KelvinKei

Notations

Traditional name

Kelvin function of the second kind

Traditional notation

$\operatorname{kei}(z)$

Mathematica StandardForm notation

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KelvinKei[z]
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Primary definition

03.15.02.0001.01

$$\operatorname{kei}(z) = \operatorname{kei}_0(z)$$

Specific values

Values at fixed points

03.15.03.0001.01

$$\operatorname{kei}(0) = -\frac{\pi}{4}$$

Values at infinities

03.15.03.0002.01

$$\lim_{x\to\infty} \operatorname{kei}(x) = 0$$

03.15.03.0003.01

$$\lim_{x\to-\infty} \operatorname{kei}(x) = \tilde{\infty}$$

General characteristics

Domain and analyticity

$\operatorname{kei}(z)$ is an analytical function of z, which is defined over the whole complex z-plane.

03.15.04.0001.01

$$z \longrightarrow \operatorname{kei}(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

03.15.04.0002.01

$$\operatorname{kei}(\bar{z}) == \overline{\operatorname{kei}(z)} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

The function kei(z) has an essential singularity at $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is a branch point.

03.15.04.0003.01

$$\mathcal{Sing}_z(\operatorname{kei}(z)) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

The function kei(z) has two branch points: $z == 0$, $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is an essential singularity.

03.15.04.0004.01

$$\mathcal{BP}_z(\operatorname{kei}(z)) == \{0, \tilde{\infty}\}$$

03.15.04.0005.01

$$\mathcal{R}_z(\operatorname{kei}(z), 0) == \log$$

03.15.04.0006.01

$$\mathcal{R}_z(\operatorname{kei}(z), \tilde{\infty}) == \log$$

Branch cuts

The function kei(z) is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$ where it is continuous from above.

03.15.04.0007.01

$$\mathcal{BC}_z(\operatorname{kei}(z)) == \{\{(-\infty, 0), -i\}\}$$

03.15.04.0008.01

$$\lim_{\epsilon \to +0} \operatorname{kei}(x + i\,\epsilon) = \operatorname{kei}(x) \;/;\; x \in \mathbb{R} \land x < 0$$

03.15.04.0009.01

$$\lim_{\epsilon \to +0} \operatorname{kei}(x - i\,\epsilon) = \operatorname{kei}(x) + 2\,i\,\pi \operatorname{bei}(x) \;/;\; x \in \mathbb{R} \land x < 0$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

03.15.06.0001.01

$$\operatorname{kei}(z) \propto \operatorname{kei}(z_0) - 2\,i\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{bei}(z_0) - \frac{2\,i\,\pi \left\lfloor \frac{\arg(z-z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0)+\pi}{2\,\pi} \right\rfloor (\operatorname{bei}_1(z_0) - \operatorname{ber}_1(z_0)) - \operatorname{kei}_1(z_0) + \operatorname{ker}_1(z_0)}{\sqrt{2}} (z - z_0) - \frac{1}{4} \left(2\,i\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor (\operatorname{ber}(z_0) - \operatorname{ber}_2(z_0)) - \operatorname{ker}(z_0) + \operatorname{ker}_2(z_0) \right) (z - z_0)^2 + \ldots \;/;\; (z \to z_0)$$

03.15.06.0002.01

$$\operatorname{kei}(z) = \sum_{k=0}^{\infty} \frac{\operatorname{kei}^{(k)}(z_0)\,(z - z_0)^k}{k!} \;/;\; |\arg(z_0)| < \pi$$

03.15.06.0003.01

$$\operatorname{kei}(z) = -\frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{k!} G_{3,7}^{3,3}\left(\frac{z_0}{4}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1-k}{4}, \frac{2-k}{4}, \frac{3-k}{4} \\ -\frac{k}{4}, \frac{2-k}{4}, \frac{2-k}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \end{matrix} \right) (z - z_0)^k \;/;\; |\arg(z_0)| < \pi$$

03.15.06.0004.01

$$\operatorname{kei}(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1+i)^k\, 2^{-\frac{3k}{2}}}{k!} \Bigg(\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{2\,j} \Bigg(\left(1 + i^k\right) \left(\operatorname{kei}_{4\,j-k}(z_0) - 2\,i\,(-1)^k\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{bei}_{k-4\,j}(z_0) \right) - i\left(1 - i^k\right) \left(\operatorname{ker}_{4\,j-k}(z_0) - 2\,i\,(-1)^k\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{ber}_{k-4\,j}(z_0) \right) \Bigg) - \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k}{2\,j+1} \Bigg(\left(1 + i^k\right) \left(\operatorname{kei}_{4\,j-k+2}(z_0) - 2\,i\,(-1)^k\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{bei}_{-4\,j+k-2}(z_0) \right) - i\left(1 - i^k\right) \left(\operatorname{ker}_{4\,j-k+2}(z_0) - 2\,i\,(-1)^k\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{ber}_{-4\,j+k-2}(z_0) \right) \Bigg) \Bigg) (z - z_0)^k$$

03.15.06.0005.01

$$\operatorname{kei}(z) \propto \left(\operatorname{kei}(z_0) - 2\,i\,\pi \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0) + \pi}{2\,\pi} \right\rfloor \operatorname{bei}(z_0) \right) (1 + O(z - z_0))$$

Expansions on branch cuts

03.15.06.0006.01

$$\operatorname{kei}(z) \propto -2\,i\,\pi \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor \operatorname{bei}(x) + \operatorname{kei}(x) - \frac{2\,i\,\pi \left\lfloor \frac{\arg(z-x)}{2\,\pi} \right\rfloor (\operatorname{bei}_1(x) - \operatorname{ber}_1(x)) - \operatorname{kei}_1(x) + \operatorname{ker}_1(x)}{\sqrt{2}} (z - x) - \frac{1}{4} \left(2\,i\,\pi \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (\operatorname{ber}(x) - \operatorname{ber}_2(x)) - \operatorname{ker}(x) + \operatorname{ker}_2(x) \right) (z - x)^2 + \ldots \;/;\; (z \to x) \land x \in \mathbb{R} \land x < 0$$

03.15.06.0007.01

$$\operatorname{kei}(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1+i)^k\,2^{-\frac{3k}{2}}}{k!}\left(\sum_{j=0}^{\left\lfloor\frac{k}{2}\right\rfloor}\binom{k}{2j}\left(\left(1+i^k\right)\left(\operatorname{kei}_{4j-k}(x)-2\,i\,(-1)^k\,\pi\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\operatorname{bei}_{k-4j}(x)\right)-i\left(1-i^k\right)\left(\operatorname{ker}_{4j-k}(x)-2\,i\,(-1)^k\,\pi\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\operatorname{ber}_{k-4j}(x)\right)\right)-\sum_{j=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\binom{k}{2j+1}\left(\left(1+i^k\right)\left(\operatorname{kei}_{4j-k+2}(x)-2\,i\,(-1)^k\,\pi\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\operatorname{bei}_{-4j+k-2}(x)\right)-i\left(1-i^k\right)\left(\operatorname{ker}_{4j-k+2}(x)-2\,i\,(-1)^k\,\pi\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\operatorname{ber}_{-4j+k-2}(x)\right)\right)\right)(z-x)^k\;/;\;x\in\mathbb{R}\land x<0$$

03.15.06.0008.01

$$\operatorname{kei}(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-2\,\pi\,i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor G_{2,6}^{1,2}\left(\frac{x}{4},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1-k}{4},\frac{3-k}{4}\\ \frac{2-k}{4},0,\frac{1}{4},\frac{1}{2},\frac{3}{4},-\frac{k}{4}\end{matrix}\right)-\frac{1}{4}\,G_{3,7}^{3,3}\left(\frac{x}{4},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1-k}{4},\frac{2-k}{4},\frac{3-k}{4}\\ -\frac{k}{4},\frac{2-k}{4},\frac{2-k}{4},0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\end{matrix}\right)\right)(z-x)^k\;/;\;x\in\mathbb{R}\land x<0$$

03.15.06.0009.01

$$\operatorname{kei}(z)\propto\left(\operatorname{kei}(x)-2\,i\,\pi\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor\operatorname{bei}(x)\right)(O(z-x)+1)\;/;\;x\in\mathbb{R}\land x<0$$

Expansions at $z == 0$

For the function itself

03.15.06.0010.01

$$\operatorname{kei}(z)\propto-\frac{\pi}{4}\left(1-\frac{z^4}{64}+\frac{z^8}{147456}+\ldots\right)-\frac{z^2}{4}\log\left(\frac{z}{2}\right)\left(1-\frac{z^4}{576}+\frac{z^8}{3686400}+\ldots\right)+\frac{z^2}{4}\left(1-\gamma+\frac{(-11+6\gamma)z^4}{3456}-\frac{(-137+60\gamma)z^8}{221184000}+\ldots\right)\;/;\;(z\to0)$$

