$$

Values at infinities

09.46.03.0011.01

$$\text{sc}^{-1}(z \mid \infty) = 0$$

09.46.03.0012.01

$$\text{sc}^{-1}(z \mid -\infty) = 0$$

09.46.03.0013.01

$$\text{sc}^{-1}(\infty \mid m) = K(m)$$

09.46.03.0014.01

$$\text{sc}^{-1}(-\infty \mid m) = -K(m)$$

General characteristics

Domain and analyticity

$\text{sc}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.46.04.0001.01

$$(z * m) \longrightarrow \text{sc}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.46.04.0002.01

$$\text{sc}^{-1}(\bar{z} \mid \bar{m}) = \overline{\text{sc}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.46.04.0003.01

$$\text{sc}^{-1}(-z \mid m) = -\text{sc}^{-1}(z \mid m)$$

Poles and essential singularities

With respect to m

The function $\mathrm{sc}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.46.04.0004.01

$$\mathcal{S}ing_m\left(\mathrm{sc}^{-1}(z \mid m)\right) = \{\}$$

With respect to z

The function $\mathrm{sc}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.46.04.0005.01

$$\mathcal{S}ing_z\left(\mathrm{sc}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to m

For fixed z, the function $\mathrm{sc}^{-1}(z \mid m)$ has two branch points: $m = \frac{z^2+1}{z^2}$, $m = \tilde{\infty}$.

09.46.04.0006.01

$$\mathcal{BP}_m\left(\mathrm{sc}^{-1}(z \mid m)\right) = \left\{\frac{z^2-1}{z^2}, \tilde{\infty}\right\}$$

09.46.04.0007.01

$$\mathcal{R}_m\left(\mathrm{sc}^{-1}(z \mid m), \frac{z^2+1}{z^2}\right) = \log$$

09.46.04.0008.01

$$\mathcal{R}_m\left(\mathrm{sc}^{-1}(z \mid m), \tilde{\infty}\right) = 2$$

With respect to z

For fixed m, the function $\mathrm{sc}^{-1}(z \mid m)$ has five branch points: $z = \pm i$, $z = \pm \frac{1}{\sqrt{m-1}}$, $z = \tilde{\infty}$.

09.46.04.0009.01

$$\mathcal{BP}_z\left(\mathrm{sc}^{-1}(z \mid m)\right) = \left\{i, -i, \frac{1}{\sqrt{m-1}}, -\frac{1}{\sqrt{m-1}}, \tilde{\infty}\right\}$$

09.46.04.0010.01

$$\mathcal{R}_z\left(\mathrm{sc}^{-1}(z \mid m), i\right) = 2$$

09.46.04.0011.01

$$\mathcal{R}_z\left(\mathrm{sc}^{-1}(z \mid m), -i\right) = 2$$

09.46.04.0012.01

$$\mathcal{R}_z\left(\mathrm{sc}^{-1}(z \mid m), \frac{1}{\sqrt{m-1}}\right) = 2$$

09.46.04.0013.01

$$\mathcal{R}_z\left(\mathrm{sc}^{-1}(z \mid m), -\frac{1}{\sqrt{m-1}}\right) = 2$$

09.46.04.0014.01

$$\mathcal{R}_z\left(\mathrm{sc}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at generic point $z == z_0$

09.46.06.0007.01

$$\mathrm{sc}^{-1}(z \mid m) \propto \mathrm{sc}^{-1}(z_0 \mid m) + \frac{1}{\mathrm{dc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right) \mathrm{nc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right)} \left(z - z_0 - \frac{z_0 \left(2 m z_0^2 - 2 z_0^2 + m - 2\right)}{2 \left(z_0^2 + 1\right) \left(m z_0^2 - z_0^2 - 1\right)} (z - z_0)^2 + \ldots\right) /; (z \to z_0)$$

