LerchPhi

Notations

Traditional name

Lerch transcendent

Traditional notation

$\Phi(z, s, a)$

Mathematica StandardForm notation

`LerchPhi[z, s, a]`

Primary definition

10.06.02.0001.01

$$\Phi(z, s, a) == \sum_{k=0}^{\infty} \frac{z^k}{\left((a+k)^2\right)^{s/2}} \;/;\; (|z| < 1 \lor (|z| == 1 \land \operatorname{Re}(s) > 1)) \bigwedge -a \notin \mathbb{N}$$

10.06.02.0002.01

$$\Phi(z, s, -n) == \sum_{k=0}^{n-1} \frac{z^k}{\left((k-n)^2\right)^{s/2}} + \sum_{k=n+1}^{\infty} \frac{z^k}{\left((k-n)^2\right)^{s/2}} \;/;\; (|z| < 1 \lor (|z| == 1 \land \operatorname{Re}(s) > 1)) \land n \in \mathbb{N}$$

10.06.02.0003.01

$$\hat{\Phi}(z, s, a) == \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \;/;\; |z| < 1 \lor (|z| == 1 \land \operatorname{Re}(s) > 1)$$

10.06.02.0004.01

$$\tilde{\Phi}(z, s, a) == \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \;/;\; (|z| < 1 \lor |z| == 1 \land \operatorname{Re}(s) > 1) \bigwedge -a \notin \mathbb{N}$$

10.06.02.0005.01

$$\tilde{\Phi}(z, s, -n) == \sum_{k=0}^{n-1} \frac{z^k}{(k-n)^s} + \sum_{k=n+1}^{\infty} \frac{z^k}{(k-n)^s} \;/;\; (|z| < 1 \lor |z| == 1 \land \operatorname{Re}(s) > 1) \land n \in \mathbb{N}$$

10.06.02.0006.01

$$\Phi(z, s, a) == \tilde{\Phi}(z, s, a) - \sum_{k=0}^{\lfloor -\operatorname{Re}(a) \rfloor} \left(\frac{z^k}{(a+k)^s} - \frac{z^k}{\left((a+k)^2\right)^{s/2}} \right) \;/;\; -a \notin \mathbb{N}^+$$

10.06.02.0007.01

$$\Phi(z, s, -n) == \tilde{\Phi}(z, s, -n) - \sum_{k=0}^{n-1} \left(\frac{z^k}{(k-n)^s} - \frac{z^k}{\left((k-n)^2\right)^{s/2}} \right) /; n \in \mathbb{N}^+$$

In *Mathematica* for definition of function $\Phi(z, s, a)$ the relations $\sum_{k=0}^{\infty} \frac{z^k}{\left((a+k)^2\right)^{s/2}} /; (|z| < 1 \lor (|z| == 1 \land \mathrm{Re}(s) > 1)) \land -a \notin \mathbb{N}$ and

$\Phi(z, s, -n) == \sum_{k=0}^{n-1} \frac{z^k}{\left((k-n)^2\right)^{s/2}} + \sum_{k=n+1}^{\infty} \frac{z^k}{\left((k-n)^2\right)^{s/2}} /; (|z| < 1 \lor (|z| == 1 \land \mathrm{Re}(s) > 1)) \land n \in \mathbb{N}$

were used. So the following identification holds:

10.06.02.0008.01

$$\Phi(z, s, a) == \tilde{\Phi}(z, s, a) /; \mathrm{Re}(a) > 0$$

Specific values

Specialized values

For fixed *z*, *s*

For $\Phi(z, s, a)$

10.06.03.0041.01

$$\Phi(z, s, n) = z^{-n} \left(\mathrm{Li}_s(z) - (2\,\theta(n-1) - 1) \sum_{k=1}^{|n|-\theta(n-1)} \frac{z^{(2\,\theta(n-1)-1)\,k}}{k^s} \right) /; n \in \mathbb{Z}$$

10.06.03.0001.01

$$\Phi(z, s, -n) == z^n \left(\mathrm{Li}_s(z) + \sum_{k=1}^{n} \frac{z^{-k}}{k^s} \right) /; n \in \mathbb{N}$$

10.06.03.0002.01

$$\Phi(z, s, -5) == z^5\, \mathrm{Li}_s(z) + z^4 + 2^{-s} z^3 + 3^{-s} z^2 + 4^{-s} z + 5^{-s}$$

10.06.03.0003.01

$$\Phi(z, s, -4) == z^4\, \mathrm{Li}_s(z) + z^3 + 2^{-s} z^2 + 3^{-s} z + 4^{-s}$$

10.06.03.0004.01

$$\Phi(z, s, -3) == z^3\, \mathrm{Li}_s(z) + z^2 + 2^{-s} z + 3^{-s}$$

10.06.03.0005.01

$$\Phi(z, s, -2) == z^2\, \mathrm{Li}_s(z) + z + 2^{-s}$$

10.06.03.0006.01

$$\Phi(z, s, -1) == z\, \mathrm{Li}_s(z) + 1$$

10.06.03.0007.01

$$\Phi(z, s, 0) == \mathrm{Li}_s(z)$$

10.06.03.0008.01

$$\Phi(z, s, n) == z^{-n} \left(\mathrm{Li}_s(z) - \sum_{k=1}^{n-1} \frac{z^k}{k^s} \right) /; n \in \mathbb{N}^+$$

10.06.03.0009.01

$$\Phi(z, s, 1) = \frac{1}{z} \operatorname{Li}_s(z)$$

10.06.03.0010.01

$$\Phi(z, s, 2) = \frac{1}{z^2} (\operatorname{Li}_s(z) - z)$$

10.06.03.0011.01

$$\Phi(z, s, 3) = \frac{1}{z^3} \left(\operatorname{Li}_s(z) - z - 2^{-s} z^2\right)$$

10.06.03.0012.01

$$\Phi(z, s, 4) = \frac{1}{z^5} \left(\operatorname{Li}_s(z) - z - 2^{-s} z^2 - 3^{-s} z^3\right)$$