03.15.06.0011.01

$$\operatorname{kei}(z)=-\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{((2k)!)^2}\left(\frac{z}{2}\right)^{4k}+\frac{z^2}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\,\psi(2k+2)}{((2k+1)!)^2}\left(\frac{z}{2}\right)^{4k}-\frac{z^2}{4}\log\left(\frac{z}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{((2k+1)!)^2}\left(\frac{z}{2}\right)^{4k}$$

03.15.06.0012.01

$$\operatorname{kei}(z)=-\frac{\pi}{4}\,{}_0F_3\left(;\frac{1}{2},\frac{1}{2},1;-\frac{z^4}{256}\right)-\frac{z^2}{4}\log\left(\frac{z}{2}\right){}_0F_3\left(;1,\frac{3}{2},\frac{3}{2};-\frac{z^4}{256}\right)+\frac{z^2}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\,\psi(2k+2)}{((2k+1)!)^2}\left(\frac{z}{2}\right)^{4k}$$

03.15.06.0013.01

$$\operatorname{kei}(z)=-\frac{\pi}{8}\left(I_0\left(\sqrt[4]{-1}\,z\right)+J_0\left(\sqrt[4]{-1}\,z\right)\right)+\frac{i}{2}\left(I_0\left(\sqrt[4]{-1}\,z\right)-J_0\left(\sqrt[4]{-1}\,z\right)\right)\log\left(\frac{z}{2}\right)+\frac{z^2}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\,\psi(2k+2)}{((2k+1)!)^2}\left(\frac{z}{2}\right)^{4k}$$

03.15.06.0014.01

$$\operatorname{kei}(z)\propto-\frac{\pi}{4}\left(1+O\left(z^4\right)\right)+\frac{z^2}{4}(1-\gamma)\left(1+O\left(z^4\right)\right)-\frac{z^2}{4}\log\left(\frac{z}{2}\right)\left(1+O\left(z^4\right)\right)\;/;\;(z\to0)$$

03.15.06.0015.01

$$\operatorname{kei}(z) \propto -\frac{\pi}{4}\left(1+O\left(z^4\right)\right)-\frac{1}{4} z^2 \log(z)\left(1+O\left(z^4\right)\right)+\frac{1}{4} z^2 (\log(2)-\gamma+1)\left(1+O\left(z^4\right)\right)$$

For small integer powers of the function

03.15.06.0016.01

$$\operatorname{kei}(z)^2 \propto \frac{1}{32}\left(\pi^2+(\log(16)-4\gamma)^2+16\left(\log\left(\frac{z}{4}\right)+2\gamma\right)\log(z)+\frac{1}{32}\left(-8\gamma(\log(16)+5)+16\gamma^2+\pi^2+8\left(2\log^2(2)+\log(32)+4\right)+8\log(z)(-4\log(2)+2\log(z)+4\gamma-5)\right)z^4+\frac{1}{221184}\left(\log^2(4096)+9\pi^2+516\log(2)+12\gamma(-24\log(2)+12\gamma-43)+12\log(z)(-24\log(2)+12\log(z)+24\gamma-43)+536\right)z^8+\ldots\right)-\frac{1}{32}\left((-4\log(2)+4\log(z)-\pi+4\gamma)(-4\log(2)+4\log(z)+\pi+4\gamma)+\frac{1}{32}\left(-8\left(6\log^2(2)+\log(2048)+4\right)+3\pi^2+8\gamma(\log(4096)-6\gamma+11)+8\log(z)(\log(4096)-6\log(z)-12\gamma+11)\right)z^4+\frac{1}{221184}\left(-4\gamma(840\log(2)+1217)+1680\gamma^2-105\pi^2+4(\log(2)(420\log(2)+1217)+838)+4\log(z)(-840\log(2)+420\log(z)+840\gamma-1217)\right)z^8+\ldots\right)-\frac{\pi z^2}{16}\left(1+\frac{z^4}{216}+\frac{z^8}{432000}+\ldots\right)+\frac{\pi z^2}{32}\left(-4\log(2)+4\log(z)+4\gamma-2+\frac{1}{864}(60\log(2)-60\log(z)-60\gamma+73)z^4+\frac{7}{204800}\left(-4\log(2)+4\log(z)-\frac{4127}{630}+4\gamma\right)z^8+\ldots\right)/;\ (z\to 0)$$

03.15.06.0017.01

$$\operatorname{kei}(z)^2 = -\frac{\pi z^2}{16} \sum_{k=0}^{\infty} \frac{64^{-k} z^{4k}}{k! \left(\frac{3}{2}\right)_k^3} + \frac{\pi z^2}{32} \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k} \left(\frac{3}{2}\right)_{2k} \left(\log(4) + 4\log(z) + \psi\left(k+\frac{3}{4}\right) + \psi\left(k+\frac{5}{4}\right) - 6\psi(2k+2)\right)}{((2k+1)!)^3} +$$
$$\frac{1}{32} \sum_{k=0}^{\infty} \frac{2^{-6k} z^{4k}}{\left(\frac{1}{2}\right)_k (k!)^3} \left(\psi\left(k+\frac{1}{2}\right)^2 + 2(\log(64) - 4\log(z) + 3\psi(k+1))\,\psi\left(k+\frac{1}{2}\right) + 2\left(2\log^2(8) + \pi^2 + 8\log\left(\frac{z}{8}\right)\log(z)\right) + \right.$$
$$\left. 3\psi(k+1)(4\log(8) - 8\log(z) + 3\psi(k+1)) - 3\psi^{(1)}(k+1) - \psi^{(1)}\left(k+\frac{1}{2}\right)\right) -$$
$$\frac{1}{16} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-4k} z^{4k} \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{\left(\frac{1}{2}\right)_k^3 (k!)^3} \left(8\log^2\left(\frac{z}{2}\right) - 12\psi(k+1)\log\left(\frac{z}{2}\right) - 12\psi\left(k+\frac{1}{2}\right)\log\left(\frac{z}{2}\right) + 4\left(\psi\left(k+\frac{1}{4}\right) + \pi\right)\log\left(\frac{z}{2}\right) + \right.$$
$$4\left(\psi\left(k+\frac{3}{4}\right) - \pi\right)\log\left(\frac{z}{2}\right) + \frac{3\pi^2}{2} + \frac{9}{2}\psi(k+1)^2 + \frac{9}{2}\psi\left(k+\frac{1}{2}\right)^2 + \frac{1}{2}\left(\psi\left(k+\frac{1}{4}\right) + \pi\right)^2 +$$
$$\frac{1}{2}\left(\psi\left(k+\frac{3}{4}\right) - \pi\right)^2 - 3\left(\psi\left(k+\frac{1}{4}\right) + \pi\right)\psi(k+1) - 3\left(\psi\left(k+\frac{3}{4}\right) - \pi\right)\psi(k+1) + 9\psi(k+1)\,\psi\left(k+\frac{1}{2}\right) -$$
$$3\left(\psi\left(k+\frac{1}{4}\right) + \pi\right)\psi\left(k+\frac{1}{2}\right) - 3\psi\left(k+\frac{1}{2}\right)\left(\psi\left(k+\frac{3}{4}\right) - \pi\right) + \left(\psi\left(k+\frac{1}{4}\right) + \pi\right)\left(\psi\left(k+\frac{3}{4}\right) - \pi\right) -$$
$$\left.\frac{3}{2}\psi^{(1)}(k+1) - \frac{3}{2}\psi^{(1)}\left(k+\frac{1}{2}\right) + \frac{1}{2}\left(\psi^{(1)}\left(k+\frac{1}{4}\right) - 2\pi^2\right) + \frac{1}{2}\left(\psi^{(1)}\left(k+\frac{3}{4}\right) - 2\pi^2\right)\right)$$

03.15.06.0018.01

$$\operatorname{kei}(z)^2 \propto \frac{1}{16}\pi^2 \left(1 + \log(z)\, O\left(z^2\right)\right)$$