09.46.06.0008.01

$$\mathrm{sc}^{-1}(z \mid m) \propto \mathrm{sc}^{-1}(z_0 \mid m) + \frac{1}{\mathrm{dc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right) \mathrm{nc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right)} \left(z - z_0 - \frac{z_0 \left(2 m z_0^2 - 2 z_0^2 + m - 2\right)}{2 \left(z_0^2 + 1\right) \left(m z_0^2 - z_0^2 - 1\right)} (z - z_0)^2 + O\left((z - z_0)^3\right)\right)$$

09.46.06.0009.01

$$\mathrm{sc}^{-1}(z \mid m) = \mathrm{sc}^{-1}(z_0 \mid m) + \frac{1}{\mathrm{dc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right) \mathrm{nc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right)} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \frac{(-1)^j (1-k)_{2(k-j)-2}}{(k-j-1)! \, (2 z_0)^{-2j+k-1}} \sum_{s=0}^{j} \binom{j}{s} \left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_{j-s} (1-m)^{j-s} \left(z_0^2 + 1\right)^{-s} \left((1-m) z_0^2 + 1\right)^{s-j} (z - z_0)^k$$

09.46.06.0010.01

$$\mathrm{sc}^{-1}(z \mid m) = \mathrm{sc}^{-1}(z_0 \mid m) + \frac{1}{\mathrm{dc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right) \mathrm{nc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right)} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} \frac{(-1)^j 2^{2j-k+1} (1-m)^j \, \Gamma\left(j + \frac{1}{2}\right) (1-k)_{2(k-j)-2} \, z_0^{2j-k+1}}{\sqrt{\pi} \, (-j+k-1)! \left((1-m) z_0^2 + 1\right)^j} \, {}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2} - j; \frac{-m z_0^2 + z_0^2 + 1}{(1-m) \left(z_0^2 + 1\right)}\right) (z - z_0)^k$$

09.46.06.0011.01

$$\mathrm{sc}^{-1}(z \mid m) = \mathrm{sc}^{-1}(z_0 \mid m) + \frac{\sqrt{z_0^2 + 1} \sqrt{(1-m) z_0^2 + 1}}{\mathrm{dc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right) \mathrm{nc}\left(\mathrm{sc}^{-1}(z_0 \mid m) \mid m\right)} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\left(\frac{\partial^{k-1} \frac{1}{\sqrt{z^2+1} \sqrt{(1-m) z^2 + 1}}}{\partial z^{k-1}}\right) /; z == z_0\right) (z - z_0)^k$$

09.46.06.0012.01

$$\mathrm{sc}^{-1}(z \mid m) \propto \mathrm{sc}^{-1}(z_0 \mid m) \left(1 + O(z - z_0)\right)$$

Expansions at $z == 0$

09.46.06.0001.02

$$\mathrm{sc}^{-1}(z \mid m) \propto z + \frac{m - 2}{6} z^3 + \frac{3 m^2 - 8 m + 8}{40} z^5 + \ldots /; (z \to 0)$$

09.46.06.0002.01

$$\operatorname{sc}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(m-1)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{1-m}\right) z^{2k+1}$$

09.46.06.0013.01

$$\operatorname{sc}^{-1}(z \mid m) \propto z\left(1 + O\left(z^2\right)\right)$$

Expansions at generic point $m = m_0$

09.46.06.0014.01

$$\operatorname{sc}^{-1}(z \mid m) \propto \operatorname{sc}^{-1}(z \mid m_0) + \frac{z^3 \sqrt{z^2+1}\,\sqrt{(1-m_0)\,z^2+1}}{6\,\operatorname{dc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right) \operatorname{nc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right)} \left(F_1\!\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; -z^2, z^2\,(m_0-1)\right)(m-m_0) + \frac{9}{20}\, z^2\, F_1\!\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; -z^2, z^2\,(m_0-1)\right)(m-m_0)^2 + \ldots\right) /; (m \to m_0)$$