10.06.03.0013.01

$$\Phi(z, s, 5) = \frac{1}{z^5} \left(\operatorname{Li}_s(z) - z - 2^{-s} z^2 - 3^{-s} z^3 - 4^{-s} z^4\right)$$

For $\tilde{\Phi}(z, s, a)$

10.06.03.0042.01

$$\tilde{\Phi}(z, s, n) = z^{-n} \left(\operatorname{Li}_s(z) - (2\,\theta(n-1) - 1) \sum_{k=1}^{|n| - \theta(n-1)} \frac{z^{(2\,\theta(n-1)-1)\,k}}{((2\,\theta(n-1) - 1)\,k)^s}\right) /;\ n \in \mathbb{Z}$$

10.06.03.0043.01

$$\tilde{\Phi}(z, s, -n) = z^n \left(\operatorname{Li}_s(z) + \sum_{k=1}^{n} \frac{z^{-k}}{(-k)^s}\right);\ n \in \mathbb{N}$$

10.06.03.0044.01

$$\tilde{\Phi}(z, s, -5) = \operatorname{Li}_s(z)\, z^5 + (-1)^{-s} z^4 + (-2)^{-s} z^3 + (-3)^{-s} z^2 + (-4)^{-s} z + (-5)^{-s}$$

10.06.03.0045.01

$$\tilde{\Phi}(z, s, -4) = \operatorname{Li}_s(z)\, z^4 + (-1)^{-s} z^3 + (-2)^{-s} z^2 + (-3)^{-s} z + (-4)^{-s}$$

10.06.03.0046.01

$$\tilde{\Phi}(z, s, -3) = \operatorname{Li}_s(z)\, z^3 + (-1)^{-s} z^2 + (-2)^{-s} z + (-3)^{-s}$$

10.06.03.0047.01

$$\tilde{\Phi}(z, s, -2) = \operatorname{Li}_s(z)\, z^2 + (-1)^{-s} z + (-2)^{-s}$$

10.06.03.0048.01

$$\tilde{\Phi}(z, s, -1) = z \operatorname{Li}_s(z) + (-1)^{-s}$$

10.06.03.0049.01

$$\tilde{\Phi}(z, s, 0) = \operatorname{Li}_s(z)$$

10.06.03.0050.01

$$\tilde{\Phi}(z, s, n) = z^{-n} \left(\operatorname{Li}_s(z) - \sum_{k=1}^{n-1} \frac{z^k}{k^s}\right) /;\ n \in \mathbb{N}^+$$

10.06.03.0051.01

$$\tilde{\Phi}(z, s, 1) = \frac{\mathrm{Li}_s(z)}{z}$$

10.06.03.0052.01

$$\tilde{\Phi}(z, s, 2) = \frac{\mathrm{Li}_s(z) - z}{z^2}$$

10.06.03.0053.01

$$\tilde{\Phi}(z, s, 3) = \frac{-2^{-s} z^2 - z + \mathrm{Li}_s(z)}{z^3}$$

10.06.03.0054.01

$$\tilde{\Phi}(z, s, 4) = \frac{-3^{-s} z^3 - 2^{-s} z^2 - z + \mathrm{Li}_s(z)}{z^5}$$

10.06.03.0055.01

$$\tilde{\Phi}(z, s, 5) = \frac{-4^{-s} z^4 - 3^{-s} z^3 - 2^{-s} z^2 - z + \mathrm{Li}_s(z)}{z^5}$$

For fixed *z*, *a*

For $\Phi(z, s, a)$

10.06.03.0014.02

$$\Phi(z, -n, a) = (1 - (-1)^n) \left((\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a)) \left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2 \right)^{n/2} + z \left((a + \lfloor -\mathrm{Re}(a) \rfloor + 1)^n + \sum_{j=0}^{n} \binom{n}{j} \mathrm{Li}_{-j}(z)\, (a + \lfloor -\mathrm{Re}(a) \rfloor + 1)^{n-j} \right) \right) \theta(-\mathrm{Re}(a))\, z^{\lfloor -\mathrm{Re}(a) \rfloor} + \left(a^n + \sum_{j=0}^{n} \binom{n}{j} \mathrm{Li}_{-j}(z)\, a^{n-j} \right) (1 - (1 - (-1)^n)\, \theta(-\mathrm{Re}(a))) \,/;\, n \in \mathbb{N}$$

10.06.03.0056.01

$$\Phi(z, -5, a) = \theta(-\mathrm{Re}(a))\, 2\, z^{-\lceil \mathrm{Re}(a) \rceil} \Bigg(\left((a - \lceil \mathrm{Re}(a) \rceil)^2 \right)^{5/2} (-\lceil \mathrm{Re}(a) \rceil + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a)) - \frac{1}{(z-1)^6} \Big(z \Big(a^5 (z-1)^5 - \lceil \mathrm{Re}(a) \rceil^5 (z-1)^5 - 5 a^4 (z-1)^4 + 5 (a (z-1) - 1) \lceil \mathrm{Re}(a) \rceil^4 (z-1)^4 - 10 \left(a^2 (z-1)^2 - 2 a (z-1) + z + 1 \right) \lceil \mathrm{Re}(a) \rceil^3 (z-1)^3 + 10 a^3 (z+1) (z-1)^3 + 10 \left(a^3 (z-1)^3 - 3 a^2 (z-1)^2 - z^2 - 4 z + 3 a \left(z^2 - 1 \right) - 1 \right) \lceil \mathrm{Re}(a) \rceil^2 (z-1)^2 - 10 a^2 \left(z^2 + 4 z + 1 \right) (z-1)^2 - 5 \left(a^4 (z-1)^4 - 4 a^3 (z-1)^3 + 6 a^2 (z+1) (z-1)^2 + z^3 + 11 z^2 + 11 z - 4 a \left(z^3 + 3 z^2 - 3 z - 1 \right) + 1 \right) \lceil \mathrm{Re}(a) \rceil (z-1) - z^4 - 26 z^3 - 66 z^2 - 26 z + 5 a \left(z^4 + 10 z^3 - 10 z - 1 \right) - 1 \Big) \Big) \Bigg) - \frac{1}{(z-1)^6} \left((2\, \theta(-\mathrm{Re}(a)) - 1) \left(-a^5 (z-1)^5 + 5 a^4 z (z-1)^4 - 10 a^3 z (z+1) (z-1)^3 + 10 a^2 z \left(z^2 + 4 z + 1 \right) (z-1)^2 - 5 a z \left(z^4 + 10 z^3 - 10 z - 1 \right) + z \left(z^4 + 26 z^3 + 66 z^2 + 26 z + 1 \right) \right) \right)$$