Asymptotic series expansions

Expansions for any *z* in exponential form

Using exponential function with branch cut-free arguments

03.15.06.0019.01

$$\operatorname{kei}(z) \propto \frac{i\, e^{-\frac{(1+i) z}{\sqrt{2}}}}{8 \sqrt{2 \pi} \sqrt{-\sqrt[4]{-1}\, z}\, \left((-1)^{3/4} z\right)^{3/2}} \Bigg(\Bigg(\sqrt{-\sqrt[4]{-1}\, z}\, \bigg(\pi \bigg(\frac{\sqrt{i z^2}\, (3-3 i)}{\sqrt{2}} + \Big(4-3 i\, e^{(1+i) \sqrt{2}\, z}\Big) z\bigg) + 4 \Big(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\Big) \Big(\log\big((-1)^{3/4} z\big) - \log(z)\Big)\bigg) - \sqrt{(-1)^{3/4} z}\, \bigg(e^{\sqrt{2}\, z} \pi \bigg(4 z - \frac{(1+i) \sqrt{-i z^2}}{\sqrt{2}}\bigg) + e^{i \sqrt{2}\, z} (-i) \pi z + 4 \Big((-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2} - e^{i \sqrt{2}\, z} z\Big) \Big(\log(z) - \log\big(-\sqrt[4]{-1}\, z\big)\Big)\bigg)\Bigg) \bigg(1 + O\bigg(\frac{1}{z^4}\bigg)\bigg) - \frac{(-1)^{3/4}}{8 z} \Bigg(\sqrt{-\sqrt[4]{-1}\, z}\, \bigg(\pi \Big(\Big(-4-3 i\, e^{2 \sqrt[4]{-1}\, z}\Big) z + 3 (-1)^{3/4} \sqrt{i z^2}\Big) + 4 \Big(\sqrt[4]{-1} \sqrt{i z^2} - e^{(1+i) \sqrt{2}\, z} z\Big) \Big(\log(z) - \log\big((-1)^{3/4} z\big)\Big)\bigg) - \frac{1}{2} \sqrt{(-1)^{3/4} z}\, \bigg(-2 e^{i \sqrt{2}\, z} \pi z + e^{\sqrt{2}\, z} (1+i) \pi \Big((4+4 i) z - i \sqrt{2} \sqrt{-i z^2}\Big) + 8 \Big(\sqrt[4]{-1}\, e^{\sqrt{2}\, z} \sqrt{-i z^2} - i\, e^{i \sqrt{2}\, z} z\Big) \Big(\log\big(-\sqrt[4]{-1}\, z\big) - \log(z)\Big)\bigg)\Bigg) \bigg(1 + O\bigg(\frac{1}{z^4}\bigg)\bigg) - \frac{9 i}{128 z^2} \Bigg(\sqrt{(-1)^{3/4} z}\, \bigg(e^{\sqrt{2}\, z} \pi \bigg(4 z - \frac{(1+i) \sqrt{-i z^2}}{\sqrt{2}}\bigg) + e^{i \sqrt{2}\, z} (-i) \pi z + 4 \Big((-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2} - e^{i \sqrt{2}\, z} z\Big) \Big(\log(z) - \log\big(-\sqrt[4]{-1}\, z\big)\Big)\bigg) + \sqrt{-\sqrt[4]{-1}\, z}\, \bigg(\pi \bigg(\frac{\sqrt{i z^2}\, (3-3 i)}{\sqrt{2}} + \Big(4-3 i\, e^{(1+i) \sqrt{2}\, z}\Big) z\bigg) + 4 \Big(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\Big) \Big(\log\big((-1)^{3/4} z\big) - \log(z)\Big)\bigg)\Bigg) \bigg(1 + O\bigg(\frac{1}{z^4}\bigg)\bigg) - \frac{75 \sqrt[4]{-1}}{1024 z^3} \Bigg(\frac{1}{2} \sqrt{(-1)^{3/4} z}\, \bigg(-2 e^{i \sqrt{2}\, z} \pi z + e^{\sqrt{2}\, z} (1+i) \pi \Big((4+4 i) z - i \sqrt{2} \sqrt{-i z^2}\Big) + 8 \Big(\sqrt[4]{-1}\, e^{\sqrt{2}\, z} \sqrt{-i z^2} - i\, e^{i \sqrt{2}\, z} z\Big) \Big(\log\big(-\sqrt[4]{-1}\, z\big) - \log(z)\Big)\bigg) + \sqrt{-\sqrt[4]{-1}\, z}\, \bigg(\pi \Big(\Big(-4-3 i\, e^{2 \sqrt[4]{-1}\, z}\Big) z + 3 (-1)^{3/4} \sqrt{i z^2}\Big) + 4 \Big(\sqrt[4]{-1} \sqrt{i z^2} - e^{(1+i) \sqrt{2}\, z} z\Big) \Big(\log(z) - \log\big((-1)^{3/4} z\big)\Big)\bigg)\Bigg) \bigg(1 + O\bigg(\frac{1}{z^4}\bigg)\bigg)\Bigg) \;/;\; (|z| \to \infty)$$

03.15.06.0020.01

$$\begin{aligned}
\operatorname{kei}(z) \propto & \frac{i\, e^{-\frac{(1+i) z}{\sqrt{2}}}}{8 \sqrt{2 \pi} \sqrt{-\sqrt[4]{-1}\, z} \left((-1)^{3/4} z\right)^{3/2}} \\
& \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1}{2}\right)_{2k}^{2} \left(\frac{i}{4 z^2}\right)^k}{(2k)!} \left(\frac{\pi}{\sqrt{2}} \left((-1)^{k+\frac{3}{4}} \sqrt{2} \left(4 - 3 i\, e^{(1+i) \sqrt{2}\, z}\right) \left(-\sqrt[4]{-1}\, z\right)^{3/2} + (-1)^k (3 - 3 i) \sqrt{i z^2} \sqrt{-\sqrt[4]{-1}\, z} - \right.\right.\right. \\
& \left.\sqrt{(-1)^{3/4} z} \left(\sqrt{2}\, e^{i \sqrt{2}\, z} (-i) z - (1+i)\, e^{\sqrt{2}\, z} \left(\sqrt{2} (-2 + 2 i) z + \sqrt{-i z^2}\right)\right)\right) - \\
& 4 \sqrt{(-1)^{3/4} z} \left(e^{i \sqrt{2}\, z} z - (-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2}\right) \left(\log\left(-\sqrt[4]{-1}\, z\right) - \log(z)\right) + \\
& \left. 4 (-1)^k \sqrt{-\sqrt[4]{-1}\, z} \left(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\right) \left(\log\left((-1)^{3/4} z\right) - \log(z)\right)\right) - \\
& \frac{(-1)^{3/4}}{2 z} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\left(\frac{1}{2}\right)_{2k+1}^{2} \left(\frac{i}{4 z^2}\right)^k}{(2k+1)!} \left(\frac{(1+i) \pi}{2} \left((-1)^{k+\frac{3}{4}} \left(4 + 3 i\, e^{2 \sqrt[4]{-1}\, z}\right) (-1+i) \left(-\sqrt[4]{-1}\, z\right)^{3/2} + (-1)^{k+\frac{3}{4}} (3 - 3 i) \right.\right. \\
& \left.\sqrt{i z^2} \sqrt{-\sqrt[4]{-1}\, z} - \sqrt{(-1)^{3/4} z} \left(e^{i \sqrt{2}\, z} (-1+i) z + e^{\sqrt{2}\, z} \left((4 + 4 i) z - i \sqrt{2} \sqrt{-i z^2}\right)\right)\right) - \\
& 4 \sqrt{(-1)^{3/4} z} \left(\sqrt[4]{-1}\, e^{\sqrt{2}\, z} \sqrt{-i z^2} - i\, e^{i \sqrt{2}\, z} z\right) \left(\log\left(-\sqrt[4]{-1}\, z\right) - \log(z)\right) + \\
& \left.\left. 4 (-1)^k \sqrt{-\sqrt[4]{-1}\, z} \left(e^{(1+i) \sqrt{2}\, z} z - \sqrt[4]{-1} \sqrt{i z^2}\right) \left(\log\left((-1)^{3/4} z\right) - \log(z)\right) + \ldots\right) \right) /;\ (|z| \to \infty) \land n \in \mathbb{N}
\end{aligned}$$

03.15.06.0021.01

$$\operatorname{kei}(z) \propto \frac{i\, e^{-\frac{(1+i) z}{\sqrt{2}}}}{8 \sqrt{2 \pi} \sqrt{-\sqrt[4]{-1}\, z}\, \left((-1)^{3/4} z\right)^{3/2}}$$

$$\Bigg(\Bigg(\Bigg(\sqrt{-\sqrt[4]{-1}\, z} \left(\pi \left(\frac{\sqrt{i z^2}\, (3-3 i)}{\sqrt{2}} + \left(4 - 3 i\, e^{(1+i) \sqrt{2}\, z}\right) z\right) + 4 \left(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\right) \left(\log\left((-1)^{3/4} z\right) - \log(z)\right)\right) -$$

$$\sqrt{(-1)^{3/4} z} \left(e^{\sqrt{2}\, z} \pi \left(4 z - \frac{(1+i) \sqrt{-i z^2}}{\sqrt{2}}\right) + e^{i \sqrt{2}\, z} (-i) \pi z + 4 \left((-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2} - e^{i \sqrt{2}\, z} z\right)\right.$$

$$\left.\left(\log(z) - \log\left(-\sqrt[4]{-1}\, z\right)\right)\right)\Bigg)\, {}_8F_3\left(\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; -\frac{16}{z^4}\right) -$$

$$\frac{(-1)^{3/4}}{8 z} \Bigg(\sqrt{-\sqrt[4]{-1}\, z} \left(\pi \left(\left(-4 - 3 i\, e^{2 \sqrt[4]{-1}\, z}\right) z + 3 (-1)^{3/4} \sqrt{i z^2}\right) + 4 \left(\sqrt[4]{-1} \sqrt{i z^2} - e^{(1+i) \sqrt{2}\, z} z\right)\right.$$

$$\left.\left(\log(z) - \log\left((-1)^{3/4} z\right)\right)\right) - \frac{1}{2} \sqrt{(-1)^{3/4} z} \left(-2 e^{i \sqrt{2}\, z} \pi z + e^{\sqrt{2}\, z} (1+i) \pi \left((4+4 i) z - i \sqrt{2} \sqrt{-i z^2}\right) + \right.$$