09.46.06.0015.01

$$\operatorname{sc}^{-1}(z \mid m) \propto \operatorname{sc}^{-1}(z \mid m_0) + \frac{z^3 \sqrt{z^2+1}\,\sqrt{(1-m_0)\,z^2+1}}{6\,\operatorname{dc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right) \operatorname{nc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right)} \left(F_1\!\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; -z^2, z^2\,(m_0-1)\right)(m-m_0) + \frac{9}{20}\, z^2\, F_1\!\left(\frac{5}{2}; \frac{1}{2}, \frac{5}{2}; \frac{7}{2}; -z^2, z^2\,(m_0-1)\right)(m-m_0)^2 + O\left((m-m_0)^3\right)\right)$$

09.46.06.0016.01

$$\operatorname{cs}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{\pi}\, z^{-2k-1}}{k!\,(2k+1)\,\Gamma\left(\frac{1}{2}-k\right)} F_1\!\left(k+\frac{1}{2}; \frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; -\frac{1}{z^2}, -\frac{1-m_0}{z^2}\right)(m-m_0)^k$$

09.46.06.0017.01

$$\operatorname{sc}^{-1}(z \mid m) = \operatorname{sc}^{-1}(z \mid m_0) - \frac{i \sqrt{z^2+1}\,\sqrt{(1-m_0)\,z^2+1}}{\operatorname{dc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right) \operatorname{nc}\left(\operatorname{sc}^{-1}(z \mid m_0) \mid m_0\right)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \operatorname{EllipticF}^{(0,k)}\left(i \sinh^{-1}(z),\, 1-m_0\right)(m-m_0)^k$$

09.46.06.0018.01

$$\operatorname{sc}^{-1}(z \mid m) \propto \operatorname{sc}^{-1}(z \mid m_0)\,(1 + O(m-m_0))$$

Expansions at $m = 0$

09.46.06.0003.02

$$\operatorname{sc}^{-1}(z \mid m) \propto \tan^{-1}(z) + \frac{\left(z^2+1\right)\tan^{-1}(z) - z}{4\left(z^2+1\right)}\, m + \frac{3\left(-5 z^3 - 3 z + 3\left(z^2+1\right)^2 \tan^{-1}(z)\right)}{64\left(z^2+1\right)^2}\, m^2 + \ldots /; (m \to 0)$$

09.46.06.0004.01

$$\operatorname{sc}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{z^{2k+1} \left(\frac{1}{2}\right)_k}{k!\,(2k+1)} \, {}_2F_1\!\left(k+\frac{1}{2}, k+1; k+\frac{3}{2}; -z^2\right) m^k /; |m| < 1$$

09.46.06.0019.01

$$\operatorname{sc}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \left(\tan^{-1}(z) - \frac{1}{2z} \sum_{j=1}^{k} \frac{(j-1)!}{\left(\frac{1}{2}\right)_j} \left(\frac{z^2}{z^2+1}\right)^j\right) m^k /; |m| < 1$$

09.46.06.0005.01

$$\operatorname{sc}^{-1}(z \mid m) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j (m-1)^k z^{2j+2k+1}}{(2j+2k+1)\, j!\, k!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k$$

09.46.06.0006.01

$$\operatorname{sc}^{-1}(z \mid m) = z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} (m-1) z^2, -z^2\right)$$

09.46.06.0020.01

$$\operatorname{sc}^{-1}(z \mid m) \propto \tan^{-1}(z)\,(1 + O(m))$$

Integral representations

On the real axis

Of the direct function

09.46.07.0001.01

$$\operatorname{sc}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{t^2+1}\, \sqrt{(1-m) t^2 + 1}}\, dt \;/;\; z \in \mathbb{R} \bigwedge (1-m) z^2 > -1$$

09.46.07.0002.01

$$\operatorname{sc}^{-1}(z \mid m) = \frac{\sqrt{(1-m) z^2 + 1}\; \operatorname{nd}\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} \int_0^z \frac{1}{\sqrt{t^2+1}\, \sqrt{(1-m) t^2 + 1}}\, dt \;/;$$

$$\neg \exists_{\tau, \{\tau \in \mathbb{R}, 0 < \tau < 1\}} \left(\operatorname{Im}\left(z^2 \tau^2 + 1\right) = 0 \bigwedge z^2 \tau^2 + 1 < 0 \bigwedge \operatorname{Im}\left((1-m) z^2 \tau^2 + 1\right) = 0 \bigwedge (1-m) z^2 \tau^2 + 1 < 0\right)$$