10.06.03.0057.01

$$\Phi(z, -4, a) = -\frac{1}{(z-1)^5} \left(a^4 (z-1)^4 - 4 a^3 z (z-1)^3 + 6 a^2 z (z+1) (z-1)^2 - 4 a z \left(z^3 + 3 z^2 - 3 z - 1 \right) + z \left(z^3 + 11 z^2 + 11 z + 1 \right) \right)$$

10.06.03.0015.02

$$\Phi(z, -3, a) = 2\left(\left((a - \lceil \mathrm{Re}(a) \rceil)^2\right)^{3/2} (-\lceil \mathrm{Re}(a) \rceil + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a)) - \frac{1}{(z-1)^4} \left(z \left(a^3 (z-1)^3 - \lceil \mathrm{Re}(a) \rceil^3 (z-1)^3 - 3 a^2 (z-1)^2 + 3 (a (z-1) - 1) \lceil \mathrm{Re}(a) \rceil^2 (z-1)^2 - 3 \left(a^2 (z-1)^2 - 2 a (z-1) + z + 1\right) \lceil \mathrm{Re}(a) \rceil (z-1) - z^2 - 4 z + 3 a \left(z^2 - 1\right) - 1\right)\right)\right) \theta(-\mathrm{Re}(a))\, z^{-\lceil \mathrm{Re}(a) \rceil} + \frac{(2\, \theta(-\mathrm{Re}(a)) - 1) \left(a^3 (z-1)^3 - 3 a^2 z (z-1)^2 + 3 a z \left(z^2 - 1\right) - z \left(z^2 + 4 z + 1\right)\right)}{(z-1)^4}$$

10.06.03.0016.01

$$\Phi(z, -2, a) = \frac{-a^2 (z-1)^2 + 2 a (z-1) z - z (1+z)}{(z-1)^3}$$

10.06.03.0017.02

$$\Phi(z, -1, a) = 2\left(\sqrt{(a - \lceil \mathrm{Re}(a) \rceil)^2}\, (-\lceil \mathrm{Re}(a) \rceil + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a)) - \frac{z (a (z-1) - \lceil \mathrm{Re}(a) \rceil (z-1) - 1)}{(z-1)^2}\right) \theta(-\mathrm{Re}(a))\, z^{-\lceil \mathrm{Re}(a) \rceil} + \frac{(2\, \theta(-\mathrm{Re}(a)) - 1) (z a - a - z)}{(z-1)^2}$$

10.06.03.0018.01

$$\Phi(z, 0, a) = \frac{1}{1-z}$$

10.06.03.0058.01

$$\Phi(z, n, a) = (1 - (1 - (-1)^n)\, \theta(-\mathrm{Re}(a)))\, a^{-n}\, {}_{n+1}F_n(1, a_1, a_2, \ldots, a_n; a_1 + 1, a_2 + 1, \ldots, a_n + 1; z) + \theta(-\mathrm{Re}(a))\, (1 - (-1)^n)\, z^{\lfloor -\mathrm{Re}(a) \rfloor} \left(z (a + \lfloor -\mathrm{Re}(a) \rfloor + 1)^{-n}\, {}_{n+1}F_n(1, b_1, b_2, \ldots, b_n; b_1 + 1, b_2 + 1, \ldots, b_n + 1; z) + \frac{(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a))}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{n/2}}\right) /;$$

$$b_n = \lfloor -\mathrm{Re}(a_n) \rfloor + a_n + 1 \bigwedge a_1 = a_2 = \ldots = a_n = a \bigwedge n \in \mathbb{N}^+$$

10.06.03.0059.01

$$\Phi(z, 1, a) = 2\left(\frac{z}{a - \lceil \mathrm{Re}(a) \rceil + 1}\, {}_2F_1(1, a - \lceil \mathrm{Re}(a) \rceil + 1; a - \lceil \mathrm{Re}(a) \rceil + 2; z) + \frac{(-\lceil \mathrm{Re}(a) \rceil + \lfloor \mathrm{Re}(a) \rfloor + 1)\, \theta(\mathrm{Im}(a))}{\sqrt{(a - \lceil \mathrm{Re}(a) \rceil)^2}}\right) \theta(-\mathrm{Re}(a))\, z^{-\lceil \mathrm{Re}(a) \rceil} + \frac{1 - 2\, \theta(-\mathrm{Re}(a))}{a}\, {}_2F_1(1, a; a + 1; z)$$

10.06.03.0060.01

$$\Phi(z, 2, a) = \frac{1}{a^2}\, {}_3F_2(1, a, a; a + 1, a + 1; z)$$

10.06.03.0061.01

$$\Phi(z,3,a)=2\left(\frac{1}{(a-\lceil\operatorname{Re}(a)\rceil+1)^3}\left(z\,{}_4F_3(1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1;a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2;z)\right)+\frac{(-\lceil\operatorname{Re}(a)\rceil+\lfloor\operatorname{Re}(a)\rfloor+1)\,\theta(\operatorname{Im}(a))}{\left((a-\lceil\operatorname{Re}(a)\rceil)^2\right)^{3/2}}\right)\theta(-\operatorname{Re}(a))\,z^{-\lceil\operatorname{Re}(a)\rceil}+\frac{1-2\,\theta(-\operatorname{Re}(a))}{a^3}\,{}_4F_3(1,a,a,a;a+1,a+1,a+1;z)$$