$$\left. 8 \left(\sqrt[4]{-1}\, e^{\sqrt{2}\, z} \sqrt{-i z^2} - i\, e^{i \sqrt{2}\, z} z\right) \left(\log\left(-\sqrt[4]{-1}\, z\right) - \log(z)\right)\right)\Bigg)$$

$${}_8F_3\left(\frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}, \frac{9}{8}, \frac{9}{8}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{16}{z^4}\right) - \frac{9 i}{128 z^2} \Bigg(\sqrt{(-1)^{3/4} z} \left(e^{\sqrt{2}\, z} \pi \left(4 z - \frac{(1+i) \sqrt{-i z^2}}{\sqrt{2}}\right) + \right.$$

$$\left. e^{i \sqrt{2}\, z} (-i) \pi z + 4 \left((-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2} - e^{i \sqrt{2}\, z} z\right) \left(\log(z) - \log\left(-\sqrt[4]{-1}\, z\right)\right)\right) + \sqrt{-\sqrt[4]{-1}\, z}$$

$$\left(\pi \left(\frac{\sqrt{i z^2}\, (3-3 i)}{\sqrt{2}} + \left(4 - 3 i\, e^{(1+i) \sqrt{2}\, z}\right) z\right) + 4 \left(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\right) \left(\log\left((-1)^{3/4} z\right) - \log(z)\right)\right)\Bigg)$$

$${}_8F_3\left(\frac{5}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8}, \frac{9}{8}, \frac{9}{8}, \frac{11}{8}, \frac{11}{8}; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; -\frac{16}{z^4}\right) - \frac{75 \sqrt[4]{-1}}{1024 z^3}$$

$$\Bigg(\frac{1}{2} \sqrt{(-1)^{3/4} z} \left(-2 e^{i \sqrt{2}\, z} \pi z + e^{\sqrt{2}\, z} (1+i) \pi \left((4+4 i) z - i \sqrt{2} \sqrt{-i z^2}\right) + \right.$$

$$\left. 8 \left(\sqrt[4]{-1}\, e^{\sqrt{2}\, z} \sqrt{-i z^2} - i\, e^{i \sqrt{2}\, z} z\right) \left(\log\left(-\sqrt[4]{-1}\, z\right) - \log(z)\right)\right) + \sqrt{-\sqrt[4]{-1}\, z}$$

$$\left(\pi \left(\left(-4 - 3 i\, e^{2 \sqrt[4]{-1}\, z}\right) z + 3 (-1)^{3/4} \sqrt{i z^2}\right) + 4 \left(\sqrt[4]{-1} \sqrt{i z^2} - e^{(1+i) \sqrt{2}\, z} z\right) \left(\log(z) - \log\left((-1)^{3/4} z\right)\right)\right)\Bigg)$$

$${}_8F_3\left(\frac{7}{8}, \frac{7}{8}, \frac{9}{8}, \frac{9}{8}, \frac{11}{8}, \frac{11}{8}, \frac{13}{8}, \frac{13}{8}; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; -\frac{16}{z^4}\right)\Bigg) \;/;\; (|z| \to \infty)$$

03.15.06.0022.01

$$\operatorname{kei}(z) \propto \frac{i\, e^{-\frac{(1+i) z}{\sqrt{2}}}}{8 \sqrt{2 \pi} \sqrt{-\sqrt[4]{-1}\, z}\, \left((-1)^{3/4} z\right)^{3/2}} \left(\left(\sqrt{-\sqrt[4]{-1}\, z} \left(\pi \left(\frac{\sqrt{i z^2}\, (3-3 i)}{\sqrt{2}} + \left(4 - 3 i\, e^{(1+i) \sqrt{2}\, z}\right) z\right) + 4 \left(e^{(1+i) \sqrt{2}\, z} z + \sqrt[4]{-1} \sqrt{i z^2}\right) \left(\log\left((-1)^{3/4} z\right) - \log(z)\right)\right) - \sqrt{(-1)^{3/4} z} \left(e^{\sqrt{2}\, z} \pi \left(4 z - \frac{(1+i) \sqrt{-i z^2}}{\sqrt{2}}\right) + e^{i \sqrt{2}\, z} (-i) \pi z + 4 \left((-1)^{3/4} e^{\sqrt{2}\, z} \sqrt{-i z^2} - e^{i \sqrt{2}\, z} z\right) \left(\log(z) - \log\left(-\sqrt[4]{-1}\, z\right)\right)\right)\right) \left(1 + O\left(\frac{1}{z^4}\right)\right) /; (|z| \to \infty)$$

03.15.06.0023.01

$$\operatorname{kei}(z) \propto \begin{cases} \frac{(-1)^{5/8} \left((-1+i) + \sqrt{2}\, e^{i \sqrt{2}\, z}\right) \sqrt{\pi}}{4\, e^{\sqrt[4]{-1}\, z} \sqrt{z}} & 4 \arg(z) \leq \pi \\ \sqrt{\frac{\pi}{2}}\, \frac{(-1)^{3/8} e^{-\sqrt[4]{-1}\, z}}{2 \sqrt{z}} \left(e^{i \sqrt{2}\, z} \left(\sqrt[4]{-1} - 2 i\, e^{\sqrt{2}\, z}\right) - 1\right) & 4 \arg(z) \leq 3 \pi \\ \sqrt{\frac{\pi}{2}}\, \frac{(-1)^{5/8} e^{-\sqrt[4]{-1}\, z}}{2 \sqrt{z}} \left((-1)^{3/4} - 2 \sqrt[4]{-1}\, e^{2 \sqrt[4]{-1}\, z} + e^{i \sqrt{2}\, z} + 2 i\, e^{\sqrt{2}\, z}\right) & \text{True} \end{cases} /; (|z| \to \infty)$$

Residue representations

03.15.06.0024.01

$$\operatorname{kei}(z) = -\frac{1}{4} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s) \left(\frac{z}{4}\right)^{-4 s}}{\Gamma(1-s)} \Gamma\left(s + \frac{1}{2}\right)^2\right)\left(-j - \frac{1}{2}\right) - \frac{1}{4} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(s + \frac{1}{2}\right)^2 \left(\frac{z}{4}\right)^{-4 s}}{\Gamma(1-s)} \Gamma(s)\right)(-j)$$

Integral representations

On the real axis

Contour integral representations

Limit representations

Generating functions

Differential equations

Ordinary linear differential equations and wronskians

03.15.13.0001.01

$$w^{(4)}(z)\, z^4 + 2\, w^{(3)}(z)\, z^3 - w''(z)\, z^2 + w'(z)\, z + z^4\, w(z) = 0 /; w(z) = c_1 \operatorname{ber}(z) + c_2 \operatorname{bei}(z) + c_3 \operatorname{ker}(z) + c_4 \operatorname{kei}(z)$$

03.15.13.0002.01

$$W_z(\mathrm{ber}(z), \mathrm{bei}(z), \mathrm{ker}(z), \mathrm{kei}(z)) = -\frac{1}{z^2}$$

03.15.13.0003.01

$$g(z)^4 g'(z)^3 w^{(4)}(z) + 2 g(z)^3 \left(g'(z)^2 - 3 g(z) g''(z)\right) g'(z)^2 w^{(3)}(z) - g(z)^2 \left(g'(z)^4 + 6 g(z) g''(z) g'(z)^2 + 4 g(z)^2 g^{(3)}(z) g'(z) - 15 g(z)^2 g''(z)^2\right) g'(z) w''(z) + g(z) \left(g'(z)^6 + g(z) g''(z) g'(z)^4 - 2 g(z)^2 g^{(3)}(z) g'(z)^3 + g(z)^2 \left(6 g''(z)^2 - g(z) g^{(4)}(z)\right) g'(z)^2 + 10 g(z)^3 g''(z) g^{(3)}(z) g'(z) - 15 g(z)^3 g''(z)^3\right) w'(z) + g(z)^4 g'(z)^7 w(z) = 0 /; w(z) = c_1 \mathrm{ber}(g(z)) + c_2 \mathrm{bei}(g(z)) + c_3 \mathrm{ker}(g(z)) + c_4 \mathrm{kei}(g(z))$$

03.15.13.0004.01

$$W_z(\mathrm{ber}(g(z)), \mathrm{bei}(g(z)), \mathrm{ker}(g(z)), \mathrm{kei}(g(z))) = -\frac{g'(z)^6}{g(z)^2}$$