09.46.07.0003.01

$$\operatorname{sc}^{-1}(z \mid m) = \operatorname{sc}^{-1}(z_0 \mid m) + \frac{\sqrt{(1-m) z^2 + 1}\; \operatorname{nd}\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} \int_{z_0}^z \frac{1}{\sqrt{t^2+1}\, \sqrt{(1-m) t^2 + 1}}\, dt \;/;$$

$$\neg \exists_{\tau, \{\tau \in \mathbb{R}, 0 < \tau < 1\}} \left(\operatorname{Im}\left((\tau (z - z_0) + z_0)^2 + 1\right) = 0 \bigwedge (\tau (z - z_0) + z_0)^2 + 1 < 0 \bigwedge \right.$$
$$\left. \operatorname{Im}\left((1-m) (\tau (z - z_0) + z_0)^2 + 1\right) = 0 \bigwedge (1-m) (\tau (z - z_0) + z_0)^2 + 1 < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.46.13.0001.01

$$w''(z) + \left(2 (1-m) z^2 - m + 2\right) z\, w'(z)^3 = 0 \;/;\; w(z) = \operatorname{sc}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.46.16.0001.01

$$\mathrm{sc}^{-1}(-z \mid m) = -\mathrm{sc}^{-1}(z \mid m)$$

Identities

Functional identities

09.46.17.0001.01

$$\left((m-1) z_1^2 z_2^2+1\right)^2 \mathrm{sc}(w(z_1)+w(z_2) \mid m)^4 + 2\left(z_1^2\left((m-1) z_2^4+\left((m-1) z_1^2+2(m-2)\right) z_2^2-1\right)-z_2^2\right) \mathrm{sc}(w(z_1)+w(z_2) \mid m)^2+\left(z_1^2-z_2^2\right)^2 = 0 \;/;\; w(z) = \mathrm{sc}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.46.20.0001.02

$$\frac{\partial \mathrm{sc}^{-1}(z \mid m)}{\partial z} = \frac{\mathrm{nd}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{z^2+1}$$

09.46.20.0002.01

$$\frac{\partial \mathrm{sc}^{-1}(z \mid m)}{\partial z} = \frac{1}{\sqrt{z^2+1}\,\sqrt{(1-m) z^2+1}} \;/;\; z \in \mathbb{R} \bigwedge (1-m) z^2 > -1$$

09.46.20.0003.02

$$\frac{\partial^2 \mathrm{sc}^{-1}(z \mid m)}{\partial z^2} = -\frac{z\left(2(m-1) z^2+m-2\right) \mathrm{nd}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{\left(z^2+1\right)^2\left((m-1) z^2-1\right)}$$

09.46.20.0011.01

$$\frac{\partial^2 \mathrm{sc}^{-1}(z \mid m)}{\partial z^2} = \frac{\sqrt{z^2+1}\;\mathrm{dn}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-(m-1) z^2}} \frac{\partial \frac{1}{\sqrt{z^2+1}\,\sqrt{(1-m) z^2+1}}}{\partial z}$$

With respect to *m*

09.46.20.0004.01

$$\frac{\partial \mathrm{sc}^{-1}(z \mid m)}{\partial m} = -\frac{\left(z^2+1\right) E\left(\mathrm{am}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right) \mid m\right)+(m-1)\left(z^2+1\right) \mathrm{sc}^{-1}(z \mid m)-m\, z\, \mathrm{nd}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{2(m-1) m\left(z^2+1\right)}$$

09.46.20.0012.01

$$\frac{\partial \mathrm{sc}^{-1}(z \mid m)}{\partial m} = \frac{z^3 \sqrt{1-(m-1) z^2}\;\mathrm{nd}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{6\sqrt{z^2+1}} F_1\left(\frac{3}{2} ; \frac{1}{2}, \frac{3}{2} ; \frac{5}{2} ;-z^2,(m-1) z^2\right)$$