10.06.03.0062.01

$$\Phi(z,4,a)=\frac{1}{a^4}\,{}_5F_4(1,a,a,a,a;a+1,a+1,a+1,a+1;z)$$

10.06.03.0063.01

$$\Phi(z,5,a)=2\left(\frac{1}{(a-\lceil\operatorname{Re}(a)\rceil+1)^5}\left(z\,{}_6F_5(1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1,a-\lceil\operatorname{Re}(a)\rceil+1;a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2,a-\lceil\operatorname{Re}(a)\rceil+2;z)\right)+\frac{(-\lceil\operatorname{Re}(a)\rceil+\lfloor\operatorname{Re}(a)\rfloor+1)\,\theta(\operatorname{Im}(a))}{\left((a-\lceil\operatorname{Re}(a)\rceil)^2\right)^{5/2}}\right)\theta(-\operatorname{Re}(a))\,z^{-\lceil\operatorname{Re}(a)\rceil}+\frac{1-2\,\theta(-\operatorname{Re}(a))}{a^5}\,{}_6F_5(1,a,a,a,a,a;a+1,a+1,a+1,a+1,a+1;z)$$

For $\tilde{\Phi}(z,s,a)$

10.06.03.0064.01

$$\tilde{\Phi}(z,-n,a)=a^n+\sum_{j=0}^{n}\binom{n}{j}\operatorname{Li}_{-j}(z)\,a^{n-j}\;/;\;n\in\mathbb{N}$$

10.06.03.0065.01

$$\tilde{\Phi}(z,-5,a)=\frac{1}{(z-1)^6}\left(-a^5(z-1)^5+5a^4z(z-1)^4-10a^3z(z+1)(z-1)^3+10a^2z\left(z^2+4z+1\right)(z-1)^2-5az\left(z^4+10z^3-10z-1\right)+z\left(z^4+26z^3+66z^2+26z+1\right)\right)$$

10.06.03.0066.01

$$\tilde{\Phi}(z,-4,a)=-\frac{1}{(z-1)^5}\left(a^4(z-1)^4-4a^3z(z-1)^3+6a^2z(z+1)(z-1)^2-4az\left(z^3+3z^2-3z-1\right)+z\left(z^3+11z^2+11z+1\right)\right)$$

10.06.03.0067.01

$$\tilde{\Phi}(z,-3,a)=\frac{-a^3(z-1)^3+3a^2z(z-1)^2-3az\left(z^2-1\right)+z\left(z^2+4z+1\right)}{(z-1)^4}$$

10.06.03.0068.01

$$\tilde{\Phi}(z,-2,a)=-\frac{a^2(z-1)^2-2az(z-1)+z(z+1)}{(z-1)^3}$$

10.06.03.0069.01

$$\tilde{\Phi}(z,-1,a)=\frac{-za+a+z}{(z-1)^2}$$

10.06.03.0070.01

$$\tilde{\Phi}(z, 0, a) = \frac{1}{1-z}$$

10.06.03.0019.02

$$\tilde{\Phi}(z, n, a) = a^{-n} \, _{n+1}F_n(1, a_1, a_2, \ldots, a_n; a_1+1, a_2+1, \ldots, a_n+1; z) \,/;\, a_1 = a_2 = \ldots = a_n = a \land n \in \mathbb{N}^+$$

10.06.03.0020.02

$$\tilde{\Phi}(z, 1, a) = \frac{1}{a} \, _2F_1(1, a; a+1; z)$$

10.06.03.0021.02

$$\tilde{\Phi}(z, 2, a) = \frac{1}{a^2} \, _3F_2(1, a, a; a+1, a+1; z)$$

10.06.03.0022.02

$$\tilde{\Phi}(z, 3, a) = \frac{1}{a^3} \, _4F_3(1, a, a, a; a+1, a+1, a+1; z)$$

10.06.03.0023.02

$$\tilde{\Phi}(z, 4, a) = \frac{1}{a^4} \, _5F_4(1, a, a, a, a; a+1, a+1, a+1, a+1; z)$$

10.06.03.0071.01

$$\tilde{\Phi}(z, 5, a) = \frac{1}{a^5} \, _6F_5(1, a, a, a, a, a; a+1, a+1, a+1, a+1, a+1; z)$$

For fixed *s*, *a*

For $\Phi(z, s, a)$

10.06.03.0072.01

$$\Phi(-1, s, a) = 2^{-s}\left(\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right)\right)$$

10.06.03.0024.01

$$\Phi(0, s, a) = \left(a^2\right)^{-s/2}$$

10.06.03.0025.01

$$\Phi(1, s, a) = \zeta(s, a)$$

For $\hat{\Phi}(z, s, a)$

10.06.03.0073.01

$$\hat{\Phi}(-1, s, a) = 2^{-s}\left(\hat{\zeta}\left(s, \frac{a}{2}\right) - \hat{\zeta}\left(s, \frac{a+1}{2}\right)\right)$$

10.06.03.0026.01

$$\hat{\Phi}(0, s, a) = a^{-s}$$

10.06.03.0027.01

$$\hat{\Phi}(1, s, a) = \hat{\zeta}(s, a)$$

For $\tilde{\Phi}(z, s, a)$

10.06.03.0074.01

$$\tilde{\Phi}(-1, s, a) = 2^{-s}\left(\tilde{\zeta}\left(s, \frac{a}{2}\right) - \tilde{\zeta}\left(s, \frac{a+1}{2}\right)\right)$$

10.06.03.0028.01

$$\tilde{\Phi}(0, s, a) = \left(a^2\right)^{-s/2}$$

10.06.03.0029.01

$$\tilde{\Phi}(1, s, a) = \tilde{\zeta}(s, a)$$

For fixed z

For $\Phi(z, s, a)$

10.06.03.0030.01

$$\Phi(z, 1, 1) = -\frac{\log(1-z)}{z}$$

10.06.03.0031.01

$$\Phi\left(z, 2, \frac{1}{2}\right) = \frac{2}{\sqrt{z}}\left(\mathrm{Li}_2\left(\sqrt{z}\right) - \mathrm{Li}_2\left(-\sqrt{z}\right)\right)$$

10.06.03.0032.01

$$\Phi\left(z, 2, \frac{3}{2}\right) = \frac{2}{z^{3/2}}\left(\mathrm{Li}_2\left(\sqrt{z}\right) - \mathrm{Li}_2\left(-\sqrt{z}\right) - 2\sqrt{z}\right)$$