03.15.13.0005.01

$$g(z)^4 g'(z)^3 h(z)^4 w^{(4)}(z) + 2 g(z)^3 g'(z)^2 \left(h(z) \left(g'(z)^2 - 3 g(z) g''(z)\right) - 2 g(z) g'(z) h'(z)\right) h(z)^3 w^{(3)}(z) + g(z)^2 g'(z) \left(-\left(g'(z)^4 + 6 g(z) g''(z) g'(z)^2 + 4 g(z)^2 g^{(3)}(z) g'(z) - 15 g(z)^2 g''(z)^2\right) h(z)^2 - 6 g(z) g'(z) \left(h'(z) g'(z)^2 + g(z) h''(z) g'(z) - 3 g(z) h'(z) g''(z)\right) h(z) + 12 g(z)^2 g'(z)^2 h'(z)^2\right) h(z)^2 w''(z) + g(z) \left(\left(g'(z)^6 + g(z) g''(z) g'(z)^4 - 2 g(z)^2 g^{(3)}(z) g'(z)^3 + g(z)^2 \left(6 g''(z)^2 - g(z) g^{(4)}(z)\right) g'(z)^2 + 10 g(z)^3 g''(z) g^{(3)}(z) g'(z) - 15 g(z)^3 g''(z)^3\right) h(z)^3 + 2 g(z) g'(z) \left(h'(z) g'(z)^4 - 3 g(z) h''(z) g'(z)^3 - 2 g(z) \left(g(z) h^{(3)}(z) - 3 h'(z) g''(z)\right) g'(z)^2 + g(z)^2 \left(9 g''(z) h''(z) + 4 h'(z) g^{(3)}(z)\right) g'(z) - 15 g(z)^2 h'(z) g''(z)^2\right) h(z)^2 + 12 g(z)^2 g'(z)^2 h'(z) \left(h'(z) g'(z)^2 + 2 g(z) h''(z) g'(z) - 3 g(z) h'(z) g''(z)\right) h(z) - 24 g(z)^3 g'(z)^3 h'(z)^3\right) h(z) w'(z) + \left(g(z)^4 h(z)^4 g'(z)^7 + g(z)^4 \left(24 h'(z)^4 - 36 h(z) h''(z) h'(z)^2 + 8 h(z)^2 h^{(3)}(z) h'(z) + h(z)^2 \left(6 h''(z)^2 - h(z) h^{(4)}(z)\right)\right) g'(z)^3 - 2 g(z)^3 h(z) \left(g'(z)^2 - 3 g(z) g''(z)\right) \left(6 h'(z)^3 - 6 h(z) h''(z) h'(z) + h(z)^2 h^{(3)}(z)\right) g'(z)^2 + g(z)^2 h(z)^2 \left(h(z) h''(z) - 2 h'(z)^2\right) \left(g'(z)^4 + 6 g(z) g''(z) g'(z)^2 + 4 g(z)^2 g^{(3)}(z) g'(z) - 15 g(z)^2 g''(z)^2\right) g'(z) - g(z) h(z)^3 h'(z) \left(g'(z)^6 + g(z) g''(z) g'(z)^4 - 2 g(z)^2 g^{(3)}(z) g'(z)^3 + g(z)^2 \left(6 g''(z)^2 - g(z) g^{(4)}(z)\right) g'(z)^2 + 10 g(z)^3 g''(z) g^{(3)}(z) g'(z) - 15 g(z)^3 g''(z)^3\right)\right) w(z) = 0 /; w(z) = c_1 h(z) \mathrm{ber}(g(z)) + c_2 h(z) \mathrm{bei}(g(z)) + c_3 h(z) \mathrm{ker}(g(z)) + c_4 h(z) \mathrm{kei}(g(z))$$

03.15.13.0006.01

$$W_z(h(z) \mathrm{ber}(g(z)), h(z) \mathrm{bei}(g(z)), h(z) \mathrm{ker}(g(z)), h(z) \mathrm{kei}(g(z))) = -\frac{h(z)^4 g'(z)^6}{g(z)^2}$$

03.15.13.0007.01

$$z^4 w^{(4)}(z) + (6 - 4 r - 4 s) z^3 w^{(3)}(z) + \left(4 r^2 + 12 (s - 1) r + 6 (s - 2) s + 7\right) z^2 w''(z) + (2 r + 2 s - 1) (-2 (s - 1) s + r (2 - 4 s) - 1) z w'(z) + \left(a^4 r^4 z^{4 r} + s^4 + 4 r s^3 + 4 r^2 s^2\right) w(z) = 0 /; w(z) = c_1 z^s \mathrm{ber}(a z^r) + c_2 z^s \mathrm{bei}(a z^r) + c_3 z^s \mathrm{ker}(a z^r) + c_4 z^s \mathrm{kei}(a z^r)$$

03.15.13.0008.01

$$W_z(z^s \mathrm{ber}(a z^r), z^s \mathrm{bei}(a z^r), z^s \mathrm{ker}(a z^r), z^s \mathrm{kei}(a z^r)) = -a^4 r^6 z^{4 r + 4 s - 6}$$

03.15.13.0009.01

$$w^{(4)}(z) - 4 (\log(r) + \log(s)) w^{(3)}(z) + 2 \left(2 \log^2(r) + 6 \log(s) \log(r) + 3 \log^2(s)\right) w''(z) + 4 (\log(r) + \log(s)) \left(-\log^2(s) - 2 \log(r) \log(s)\right) w'(z) + \left(a^4 \log^4(r) r^{4 z} + \log^4(s) + 4 \log(r) \log^3(s) + 4 \log^2(r) \log^2(s)\right) w(z) = 0 /; w(z) = c_1 s^z \mathrm{ber}(a r^z) + c_2 s^z \mathrm{bei}(a r^z) + c_3 s^z \mathrm{ker}(a r^z) + c_4 s^z \mathrm{kei}(a r^z)$$

03.15.13.0010.01

$$W_z(s^z \mathrm{ber}(a r^z), s^z \mathrm{bei}(a r^z), s^z \mathrm{ker}(a r^z), s^z \mathrm{kei}(a r^z)) = -a^4 r^{4 z} s^{4 z} \log^6(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

03.15.16.0001.01

$$\operatorname{kei}(-z) = \operatorname{kei}(z) + \operatorname{bei}(z)\,(\log(z) - \log(-z))$$

03.15.16.0002.01

$$\operatorname{kei}(i\,z) = -\operatorname{kei}(z) - \frac{1}{2}\pi\operatorname{ber}(z) + (\log(i\,z) - \log(z))\operatorname{bei}(z)$$

03.15.16.0003.01

$$\operatorname{kei}(-i\,z) = -\operatorname{kei}(z) - \frac{1}{2}\pi\operatorname{ber}(z) + (\log(-i\,z) - \log(z))\operatorname{bei}(z)$$

03.15.16.0004.01

$$\operatorname{kei}\left(\frac{1}{\sqrt[4]{-1}}\,z\right) = -\operatorname{kei}\left(\sqrt[4]{-1}\,z\right) - \frac{1}{2}\pi\operatorname{ber}\left(\sqrt[4]{-1}\,z\right) + \operatorname{bei}\left(\sqrt[4]{-1}\,z\right)\left(\log\left(-(-1)^{3/4}\,z\right) - \log\left(\sqrt[4]{-1}\,z\right)\right)$$

03.15.16.0005.01

$$\operatorname{kei}\left((-1)^{-3/4}\,z\right) = \operatorname{kei}\left(\sqrt[4]{-1}\,z\right) + \operatorname{bei}\left(\sqrt[4]{-1}\,z\right)\left(\log\left(\sqrt[4]{-1}\,z\right) - \log\left(-\sqrt[4]{-1}\,z\right)\right)$$

03.15.16.0006.01

$$\operatorname{kei}\left((-1)^{3/4}\,z\right) = -\operatorname{kei}\left(\sqrt[4]{-1}\,z\right) - \frac{1}{2}\pi\operatorname{ber}\left(\sqrt[4]{-1}\,z\right) + \operatorname{bei}\left(\sqrt[4]{-1}\,z\right)\left(\log\left((-1)^{3/4}\,z\right) - \log\left(\sqrt[4]{-1}\,z\right)\right)$$

03.15.16.0007.01

$$\operatorname{kei}\left(\sqrt[4]{z^4}\right) = \frac{\sqrt{z^4}\,\left(4\operatorname{kei}(z) + \operatorname{bei}(z)\left(4\log(z) - \log\left(z^4\right)\right)\right) + \pi\left(\sqrt{z^4} - z^2\right)\operatorname{ber}(z)}{4\,z^2}$$

Addition formulas

03.15.16.0008.01

$$\operatorname{kei}(z_1 - z_2) = \sum_{k=-\infty}^{\infty}\left(\operatorname{ber}_k(z_2)\operatorname{kei}_k(z_1) + \operatorname{bei}_k(z_2)\operatorname{ker}_k(z_1)\right) /;\ \left|\frac{z_2}{z_1}\right| < 1$$

03.15.16.0009.01

$$\operatorname{kei}(z_1 + z_2) = \sum_{k=-\infty}^{\infty}\left(\operatorname{ber}_k(z_2)\operatorname{kei}_{-k}(z_1) + \operatorname{bei}_k(z_2)\operatorname{ker}_{-k}(z_1)\right) /;\ \left|\frac{z_2}{z_1}\right| < 1$$