09.46.20.0013.01

$$\frac{\partial \mathrm{sc}^{-1}(z \mid m)}{\partial m} = -\frac{i \sqrt{1-(m-1) z^2}\;\mathrm{nd}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+1}} \frac{\partial F\left(i \sinh^{-1}(z) \mid 1-m\right)}{\partial m}$$

09.46.20.0014.01

$$\frac{\partial \operatorname{sc}^{-1}(z \mid m)}{\partial m} = \frac{1}{2\,(m-1)\,m\,\sqrt{z^2+1}} \left(\left((m-1)\sqrt{z^2+1}\; z - i\sqrt{-m\,z^2+z^2+1}\; E\!\left(i \sinh^{-1}(z) \,\middle|\, 1-m\right) - m\sqrt{1-(m-1)\,z^2}\; \operatorname{sc}^{-1}(z \mid m)\right) \operatorname{nd}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)\right)$$

09.46.20.0015.01

$$\frac{\partial \operatorname{sc}^{-1}(z \mid m)}{\partial m} = \frac{1}{2\,(m-1)\,m\,\sqrt{z^2+1}} \left(\left(\sqrt{1-(m-1)\,z^2}\; F_1\!\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; -z^2, (m-1)\,z^2\right) z + (m-1)\sqrt{z^2+1}\; z - m\sqrt{-m\,z^2+z^2+1}\; \operatorname{sc}^{-1}(z \mid m)\right) \operatorname{nd}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)\right)$$

09.46.20.0005.02

$$\frac{\partial \operatorname{sc}^{-1}(z \mid m)}{\partial m} = \frac{\operatorname{nd}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)}{2\,m} \left(z - \frac{\sqrt{1-(m-1)\,z^2}\; E\!\left(\sin^{-1}\!\left(\sqrt{m-1}\; z\right) \,\middle|\, \frac{1}{1-m}\right)}{\sqrt{m-1}\;\sqrt{z^2+1}}\right)$$

09.46.20.0006.02

$$\frac{\partial^2 \operatorname{sc}^{-1}(z \mid m)}{\partial m^2} = \Big(3\left(z^2+1\right)\left((m-1)\,z^2-1\right) \operatorname{sc}^{-1}(z \mid m)\,(m-1)^2 + \left(z^2+1\right)\left((m-1)\,z^2-1\right)\left((4\,m-2)\,E\!\left(\operatorname{am}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right) + (m-1)\,F\!\left(\operatorname{am}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right)\right) - m\,z\left(4\,m^2\,z^2+z^2-m\left(5\,z^2+3\right)+1\right) \operatorname{nd}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)\Big) \Big/ \left(4\,(m-1)^2\,m^2\left(z^2+1\right)\left((m-1)\,z^2-1\right)\right)$$

09.46.20.0016.01

$$\frac{\partial^2 \operatorname{sc}^{-1}(z \mid m)}{\partial m^2} = -\left(\left(1-(m-1)\,z^2\right)\left(-4\,i\,(2\,m-1)\,E\!\left(i \sinh^{-1}(z) \,\middle|\, 1-m\right)\left(1-(m-1)\,z^2\right)^{3/2} - 2\,m\,(3\,m-1) \operatorname{sc}^{-1}(z \mid m)\left(1-(m-1)\,z^2\right)^{3/2} - 2\,(m-1)\,z\sqrt{z^2+1}\left(2\,(m-1)\,(2\,m-1)\,z^2-3\,m+2\right)\right) \operatorname{nd}\!\left(\operatorname{sc}^{-1}(z \mid m) \,\middle|\, m\right)\right) \Big/ \left(8\,(m-1)^2\,m^2\sqrt{z^2+1}\left((m-1)\,z^2-1\right)^2\right)$$