For fixed s

For $\Phi(z, s, a)$

10.06.03.0033.01

$$\Phi(1, s, 1) = \zeta(s) \;/;\; \mathrm{Re}(s) > 1$$

10.06.03.0034.01

$$\Phi\left(1, s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s) \;/;\; \mathrm{Re}\,(s) > 1$$

10.06.03.0035.01

$$\Phi(-1, s, 1) = \left(1 - 2^{1-s}\right)\zeta(s)$$

For fixed a

For $\Phi(z, s, a)$

10.06.03.0036.01

$$\Phi(0, 1, a) = \frac{1}{\sqrt{a^2}}$$

10.06.03.0037.01

$$\Phi(1, 1, a) = \tilde{\infty}$$

Values at fixed points

For $\Phi(z, s, a)$

10.06.03.0038.01

$$\Phi\left(-1, 2, \frac{1}{2}\right) == 4\,C$$

10.06.03.0039.01

$$\Phi(1, 2, 1) == \frac{\pi^2}{6}$$

10.06.03.0040.01

$$\Phi\left(\frac{1-i}{2}, 2, 1\right) == (1-i)\left(-\frac{i}{8}\log^2(2) - \frac{\pi}{8}\log(2) + \frac{5\,i\,\pi^2}{96} + C\right)$$

General characteristics

Domain and analyticity

Domain and analyticity

For $\Phi(z, s, a)$

$\Phi(z, s, a)$ is an analytical function of z, s, a which is defined in $\mathbb{C}^3$. For fixed a, z, it is an entire function of s.

10.06.04.0001.01

$$(z * s * a) \longrightarrow \Phi(z, s, a) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

For $\Phi(z, s, a)$

10.06.04.0002.01

$$\Phi(\bar{z}, \bar{s}, \bar{a}) == \overline{\Phi(z, s, a)} \;/;\; z \notin (1, \infty)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to a

For $\Phi(z, s, a)$

For fixed z, s, the function $\Phi(z, s, a)$ does not have poles and essential singularities.

10.06.04.0003.01

$$\mathcal{S}ing_a(\Phi(z, s, a)) = \{\}$$

For $\hat{\Phi}(z, s, a)$

For fixed z, s /; $s \notin \mathbb{N}^+$, the function $\hat{\Phi}(z, s, a)$ does not have poles and essential singularities.

10.06.04.0004.01

$$\mathcal{S}ing_a\left(\hat{\Phi}(z, s, a)\right) = \{\} /; s \notin \mathbb{N}^+$$

For fixed z and positive integer s, the function $\hat{\Phi}(z, s, a)$has an infinite set of singular points:

a) $a = -n$ /; $s = 1 \bigwedge n \in \mathbb{N}$, are the simple poles with residues z^n;

b) $a = -n$ /; $s > 1 \bigwedge n \in \mathbb{N}$, are the poles of order s with residues 0;

c) $a = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

10.06.04.0005.01

$$\mathcal{S}ing_a\left(\hat{\Phi}(z, s, a)\right) = \{\{\{-n, s\} /; n \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\} /; s \notin \mathbb{N}^+$$

10.06.04.0006.01

$$\text{res}_a\left(\hat{\Phi}(z, s, a)\right)(-n) = z^n \delta_{s,1} /; n \in \mathbb{N}$$

With respect to *s*

For $\Phi(z, s, a)$

For fixed z, a, the function $\Phi(z, s, a)$ has only one singular point at $s = \tilde{\infty}$. It is an essential singular point.

10.06.04.0007.01

$$\mathcal{S}ing_s(\Phi(z, s, a)) = \{\{\tilde{\infty}, \infty\}\}$$

For $\hat{\Phi}(z, s, a)$

For fixed z, a, the function $\hat{\Phi}(z, s, a)$ has only one singular point at $s = \tilde{\infty}$. It is an essential singular point.

10.06.04.0008.01

$$\mathcal{S}ing_s\left(\hat{\Phi}(z, s, a)\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to *z*

For $\Phi(z, s, a)$

For fixed s, a, the function $\Phi(z, s, a)$ does not have poles and essential singularities.

10.06.04.0009.01

$$\mathcal{S}ing_z(\Phi(z, s, a)) = \{\}$$

For $\hat{\Phi}(z, s, a)$

For fixed s, a, the function $\hat{\Phi}(z, s, a)$ does not have poles and essential singularities.

10.06.04.0010.01

$$\mathcal{S}ing_z\left(\hat{\Phi}(z, s, a)\right) = \{\}$$

Branch points

With respect to a

For $\Phi(z, s, a)$

For fixed z, s, the function $\Phi(z, s, a)$ does not have branch points.

10.06.04.0011.01

$$\mathcal{BP}_a(\Phi(z, s, a)) = \{\}$$

For $\hat{\Phi}(z, s, a)$

For fixed z and noninteger s, the function $\hat{\Phi}(z, s, a)$ has infinitely many branch points: $a = -n /; n \in \mathbb{N}$ and $a = \tilde{\infty}$. All these are power-type branch points.

For fixed z and integer s, the function $\hat{\Phi}(z, s, a)$ does not have branch points.

10.06.04.0012.01

$$\mathcal{BP}_a\left(\hat{\Phi}(z, s, a)\right) = \{\{-n /; n \in \mathbb{N}\}, \tilde{\infty}\}$$

10.06.04.0013.01

$$\mathcal{R}_a\left(\hat{\Phi}(z, s, a), -n\right) = \log /; n \in \mathbb{N} \land s \notin \mathbb{Z} \land s \notin \mathbb{Q}$$

10.06.04.0014.01

$$\mathcal{R}_a\left(\hat{\Phi}(z, s, a), -n\right) = q /; n \in \mathbb{N} \bigwedge s = \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

10.06.04.0015.01

$$\mathcal{R}_a\left(\hat{\Phi}(z, s, a), \tilde{\infty}\right) = \log /; s \notin \mathbb{Z} \land s \notin \mathbb{Q}$$

10.06.04.0016.01

$$\mathcal{R}_a\left(\hat{\Phi}(z, s, a), \tilde{\infty}\right) = q /; s = \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

With respect to s

For $\Phi(z, s, a)$

For fixed z, a, the function $\Phi(z, s, a)$ does not have branch points.