Multiple arguments

03.15.16.0010.01

$$\operatorname{kei}(z_1\,z_2) = \sum_{k=0}^{\infty}\frac{\left(1 - z_1^2\right)^k\left(\frac{z_2}{2}\right)^k}{k!}\left(\cos\left(\frac{3\,k\,\pi}{4}\right)\operatorname{kei}_k(z_2) + \operatorname{ker}_k(z_2)\sin\left(\frac{3\,k\,\pi}{4}\right)\right) /;\ \left|z_1^2 - 1\right| < 1$$

Related transformations

Involving ker(z)

03.15.16.0011.01

$$\operatorname{kei}(z)+i\operatorname{ker}(z)=i\,J_0\!\left(\sqrt[4]{-1}\;z\right)\left(\frac{i\,\pi}{4}-\log(z)+\log\!\left(\sqrt[4]{-1}\;z\right)\right)-\frac{1}{2}\,(\pi\,i)\,Y_0\!\left(\sqrt[4]{-1}\;z\right)$$

03.15.16.0012.01

$$\operatorname{kei}(z)-i\operatorname{ker}(z)=-i\,K_0\!\left(\sqrt[4]{-1}\;z\right)-i\,I_0\!\left(\sqrt[4]{-1}\;z\right)\left(-\frac{1}{4}\,(\pi\,i)-\log(z)+\log\!\left(\sqrt[4]{-1}\;z\right)\right)$$

Differentiation

Low-order differentiation

03.15.20.0001.01

$$\frac{\partial\operatorname{kei}(z)}{\partial z}=\frac{\operatorname{kei}_1(z)-\operatorname{ker}_1(z)}{\sqrt{2}}$$

03.15.20.0002.01

$$\frac{\partial^2\operatorname{kei}(z)}{\partial z^2}=\frac{1}{2}\,(\operatorname{ker}(z)-\operatorname{ker}_2(z))$$

Symbolic differentiation

03.15.20.0003.01

$$\frac{\partial^n\operatorname{kei}(z)}{\partial z^n}=2^{-\frac{3n}{2}-1}\,(i-1)^n$$

$$\left(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2\,k}\left((1+i^n)\operatorname{kei}_{4k-n}(z)-i\,(1-i^n)\operatorname{ker}_{4k-n}(z)\right)+\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2\,k+1}\left(i\,(1-i^n)\operatorname{ker}_{4k-n+2}(z)-(1+i^n)\operatorname{kei}_{4k-n+2}(z)\right)\right)/;\;n\in\mathbb{N}$$

03.15.20.0004.01

$$\frac{\partial^n\operatorname{kei}(z)}{\partial z^n}=2^{-\frac{3n}{2}-1}\,(i-1)^n\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\frac{n+1}{2\,k+1}\binom{n}{2\,k}\left((1+i^n)\operatorname{kei}_{4k-n}(z)+\left(-i+i^{n+1}\right)\operatorname{ker}_{4k-n}(z)\right)+\right.$$

$$\left.\frac{\sqrt{2}\;(1+i)\,(4\,k-n+1)}{z}\binom{n}{2\,k+1}\left(\left(1-i^{n+1}\right)\operatorname{kei}_{4k-n+1}(z)+(-i+i^n)\operatorname{ker}_{4k-n+1}(z)\right)\right)/;\;n\in\mathbb{N}$$

03.15.20.0005.01

$$\frac{\partial^n\operatorname{kei}(z)}{\partial z^n}=-\frac{1}{4}\,G_{3,7}^{3,3}\!\left(\frac{z}{4},\,\frac{1}{4}\,\middle|\,\begin{array}{c}\frac{1-n}{4},\,\frac{2-n}{4},\,\frac{3-n}{4}\\ -\frac{n}{4},\,\frac{2-n}{4},\,\frac{2-n}{4},\,0,\,\frac{1}{4},\,\frac{1}{2},\,\frac{3}{4}\end{array}\right)/;\;n\in\mathbb{N}$$

Fractional integro-differentiation

03.15.20.0006.01

$$\frac{\partial^\alpha\operatorname{kei}(z)}{\partial z^\alpha}=i\,z^{2-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{-4k}\,(4\,k+2)!\,(\log(2)+\psi(2\,k+2))}{((2\,k+1)!)^2\,\Gamma(4\,k-\alpha+3)}\,z^{4k}+$$

$$\frac{i\,z^{2-\alpha}}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{-4k}\,\mathcal{FC}_{\log}^{(\alpha)}(z,\,4\,k+2)}{((2\,k+1)!)^2}\,z^{4k}-\frac{\pi\,z^{-\alpha}}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{-4k}\,(4\,k)!\,z^{4k}}{((2\,k)!)^2\,\Gamma(4\,k-\alpha+1)}$$

03.15.20.0007.01

$$\frac{\partial^{\alpha}\operatorname{kei}(z)}{\partial z^{\alpha}} = 2^{2\alpha-\frac{7}{2}}\, i\, \pi^2 \log(2)\, z^{2-\alpha}\, {}_2\tilde{F}_5\left(\frac{3}{4}, \frac{5}{4}; \frac{3}{2}, \frac{3}{4}-\frac{\alpha}{4}, 1-\frac{\alpha}{4}, \frac{5}{4}-\frac{\alpha}{4}, \frac{3}{2}-\frac{\alpha}{4}; -\frac{z^4}{256}\right) - 2^{2\alpha-\frac{3}{2}}\, \pi^3\, z^{-\alpha}\, {}_2\tilde{F}_5\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1-\alpha}{4}, \frac{2-\alpha}{4}, \frac{3-\alpha}{4}, 1-\frac{\alpha}{4}; -\frac{z^4}{256}\right) + i\, z^{2-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{-4k}\, (4k+2)!\, \psi(2k+2)}{((2k+1)!)^2\, \Gamma(4k-\alpha+3)}\, z^{4k} + \frac{i\, z^{2-\alpha}}{4} \sum_{k=0}^{\infty} \frac{(-1)^k\, 2^{-4k}\, \mathcal{FC}_{\log}^{(\alpha)}(z, 4k+2)}{((2k+1)!)^2}\, z^{4k}$$

Integration

Indefinite integration

03.15.21.0001.01

$$\int \operatorname{kei}(a\, z)\, dz = -\frac{1}{16}\, z\, G_{1,5}^{3,1}\left(\frac{a\, z}{4}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{3}{4} \\ 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, 0 \end{matrix}\right)$$

Definite integration

03.15.21.0002.01

$$\int_0^{\infty} t^{\alpha-1}\, e^{-p\, t}\, \operatorname{kei}(t)\, dt = \frac{1}{3}\, 2^{\alpha-3} \left(2\, p\, \Gamma\left(\frac{\alpha+1}{2}\right)^2 \left(p^2 (\alpha+1)^2 \cos\left(\frac{1}{4}\, \pi\, (\alpha+1)\right) {}_4F_3\left(\frac{\alpha}{4}+\frac{3}{4}, \frac{\alpha}{4}+\frac{3}{4}, \frac{\alpha}{4}+\frac{5}{4}, \frac{\alpha}{4}+\frac{5}{4}; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; -p^4\right) + 6\cos\left(\frac{1}{4}\,(\pi-\pi\,\alpha)\right) {}_4F_3\left(\frac{\alpha}{4}+\frac{1}{4}, \frac{\alpha}{4}+\frac{1}{4}, \frac{\alpha}{4}+\frac{3}{4}, \frac{\alpha}{4}+\frac{3}{4}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -p^4\right)\right) - 3\,\Gamma\left(\frac{\alpha}{2}\right)^2 \left(p^2\, \alpha^2 \cos\left(\frac{\pi\,\alpha}{4}\right) {}_4F_3\left(\frac{\alpha}{4}+\frac{1}{2}, \frac{\alpha}{4}+\frac{1}{2}, \frac{\alpha}{4}+1, \frac{\alpha}{4}+1; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; -p^4\right) + 2\sin\left(\frac{\pi\,\alpha}{4}\right) {}_4F_3\left(\frac{\alpha}{4}+\frac{1}{2}, \frac{\alpha}{4}+\frac{1}{2}, \frac{\alpha}{4}, \frac{\alpha}{4}; \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; -p^4\right)\right)\right) /; \operatorname{Re}(\alpha) > 0 \bigwedge \operatorname{Re}(p) > -\frac{1}{\sqrt{2}}$$

Integral transforms

Laplace transforms

03.15.22.0001.01

$$\mathcal{L}_t[\operatorname{kei}(t)](z) = \frac{1}{4\sqrt[4]{z^4+1}} \left(4\, z\, \sqrt[4]{z^4+1}\; {}_3F_2\left(\frac{1}{2}, 1, 1; \frac{3}{4}, \frac{5}{4}; -z^4\right) - \sqrt{2}\, \pi \left(\cos\left(\frac{1}{2}\tan^{-1}(z^2)\right) + \sin\left(\frac{1}{2}\tan^{-1}(z^2)\right)\right)\right) /; \operatorname{Re}(z) > -\frac{1}{\sqrt{2}}$$