09.46.20.0017.01

$$\frac{\partial^3 \mathrm{sc}^{-1}(z\,|\,m)}{\partial m^3} = \Bigg(-15\left(z^2+1\right)\left((m-1)z^2-1\right)^2 \mathrm{sc}^{-1}(z\,|\,m)(m-1)^3 - \left(z^2+1\right)\left((m-1)z^2-1\right)\Bigg(\left(23m^2-23m+8\right)\left((m-1)z^2-1\right)E\left(\mathrm{am}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)\,\middle|\,m\right) + (m-1)\left(m\sqrt{\frac{-m z^2+z^2+1}{z^2+1}}\,z + \left(11m^2z^2-18mz^2+7z^2-11m+7\right)F\left(\mathrm{am}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)\,\middle|\,m\right)\right)\Bigg) + m\,z\left(23m^4z^4-m^3\left(59z^2+35\right)z^2+5\left(z^2+1\right)^2+3m^2\left(18z^4+20z^2+5\right)-m\left(23z^4+35z^2+12\right)\right)\mathrm{nd}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)\Bigg)\Big/\left(8(m-1)^3m^3\left(z^2+1\right)\left((m-1)z^2-1\right)^2\right)$$

Symbolic differentiation

With respect to *z*

09.46.20.0018.01

$$\frac{\partial^n \mathrm{sc}^{-1}(z\,|\,m)}{\partial z^n} = \delta_n\,\mathrm{sc}^{-1}(z\,|\,m) - \frac{\mathrm{nd}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)}{z^2+1}\sum_{j=0}^{n-1}\frac{(-1)^{j-1}(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{j-k}(1-m)^{j-k}\left(z^2+1\right)^{-k}\left((1-m)z^2+1\right)^{k-j} \;/;\; n\in\mathbb{N}$$

09.46.20.0019.01

$$\frac{\partial^n \mathrm{sc}^{-1}(z\,|\,m)}{\partial z^n} = \delta_n\,\mathrm{sc}^{-1}(z\,|\,m) - \frac{\mathrm{nd}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)}{z^2+1}\sum_{j=0}^{n-1}\frac{(-1)^{j-1}2^{2j-n+1}(1-m)^j z^{2j-n+1}\left((1-m)z^2+1\right)^{-j}\left(\frac{1}{2}\right)_j(1-n)_{2(n-j)-2}}{(n-j-1)!}\,{}_2F_1\left(\frac{1}{2},-j;\frac{1}{2}-j;\frac{-m z^2+z^2+1}{(1-m)\left(z^2+1\right)}\right) \;/;\; n\in\mathbb{N}$$

09.46.20.0020.01

$$\frac{\partial^n \mathrm{sc}^{-1}(z\,|\,m)}{\partial z^n} = \delta_n\,\mathrm{sc}^{-1}(z\,|\,m) + \frac{\sqrt{1-(m-1)z^2}\;\mathrm{nd}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)}{\sqrt{z^2+1}}\,\frac{\partial^{n-1}\frac{1}{\sqrt{z^2+1}\sqrt{(1-m)z^2+1}}}{\partial z^{n-1}} \;/;\; n\in\mathbb{N}$$

09.46.20.0007.01

$$\frac{\partial^n \mathrm{sc}^{-1}(z\,|\,m)}{\partial z^n} = \frac{2^{n-1}\pi z^{n-1}(n-1)!\,\mathrm{nd}\left(\mathrm{sc}^{-1}(z\,|\,m)\,\middle|\,m\right)}{z^2+1}\sum_{j=0}^{n-1}\frac{(1-m)^{n-j-1}\left(z^2+1\right)^{-j}\left((1-m)z^2+1\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right)\Gamma\left(j-n+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{1-j}{2},-\frac{j}{2};\frac{1}{2}-j;1+\frac{1}{z^2}\right){}_2F_1\left(\frac{j-n+2}{2},\frac{j-n+1}{2};j-n+\frac{3}{2};1+\frac{1}{(1-m)z^2}\right) \;/;\; n\in\mathbb{N}^+$$