10.06.04.0017.01

$$\mathcal{BP}_s(\Phi(z, s, a)) = \{\}$$

For $\hat{\Phi}(z, s, a)$

For fixed z, a, the function $\hat{\Phi}(z, s, a)$ does not have branch points.

10.06.04.0018.01

$$\mathcal{BP}_s\left(\hat{\Phi}(z, s, a)\right) == \{\}$$

With respect to z

For $\Phi(z, s, a)$

For fixed s, a, the function $\Phi(z, s, a)$ has two branch points: $z == 1$, $z == \tilde{\infty}$.

10.06.04.0019.01

$$\mathcal{BP}_z(\Phi(z, s, a)) == \{1, \tilde{\infty}\}$$

10.06.04.0020.01

$$\mathcal{R}_z(\Phi(z, s, a), 1) == \log\ /;\ s \notin \mathbb{Z} \bigwedge s \notin \mathbb{Q}$$

10.06.04.0021.01

$$\mathcal{R}_z(\Phi(z, s, a), 1) == q\ /;\ s == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

10.06.04.0022.01

$$\mathcal{R}_z(\Phi(z, s, a), \tilde{\infty}) == \log$$

For $\hat{\Phi}(z, s, a)$

For fixed s, a, the function $\Phi(z, s, a)$ has two branch points: $z == 1$, $z == \tilde{\infty}$.

10.06.04.0023.01

$$\mathcal{BP}_z\left(\hat{\Phi}(z, s, a)\right) == \{1, \tilde{\infty}\}$$

10.06.04.0024.01

$$\mathcal{R}_z\left(\hat{\Phi}(z, s, a), 1\right) == \log\ /;\ s \notin \mathbb{Z} \bigwedge s \notin \mathbb{Q}$$

10.06.04.0025.01

$$\mathcal{R}_z\left(\hat{\Phi}(z, s, a), 1\right) == q\ /;\ s == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

10.06.04.0026.01

$$\mathcal{R}_z\left(\hat{\Phi}(z, s, a), \tilde{\infty}\right) == \log$$

Branch cuts

With respect to *a*

For $\Phi(z, s, a)$

For fixed z and fixed s /; $\frac{s}{2} \notin \mathbb{Z}$, the function $\Phi(z, s, a)$ has infinitely many branch cuts parallel to the imaginary axis at $\mathrm{Re}(a) = -n$ /; $n \in \mathbb{N}$. In these cases the function $\Phi(z, s, a)$ is continuous from the left on the interval $(-i\,\infty - n, -n)$ /; $n \in \mathbb{N}$ and from the right on the interval $(-n, -n + i\,\infty)$ /; $n \in \mathbb{N}$.

10.06.04.0027.01

$$\mathcal{BC}_a(\Phi(z, s, a)) = \{\{\{(-i\,\infty - n, -n), 1\} /; n \in \mathbb{N}\}, \{\{(-n, -n + i\,\infty), -1\} /; n \in \mathbb{N}\}\}$$

10.06.04.0028.01

$$\lim_{\epsilon \to +0} \Phi(z, s, -n - i\,x - \epsilon) = \Phi(z, s, -n - i\,x) /; n \in \mathbb{N} \land x > 0$$

10.06.04.0029.01

$$\lim_{\epsilon \to +0} \Phi(z, s, -n - i\,x + \epsilon) = \Phi(z, s, -n - i\,x) + 2\,e^{\frac{i\pi s}{2}}\,i \sin\left(\frac{\pi s}{2}\right) z^n \left(-x^2\right)^{-\frac{s}{2}} /; n \in \mathbb{N} \land x > 0$$

10.06.04.0030.01

$$\lim_{\epsilon \to +0} \Phi(z, s, -n + i\,x + \epsilon) = \Phi(z, s, -n + i\,x) /; n \in \mathbb{N} \land x > 0$$

10.06.04.0031.01

$$\lim_{\epsilon \to +0} \Phi(z, s, -n + i\,x - \epsilon) = \Phi(z, s, -n + i\,x) + 2\,e^{\frac{i\pi s}{2}}\,i \sin\left(\frac{\pi s}{2}\right) z^n \left(-x^2\right)^{-\frac{s}{2}} /; n \in \mathbb{N} \land x > 0$$

For $\hat{\Phi}(z, s, a)$

For fixed z, s, the function $\hat{\Phi}(z, s, a)$ is a single-valued function on the a-plane cut along the interval $(-\infty, 0)$, where $\hat{\Phi}(z, s, a)$ is continuous from above. This interval includes an infinite set of branch cut lines of power type along $(-\infty, -n)$ /; $n \in \mathbb{N}$.

For integer s, the function $\hat{\Phi}(z, s, a)$ does not have branch cuts.

10.06.04.0032.01

$$\mathcal{BC}_a\left(\hat{\Phi}(z, s, a)\right) = \{\{(-\infty, 0), -i\}\}$$

10.06.04.0033.01

$$\lim_{\epsilon \to +0} \hat{\Phi}(z, s, x + i\,\epsilon) = \hat{\Phi}(z, s, x) /; x < 0$$

10.06.04.0034.01

$$\lim_{\epsilon \to +0} \hat{\Phi}(z, s, x - i\,\epsilon) = \hat{\Phi}(z, s, x) + \left(e^{2 i \pi s} - 1\right) \sum_{k=0}^{\lfloor -x \rfloor} \frac{z^k}{(k + x)^s} /; x < 0$$

For $\tilde{\Phi}(z, s, a)$

For fixed z, s, the function $\tilde{\Phi}(z, s, a)$ is a single-valued function on the a-plane cut along the interval $(-\infty, 0)$, where $\tilde{\Phi}(z, s, a)$ is continuous from above. This interval includes an infinite set of branch cut lines of power type along $(-\infty, -n)$ /; $n \in \mathbb{N}$.

For integer s, the function $\tilde{\Phi}(z, s, a)$ does not have branch cuts.