Mellin transforms

03.15.22.0002.01

$$\mathcal{M}_t[\operatorname{kei}(t)](z) = -2^{z-2}\,\Gamma\left(\frac{z}{2}\right)^2 \sin\left(\frac{\pi z}{4}\right) /;\ \operatorname{Re}(z) > 0$$

Representations through more general functions

Through hypergeometric functions

Involving hypergeometric *U*

03.15.26.0001.01

$$\operatorname{kei}(z) = \frac{1}{2} e^{-(-1)^{3/4} z}\, i \sqrt{\pi}\, U\left(\frac{1}{2}, 1, 2(-1)^{3/4} z\right) - \frac{1}{2} e^{-\sqrt[4]{-1}\, z}\, i \sqrt{\pi}\, U\left(\frac{1}{2}, 1, 2\sqrt[4]{-1}\, z\right) - \frac{1}{8}\left(-4 i \log(z) + 4 i \log\left(\sqrt[4]{-1}\, z\right) + \pi\right) {}_0F_1\left(; 1; \frac{i z^2}{4}\right) - \frac{1}{8}\left(4 i \log(z) - 4 i \log\left((-1)^{3/4} z\right) + \pi\right) {}_0F_1\left(; 1; -\frac{i z^2}{4}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

03.15.26.0002.01

$$\operatorname{kei}(z) = -\frac{1}{4} G_{0,4}^{3,0}\left(\frac{z^4}{256} \,\middle|\, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0\right) /;\ -\frac{\pi}{4} < \arg(z) \le \frac{\pi}{4}$$

Classical cases for powers of kei

03.15.26.0003.01

$$\operatorname{kei}\left(\sqrt[4]{z}\right)^2 = \frac{1}{16\sqrt{\pi}} G_{0,4}^{4,0}\left(\frac{z}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{1}{8}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0004.01

$$\operatorname{kei}(z)^2 = \frac{1}{16\sqrt{\pi}} G_{0,4}^{4,0}\left(\frac{z^4}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{1}{8}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z^4}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right) /;\ -\frac{\pi}{4} < \arg(z) \le \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving bei

03.15.26.0005.01

$$\operatorname{bei}\left(\sqrt[4]{z}\right) \operatorname{kei}\left(\sqrt[4]{z}\right) = \frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{1}{8\sqrt{2\pi}} G_{2,6}^{3,2}\left(\frac{z}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0006.01

$$\operatorname{bei}(z) \operatorname{kei}(z) = \frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z^4}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{1}{8\sqrt{2\pi}} G_{2,6}^{3,2}\left(\frac{z^4}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right) /;\ 0 \le \arg(z) \le \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving ber

03.15.26.0007.01

$$\operatorname{ber}\left(\sqrt[4]{z}\right)\operatorname{kei}\left(\sqrt[4]{z}\right) = -\frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{64} \,\middle|\, 0, \tfrac{1}{2}, 0, 0\right) - \frac{1}{8\sqrt{2\pi}}\, G_{2,6}^{3,2}\left(\frac{z}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \tfrac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0008.01

$$\operatorname{ber}(z)\operatorname{kei}(z) = -\frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z^4}{64} \,\middle|\, 0, \tfrac{1}{2}, 0, 0\right) - \frac{1}{8\sqrt{2\pi}}\, G_{2,6}^{3,2}\left(\frac{z^4}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \tfrac{1}{2} \end{matrix}\right) /;\ 0 \leq \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving powers of ker

03.15.26.0009.01

$$\operatorname{kei}\left(\sqrt[4]{z}\right)^2 + \operatorname{ker}\left(\sqrt[4]{z}\right)^2 = \frac{1}{8\sqrt{\pi}}\, G_{0,4}^{4,0}\left(\frac{z}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0010.01

$$\operatorname{kei}\left(\sqrt[4]{z}\right)^2 - \operatorname{ker}\left(\sqrt[4]{z}\right)^2 = -\frac{1}{4}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0011.01

$$\operatorname{kei}(z)^2 + \operatorname{ker}(z)^2 = \frac{1}{8\sqrt{\pi}}\, G_{0,4}^{4,0}\left(\frac{z^4}{64} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) /;\ -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

03.15.26.0012.01

$$\operatorname{kei}(z)^2 - \operatorname{ker}(z)^2 = -\frac{1}{4}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z^4}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right) /;\ -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving ker

03.15.26.0013.01

$$\operatorname{kei}\left(\sqrt[4]{z}\right)\operatorname{ker}\left(\sqrt[4]{z}\right) = -\frac{1}{8}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z}{16} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0 \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0014.01

$$\operatorname{kei}(z)\operatorname{ker}(z) = -\frac{1}{8}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z^4}{16}\,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix}\right) /; -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving ber, bei and ker

03.15.26.0015.01

$$\operatorname{bei}\left(\sqrt[4]{z}\right)\operatorname{kei}\left(\sqrt[4]{z}\right) + \operatorname{ber}\left(\sqrt[4]{z}\right)\operatorname{ker}\left(\sqrt[4]{z}\right) = \frac{1}{4}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{64}\,\middle|\, 0, 0, 0, \frac{1}{2}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0016.01

$$\operatorname{bei}\left(\sqrt[4]{z}\right)\operatorname{kei}\left(\sqrt[4]{z}\right) - \operatorname{ber}\left(\sqrt[4]{z}\right)\operatorname{ker}\left(\sqrt[4]{z}\right) = -\frac{1}{2^{5/2}\sqrt{\pi}}\, G_{2,6}^{3,2}\left(\frac{z}{16}\,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0017.01

$$\operatorname{ber}\left(\sqrt[4]{z}\right)\operatorname{kei}\left(\sqrt[4]{z}\right) + \operatorname{bei}\left(\sqrt[4]{z}\right)\operatorname{ker}\left(\sqrt[4]{z}\right) = -\frac{1}{2^{5/2}\sqrt{\pi}}\, G_{2,6}^{3,2}\left(\frac{z}{16}\,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0018.01

$$\operatorname{bei}\left(\sqrt[4]{z}\right)\operatorname{ker}\left(\sqrt[4]{z}\right) - \operatorname{ber}\left(\sqrt[4]{z}\right)\operatorname{kei}\left(\sqrt[4]{z}\right) = \frac{1}{4}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{64}\,\middle|\, 0, \frac{1}{2}, 0, 0\right)$$

Brychkov Yu.A. (2006)

03.15.26.0019.01

$$\operatorname{bei}(z)\operatorname{kei}(z) + \operatorname{ber}(z)\operatorname{ker}(z) = \frac{1}{4}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z^4}{64}\,\middle|\, 0, 0, 0, \frac{1}{2}\right) /; -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

03.15.26.0020.01

$$\operatorname{bei}(z)\operatorname{kei}(z) - \operatorname{ber}(z)\operatorname{ker}(z) = -\frac{1}{2^{5/2}\sqrt{\pi}}\, G_{2,6}^{3,2}\left(\frac{z^4}{16}\,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right) /; -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

03.15.26.0021.01

$$\operatorname{ber}(z)\operatorname{kei}(z) + \operatorname{bei}(z)\operatorname{ker}(z) = -\frac{1}{2^{5/2}\sqrt{\pi}}\, G_{2,6}^{3,2}\left(\frac{z^4}{16}\,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2} \end{matrix}\right) /; -\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$$

Brychkov Yu.A. (2006)

03.15.26.0022.01

$$\operatorname{bei}(z)\operatorname{ker}(z)-\operatorname{ber}(z)\operatorname{kei}(z)=\frac{1}{4}\sqrt{\pi}\,G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,\tfrac{1}{2},0,0\right)/;-\frac{\pi}{4}<\arg(z)\leq\frac{\pi}{4}\bigvee\frac{3\pi}{4}<\arg(z)\leq\pi\bigvee-\pi<\arg(z)\leq-\frac{3\pi}{4}$$

Brychkov Yu.A. (2006)

Classical cases involving Bessel J

03.15.26.0023.01

$$J_0\!\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{1}{8}\sqrt{\pi}\left(-i\,G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)-i\,G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)\right)\right)/;-\frac{\pi}{4}<\arg(z)\leq\frac{\pi}{4}$$

Classical cases involving Bessel I

03.15.26.0024.01

$$I_0\!\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{1}{8}\sqrt{\pi}\left(i\,G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(i\,G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)+G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)\right)\right)/;-\frac{\pi}{4}<\arg(z)\leq\frac{\pi}{4}$$

Classical cases involving Bessel K

03.15.26.0025.01

$$K_0\!\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{i}{16\sqrt{\pi}}\,G_{0,4}^{4,0}\!\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-\frac{i}{8\sqrt{2\pi}}\,G_{2,6}^{6,0}\!\left(\frac{1}{2}\sqrt[4]{-1}\,z,\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix}\right)/;-\pi<\arg(z)\leq 0$$