With respect to *m*

09.46.20.0008.02

$$\frac{\partial^n \operatorname{sc}^{-1}(z \mid m)}{\partial m^n} = \frac{z^{2n+1} \left(\frac{1}{2}\right)_n \sqrt{(1-m)z^2+1} \operatorname{nd}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)}{(2n+1)\sqrt{z^2+1}} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; -z^2, (m-1)z^2\right) /; n \in \mathbb{N}$$

09.46.20.0021.01

$$\frac{\partial^n \operatorname{sc}^{-1}(z \mid m)}{\partial m^n} = -\frac{i\sqrt{1-(m-1)z^2} \operatorname{nd}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+1}} \frac{\partial^n F\left(i \sinh^{-1}(z) \mid 1-m\right)}{\partial m^n} /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

09.46.20.0009.01

$$\frac{\partial^\alpha \operatorname{sc}^{-1}(z \mid m)}{\partial z^\alpha} = z^{1-\alpha} \sqrt{\pi}\, \tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;; \end{matrix} -z^2, (m-1)z^2\right) /; z \in \mathbb{R} \bigwedge (m-1)z^2 < 1$$

With respect to *m*

09.46.20.0010.01

$$\frac{\partial^\alpha \operatorname{sc}^{-1}(z \mid m)}{\partial m^\alpha} = \frac{z\, m^{-\alpha} \sqrt{\pi}}{2\sqrt{z^2+1}} \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha; \end{matrix} \frac{z^2}{z^2+1}, \frac{m z^2}{z^2+1}\right) /; z \in \mathbb{R} \bigwedge (m-1)z^2 < 1$$

Integration

Indefinite integration

Involving only one direct function

09.46.21.0001.01

$$\int \operatorname{sc}^{-1}(z \mid m)\, dz = \operatorname{sc}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\operatorname{dc}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}} + \operatorname{nc}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)\right)$$

Involving only one direct function with respect to *m*

09.46.21.0002.01

$$\int \operatorname{sc}^{-1}(z \mid m)\, dm = \frac{2}{z}\left(i\sqrt{1-m}\, z\, E\left(i \sinh^{-1}\left(\sqrt{1-m}\, z\right) \middle| \frac{1}{1-m}\right) + \sqrt{z^2+1}\sqrt{1-m z^2+z^2} - 1\right) /; z \in \mathbb{R} \bigwedge (1-m)z^2 > -1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.46.26.0001.01

$$\operatorname{sc}^{-1}(z \mid m) = z\, F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} (m-1)z^2, -z^2\right)$$

Through other functions

Involving some hypergeometric-type functions

09.46.26.0002.01

$$\mathrm{sc}^{-1}(z \mid m) = z\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2, (m-1)\, z^2\right) /; z \in \mathbb{R} \bigwedge (m-1)\, z^2 < 1$$

09.46.26.0003.01

$$\mathrm{sc}^{-1}(z \mid m) = \frac{z}{\sqrt{z^2+1}}\, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{z^2}{z^2+1}, \frac{m\, z^2}{z^2+1}\right) /; z \in \mathbb{R} \bigwedge (m-1)\, z^2 < 1$$

Representations through equivalent functions

With inverse function

09.46.27.0001.01

$$\mathrm{sc}\left(\mathrm{sc}^{-1}(z \mid m) \mid m\right) = z$$

With related functions

Involving cd^{-1}

09.46.27.0002.01

$$\mathrm{sc}^{-1}(z \mid m) = i\left(\mathrm{cd}^{-1}(i\, z \mid 1-m) - K(1-m)\right) /; z \in \mathbb{R} \land m \in \mathbb{R}$$

Involving cn^{-1}

09.46.27.0003.01

$$\mathrm{sc}^{-1}(z \mid m) = i\, \mathrm{cn}^{-1}\left(\sqrt{z^2+1} \;\middle|\; 1-m\right) /; 0 < z < 1 \land m > 1$$

Involving cs^{-1}

09.46.27.0004.01

$$\mathrm{sc}^{-1}(z \mid m) = \mathrm{cs}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /; z > 0 \land m < 1$$