10.06.04.0035.01

$$\mathcal{BC}_a\left(\tilde{\Phi}(z, s, a)\right) = \{\{(-\infty, 0), -i\}\}$$

10.06.04.0036.01

$$\lim_{\epsilon \to +0} \tilde{\Phi}(z, s, x + i\,\epsilon) = \tilde{\Phi}(z, s, x) \;/;\; x < 0$$

10.06.04.0037.01

$$\lim_{\epsilon \to +0} \tilde{\Phi}(z, s, x - i\,\epsilon) = \tilde{\Phi}(z, s, x) + \left(e^{2 i \pi s} - 1\right) \sum_{k=0}^{\lfloor -x \rfloor} \frac{z^k}{(k + x)^s} \;/;\; x < 0$$

With respect to s

For $\Phi(z, s, a)$

For fixed z, a, the function $\Phi(z, s, a)$ does not have branch cuts.

10.06.04.0038.01

$$\mathcal{BC}_s(\Phi(z, s, a)) = \{\}$$

For $\hat{\Phi}(z, s, a)$

For fixed z, a, the function $\hat{\Phi}(z, s, a)$ does not have branch cuts.

10.06.04.0039.01

$$\mathcal{BC}_s\left(\hat{\Phi}(z, s, a)\right) = \{\}$$

For $\tilde{\Phi}(z, s, a)$

For fixed z, a, the function $\tilde{\Phi}(z, s, a)$ does not have branch cuts.

10.06.04.0040.01

$$\mathcal{BC}_s\left(\tilde{\Phi}(z, s, a)\right) = \{\}$$

With respect to z

For $\Phi(z, s, a)$

For fixed s, a, the function $\Phi(z, s, a)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.06.04.0041.01

$$\mathcal{BC}_z(\Phi(z, s, a)) = \{\{\{1, \infty\}, i\}\}$$

10.06.04.0042.01

$$\lim_{\epsilon \to +0} \Phi(x - i\,\epsilon, s, a) = \Phi(x, s, a) \;/;\; x > 1$$

10.06.04.0043.01

$$\lim_{\epsilon \to +0} \Phi(x + i\,\epsilon, s, a) = \Phi(x, s, a) + \frac{2\,i\,\pi\,x^{-a}}{\Gamma(s)} \log^{s-1}(x) \;/;\; x > 1$$

For $\hat{\Phi}(z, s, a)$

For fixed s, a, the function $\hat{\Phi}(z, s, a)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.06.04.0044.01

$$\mathcal{BC}_z\left(\hat{\Phi}(z, s, a)\right) = \{\{\{1, \infty\}, i\}\}$$

10.06.04.0045.01

$$\lim_{\epsilon \to +0} \hat{\Phi}(x - i\,\epsilon, s, a) = \hat{\Phi}(x, s, a) \;/;\; x > 1$$

10.06.04.0046.01

$$\lim_{\epsilon \to +0} \hat{\Phi}(x + i\,\epsilon, s, a) = \hat{\Phi}(x, s, a) + \frac{2\,i\,\pi\,x^{-a}}{\Gamma(s)} \log^{s-1}(x) \;/;\; x > 1$$

For $\tilde{\Phi}(z, s, a)$

For fixed s, a, the function $\tilde{\Phi}(z, s, a)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.06.04.0047.01

$$\mathcal{BC}_z\left(\tilde{\Phi}(z, s, a)\right) = \{\{\{1, \infty\}, i\}\}$$

10.06.04.0048.01

$$\lim_{\epsilon \to +0} \tilde{\Phi}(x - i\,\epsilon, s, a) = \tilde{\Phi}(x, s, a) \;/;\; x > 1$$

10.06.04.0049.01

$$\lim_{\epsilon \to +0} \tilde{\Phi}(x + i\,\epsilon, s, a) = \tilde{\Phi}(x, s, a) + \frac{2\,i\,\pi\,x^{-a}}{\Gamma(s)} \log^{s-1}(x) \;/;\; x > 1$$

Series representations

Generalized power series

Expansions at $z == z_0$

For $\Phi(z, s, a)$

10.06.06.0017.01

$$\Phi(z, s, a) \propto \Phi(z_0, s, a) + \left(z_0^{\max(\lfloor -\text{Re}(a) \rfloor, 0)} (\Phi(z_0, s-1, a+\max(\lfloor -\text{Re}(a) \rfloor, 0)+1) - a\,\Phi(z_0, s, a+\max(\lfloor -\text{Re}(a) \rfloor, 0)+1)) + \sum_{k=0}^{\lfloor -\text{Re}(a) \rfloor - 1} \frac{(k+1)\, z_0^k}{\left((a+k+1)^2\right)^{s/2}} \right) (z-z_0) + \frac{1}{2} \left(z_0^{\max(\lfloor -\text{Re}(a) \rfloor, 0)-1} (\Phi(z_0, s-2, a+\max(0, \lfloor -\text{Re}(a) \rfloor)+1) - (2a+1)\,\Phi(z_0, s-1, a+\max(0, \lfloor -\text{Re}(a) \rfloor)+1) + (a+1)\, a\, \Phi(z_0, s, a+\max(0, \lfloor -\text{Re}(a) \rfloor)+1)) + \sum_{k=0}^{\lfloor -\text{Re}(a) \rfloor - 2} \frac{(k+1)(k+2)\, z_0^k}{\left((a+k+2)^2\right)^{s/2}} \right) (z-z_0)^2 + \ldots \;/;\; (z \to z_0)$$

10.06.06.0018.01

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{\Phi^{(k,0,0)}(z_0, s, a)}{k!} (z-z_0)^k$$

10.06.06.0019.01

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z_0^{1-k+\max(\lfloor -\text{Re}(a) \rfloor, 0)}}{k!} \left(\sum_{j=1}^{k} \sum_{p=0}^{j} S_k^{(j)} \binom{j}{p} \Phi(z_0, s-p, a+\max(\lfloor -\text{Re}(a) \rfloor, 0)+1)\, (-a)^{j-p} + \sum_{j=0}^{\lfloor -\text{Re}(a) \rfloor - k} \frac{(j+1)_k\, z_0^j}{\left((a+j+k)^2\right)^{s/2}} \right) (z-z_0)^k$$