Classical cases involving ${}_0F_1$

03.15.26.0026.01

$${}_0F_1\!\left(;1;\frac{i\sqrt{z}}{4}\right)\operatorname{kei}\!\left(\sqrt[4]{z}\right)=\frac{1}{8}\sqrt{\pi}\left(i\,G_{0,4}^{2,0}\!\left(\frac{z}{64}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\!\left(\frac{z}{64}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(i\,G_{2,6}^{3,2}\!\left(\frac{z}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)+G_{2,6}^{3,2}\!\left(\frac{z}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)\right)\right)$$

03.15.26.0027.01

$${}_0F_1\!\left(;1;\frac{i\,z^2}{4}\right)\operatorname{kei}(z)=\frac{1}{8}\sqrt{\pi}\left(i\,G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\!\left(\frac{z^4}{64}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(i\,G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)+G_{2,6}^{3,2}\!\left(\frac{z^4}{16}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)\right)\right)/;-\frac{\pi}{4}<\arg(z)\leq\frac{\pi}{4}$$

Generalized cases for the direct function itself

03.15.26.0028.01

$$\operatorname{kei}(z) = -\frac{1}{4} G_{0,4}^{3,0}\left(\frac{z}{4}, \frac{1}{4} \,\middle|\, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

Generalized cases for powers of kei

03.15.26.0029.01

$$\operatorname{kei}(z)^2 = \frac{1}{16\sqrt{\pi}} G_{0,4}^{4,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{\sqrt{\pi}}{2^{7/2}} G_{2,6}^{5,0}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving bei

03.15.26.0030.01

$$\operatorname{bei}(z)\operatorname{kei}(z) = \frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - \frac{1}{8\sqrt{2\pi}} G_{2,6}^{3,2}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving ber

03.15.26.0031.01

$$\operatorname{ber}(z)\operatorname{kei}(z) = -\frac{1}{8}\sqrt{\pi}\, G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, \tfrac{1}{2}, 0, 0\right) - \frac{1}{8\sqrt{2\pi}} G_{2,6}^{3,2}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving powers of ker

03.15.26.0032.01

$$\operatorname{kei}(z)^2 + \operatorname{ker}(z)^2 = \frac{1}{8\sqrt{\pi}} G_{0,4}^{4,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0033.01

$$\operatorname{kei}(z)^2 - \operatorname{ker}(z)^2 = -\frac{1}{4}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving ker

03.15.26.0034.01

$$\operatorname{kei}(z)\operatorname{ker}(z) = -\frac{1}{8}\sqrt{\frac{\pi}{2}}\, G_{2,6}^{5,0}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix}\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving ber, bei and ker

03.15.26.0035.01

$$\operatorname{bei}(z)\operatorname{kei}(z)+\operatorname{ber}(z)\operatorname{ker}(z)=\frac{1}{4}\sqrt{\pi}\,G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,0,0,\tfrac{1}{2}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0036.01

$$\operatorname{bei}(z)\operatorname{kei}(z)-\operatorname{ber}(z)\operatorname{ker}(z)=-\frac{1}{4\sqrt{2\pi}}\,G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0037.01

$$\operatorname{bei}(z)\operatorname{ker}(z)+\operatorname{ber}(z)\operatorname{kei}(z)=-\frac{1}{4\sqrt{2\pi}}\,G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)$$

Brychkov Yu.A. (2006)

03.15.26.0038.01

$$\operatorname{bei}(z)\operatorname{ker}(z)-\operatorname{ber}(z)\operatorname{kei}(z)=\frac{1}{4}\sqrt{\pi}\,G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,\tfrac{1}{2},0,0\right)$$

Brychkov Yu.A. (2006)

Generalized cases involving Bessel *J*

03.15.26.0039.01

$$J_0\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{1}{8}\sqrt{\pi}\left(-i\,G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)-i\,G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)\right)\right)$$

Generalized cases involving Bessel *I*

03.15.26.0040.01

$$I_0\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{1}{8}\sqrt{\pi}\left(i\,G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,\tfrac{1}{2},0,0\right)-\frac{1}{\sqrt{2}\,\pi}\left(i\,G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\end{matrix}\right)+G_{2,6}^{3,2}\left(\frac{z}{2},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right)\right)\right)$$

Generalized cases involving Bessel *K*

03.15.26.0041.01

$$K_0\left(\sqrt[4]{-1}\,z\right)\operatorname{kei}(z)=\frac{i}{16\sqrt{\pi}}\,G_{0,4}^{4,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,0,0,\tfrac{1}{2}\right)-\frac{i}{8\sqrt{2\pi}}\,G_{2,6}^{6,0}\left(\frac{1}{2}\sqrt[4]{-1}\,z,\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix}\right)/;\ -\pi<\arg(z)\le\frac{3\pi}{4}$$

Generalized cases involving ${}_0F_1$

03.15.26.0042.01

$$ {}_0F_1\left(;1;\frac{i z^2}{4}\right)\operatorname{kei}(z) = \frac{1}{8}\sqrt{\pi}\left(i\, G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, 0, 0, \tfrac{1}{2}\right) - G_{0,4}^{2,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, \tfrac{1}{2}, 0, 0\right) - \frac{1}{\sqrt{2}\,\pi}\left(i\, G_{2,6}^{3,2}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2} \end{matrix}\right) + G_{2,6}^{3,2}\left(\frac{z}{2}, \frac{1}{4} \,\middle|\, \begin{matrix} \tfrac{1}{4}, \tfrac{3}{4} \\ 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \tfrac{1}{2} \end{matrix}\right)\right)\right) $$

Representations through equivalent functions

With related functions

03.15.27.0001.01

$$ \operatorname{kei}(z) = -\frac{1}{4}\, i\left(2 K_0\left(\sqrt[4]{-1}\, z\right) + \pi\, Y_0\left(\sqrt[4]{-1}\, z\right) - 4 i\left(\log(z) - \log\left(\sqrt[4]{-1}\, z\right)\right)\operatorname{bei}(z) - i\pi\operatorname{ber}(z)\right) $$

03.15.27.0002.01

$$ \operatorname{kei}(z) == -\frac{1}{8}\, i\left(4 K_0\left(\sqrt[4]{-1}\, z\right) + 2\pi\, Y_0\left(\sqrt[4]{-1}\, z\right) + \left(-i\pi - 4\log(z) + 4\log\left(\sqrt[4]{-1}\, z\right)\right) I_0\left(\sqrt[4]{-1}\, z\right) + \left(-i\pi + 4\log(z) - 4\log\left(\sqrt[4]{-1}\, z\right)\right) J_0\left(\sqrt[4]{-1}\, z\right)\right) $$

03.15.27.0003.01

$$ \operatorname{kei}(z) = \begin{cases} -\pi I_0\left(\sqrt[4]{-1}\, z\right) + \frac{3}{4}\pi J_0\left(\sqrt[4]{-1}\, z\right) - \frac{1}{2} i K_0\left(\sqrt[4]{-1}\, z\right) - \frac{1}{4} i\pi\, Y_0\left(\sqrt[4]{-1}\, z\right) & \frac{3\pi}{4} < \arg(z) \le \pi \\ -\frac{1}{2} i K_0\left(\sqrt[4]{-1}\, z\right) - \frac{1}{4}\pi\left(J_0\left(\sqrt[4]{-1}\, z\right) + i\, Y_0\left(\sqrt[4]{-1}\, z\right)\right) & \text{True} \end{cases} $$

03.15.27.0004.01

$$ \operatorname{kei}(z) + i\operatorname{ker}(z) = \frac{i}{4}\left(\left(i\pi - 4\log(z) + 4\log\left(\sqrt[4]{-1}\, z\right)\right) J_0\left(\sqrt[4]{-1}\, z\right) - 2\pi\, Y_0\left(\sqrt[4]{-1}\, z\right)\right) $$

03.15.27.0005.01

$$ \operatorname{kei}(z) + i\operatorname{ker}(z) = \begin{cases} -\frac{1}{2} i\pi\left(3 i J_0\left(\sqrt[4]{-1}\, z\right) + Y_0\left(\sqrt[4]{-1}\, z\right)\right) & \frac{3\pi}{4} < \arg(z) \le \pi \\ -\frac{1}{2} i\pi\left(Y_0\left(\sqrt[4]{-1}\, z\right) - i J_0\left(\sqrt[4]{-1}\, z\right)\right) & \text{True} \end{cases} $$

03.15.27.0006.01

$$ \operatorname{kei}(z) - i\operatorname{ker}(z) = \frac{1}{4}\, i\, I_0\left(\sqrt[4]{-1}\, z\right)\left(i\pi + 4\log(z) - 4\log\left(\sqrt[4]{-1}\, z\right)\right) - i K_0\left(\sqrt[4]{-1}\, z\right) $$

03.15.27.0007.01

$$ \operatorname{kei}(z) - i\operatorname{ker}(z) = \begin{cases} -2\pi I_0\left(\sqrt[4]{-1}\, z\right) - i K_0\left(\sqrt[4]{-1}\, z\right) & \frac{3\pi}{4} < \arg(z) \le \pi \\ -i K_0\left(\sqrt[4]{-1}\, z\right) & \text{True} \end{cases} $$

Theorems

History

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