Involving dc^{-1}

09.46.27.0005.01

$$\mathrm{sc}^{-1}(z \mid m) = i\left(K(1-m) - \frac{1}{\sqrt{1-m}}\, \mathrm{dc}^{-1}\left(-i\, z \;\middle|\; \frac{1}{1-m}\right)\right) /; z \in \mathbb{R} \land 0 < m < 1$$

Involving dn^{-1}

09.46.27.0006.01

$$\mathrm{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\, \mathrm{dn}^{-1}\left(i\, z \;\middle|\; 1 - \frac{1}{1-m}\right) - i\, K(1-m) /; z < 1 \land m < 0$$

Involving ds^{-1}

09.46.27.0007.01

$$\mathrm{sc}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}}\, \mathrm{ds}^{-1}\left(-\frac{i}{z\sqrt{m}} \;\middle|\; \frac{m-1}{m}\right) /; z > 0 \land m > 1$$

Involving nc^{-1}

09.46.27.0008.01

$$\text{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \text{nc}^{-1}\left(i\, z \,\middle|\, \frac{1}{m}\right) - i\, K(1-m) \;/;\; -1 < z < 1 \land m > 0$$

Involving nd^{-1}

09.46.27.0009.01

$$\text{sc}^{-1}(z \mid m) = \text{nd}^{-1}(i\, z \mid m) - i\, K(1-m) \;/;\; z \in \mathbb{R} \land m \in \mathbb{R}$$

Involving ns^{-1}

09.46.27.0010.01

$$\text{sc}^{-1}(z \mid m) = -i\, \text{ns}^{-1}\left(-\frac{i}{z} \,\middle|\, 1-m\right) /;\; 0 < z < 1 \land m > 0$$

Involving sd^{-1}

09.46.27.0011.01

$$\text{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \text{sd}^{-1}\left(z \sqrt{m} \,\middle|\, \frac{1}{m}\right) /;\; z < 1 \land m < 0$$

Involving sn^{-1}

09.46.27.0012.01

$$\text{sc}^{-1}(z \mid m) = -i\, \text{sn}^{-1}(i\, z \mid 1-m)$$

Involving elliptic integrals

09.46.27.0013.01

$$\text{sc}^{-1}(z \mid m) = -i\, F\left(i \sinh^{-1}(z) \,\middle|\, 1-m\right) /;\; |z| < 1$$

09.46.27.0015.01

$$\text{sc}^{-1}(z \mid m) = -\frac{i \sqrt{(1-m) z^2 + 1}\; \text{nd}\left(\text{sc}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} F\left(i \sinh^{-1}(z) \,\middle|\, 1-m\right) /;$$
$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left(z^2 \tau^2 + 1\right) = 0 \bigwedge z^2 \tau^2 + 1 < 0 \bigwedge \text{Im}\left((1-m) z^2 \tau^2 + 1\right) = 0 \bigwedge (1-m) z^2 \tau^2 + 1 < 0\right)$$

09.46.27.0016.01

$$\text{sc}^{-1}(z \mid m) = \text{sc}^{-1}(z_0 \mid m) - \frac{i \sqrt{(1-m) z^2 + 1}\; \text{nd}\left(\text{sc}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} \left(F\left(i \sinh^{-1}(z) \,\middle|\, 1-m\right) - F\left(i \sinh^{-1}(z_0) \,\middle|\, 1-m\right)\right) /;$$
$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left((\tau (z-z_0) + z_0)^2 + 1\right) = 0 \bigwedge (\tau (z-z_0) + z_0)^2 + 1 < 0 \bigwedge \right.$$
$$\left. \text{Im}\left((1-m)(\tau (z-z_0) + z_0)^2 + 1\right) = 0 \bigwedge (1-m)(\tau (z-z_0) + z_0)^2 + 1 < 0\right)$$

Involving other related functions

09.46.27.0014.01

$$\text{sc}^{-1}(z \mid m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) \;/;\; \{a, b, z_1\} = \left\{2-m, 1-m, \frac{1}{z^2}\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m < 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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