10.06.06.0020.01

$$\Phi(z, s, a) \propto \Phi(z_0, s, a)\,(1 + O(z-z_0))$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0021.01

$$\hat{\Phi}(z, s, a) \propto \hat{\Phi}(z_0, s, a) + \frac{\hat{\Phi}(z_0, s-1, a) - a\,\hat{\Phi}(z_0, s, a)}{z_0} (z-z_0) + \frac{1}{2 z_0^2} \left(\hat{\Phi}(z_0, s-2, a) - (2a+1)\,\hat{\Phi}(z_0, s-1, a) + a\,(a+1)\,\hat{\Phi}(z_0, s, a) \right) (z-z_0)^2 + \ldots \;/;\; (z \to z_0)$$

10.06.06.0022.01

$$\hat{\Phi}(z, s, a) = \sum_{k=0}^{\infty} \frac{\hat{\Phi}^{(k,0,0)}(z_0, s, a)}{k!} (z-z_0)^k$$

10.06.06.0023.01

$$\hat{\Phi}(z, s, a) = \sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!} \sum_{j=1}^{k} \sum_{p=0}^{j} S_k^{(j)} \binom{j}{p} \left(\Phi(z_0, s-p, a+\max(\lfloor -\text{Re}(a) \rfloor, 0)+1)\, z_0^{\max(\lfloor -\text{Re}(a)\rfloor, 0)+1} + \sum_{i=0}^{\lfloor -\text{Re}(a) \rfloor} \frac{z_0^i}{(a+i)^{s-p}} \right) (-a)^{j-p} (z-z_0)^k$$

10.06.06.0024.01

$$\hat{\Phi}(z, s, a) \propto \hat{\Phi}(z_0, s, a)\,(1 + O(z - z_0))$$

Expansions at $z == 0$

For $\Phi(z, s, a)$

10.06.06.0001.01

$$\Phi(z, s, a) \propto (a^2)^{-\frac{s}{2}} + ((a+1)^2)^{-\frac{s}{2}} z + ((a+2)^2)^{-\frac{s}{2}} z^2 + \ldots \;/;\; (z \to 0) \bigwedge -a \notin \mathbb{N}$$

10.06.06.0025.01

$$\Phi(z, s, a) \propto (a^2)^{-\frac{s}{2}} + ((a+1)^2)^{-\frac{s}{2}} z + ((a+2)^2)^{-\frac{s}{2}} z^2 + O(z^3) \;/;\; -a \notin \mathbb{N}$$

10.06.06.0002.01

$$\Phi(z, s, a) == \sum_{k=0}^{\infty} \frac{z^k}{((a+k)^2)^{s/2}} \;/;\; (|z| < 1 \vee |z| == 1 \wedge \text{Re}(s) > 1) \bigwedge -a \notin \mathbb{N}$$

10.06.06.0003.02

$$\Phi(z, s, a) \propto (a^2)^{-\frac{s}{2}} (1 + O(z))$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0004.01

$$\hat{\Phi}(z, s, a) \propto a^{-s} + (a+1)^{-s} z + (a+2)^{-s} z^2 + \ldots \;/;\; (z \to 0)$$

10.06.06.0026.01

$$\hat{\Phi}(z, s, a) \propto a^{-s} + (a+1)^{-s} z + (a+2)^{-s} z^2 + O(z^3)$$

10.06.06.0005.01

$$\hat{\Phi}(z, s, a) == \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \;/;\; |z| < 1 \vee |z| == 1 \wedge \text{Re}(s) > 1$$

10.06.06.0006.02

$$\hat{\Phi}(z, s, a) \propto a^{-s} (1 + O(z))$$

Expansions at $z == 1$

For $\hat{\Phi}(z, s, a)$

10.06.06.0027.01

$$\hat{\Phi}(z, s, a) \propto \hat{\zeta}(s, a) + (z-1)\,(\zeta(s-1, a+1) - a\,\zeta(s, a+1)) + \ldots \;/;\; (z \to 1) \wedge \text{Re}\, s > 1$$

10.06.06.0007.01

$$\hat{\Phi}(z, s, a) \propto \hat{\zeta}(s, a) + (z-1)\,(\zeta(s-1, a+1) - a\,\zeta(s, a+1)) + O((z-1)^2) \;/;\; (z \to 1) \wedge \text{Re}\, s > 1$$

10.06.06.0008.01

$$\hat{\Phi}(z, s, a) \propto \Gamma(1-s)\,(1-z)^{s-1}\,(1+O(z-1)) \;/;\; (z \to 1) \land \operatorname{Re} s < 1$$

10.06.06.0009.02

$$\hat{\Phi}(z, 1, a) \propto -\log(1-z)\,(1+O(z-1))$$

Expansions at $s == s_0$

For $\Phi(z, s, a)$

10.06.06.0028.01

$$\Phi(z, s, a) \propto \Phi(z, s_0, a) + \Phi^{(0,1,0)}(z, s_0, a)\,(s-s_0) + \frac{1}{2}\,\Phi^{(0,2,0)}(z, s_0, a)\,(s-s_0)^2 + \ldots \;/;\; (s \to s_0)$$

10.06.06.0029.01

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{\Phi^{(0,k,0)}(z, s_0, a)}{k!}\,(s-s_0)^k$$

10.06.06.0030.01

$$\Phi(z, s, a) \propto \Phi(z, s_0, a)\,(1+O(s-s_0))$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0031.01

$$\hat{\Phi}(z, s, a) \propto \hat{\Phi}(z, s_0, a) + \hat{\Phi}^{(0,1,0)}(z, s_0, a)\,(s-s_0) + \frac{1}{2}\,\hat{\Phi}^{(0,2,0)}(z, s_0, a)\,(s-s_0)^2 + \ldots \;/;\; (s \to s_0)$$

10.06.06.0032.01

$$\hat{\Phi}(z, s, a) = \sum_{k=0}^{\infty} \frac{\hat{\Phi}^{(0,k,0)}(z, s_0, a)}{k!}\,(s-s_0)^k$$

10.06.06.0033.01

$$\hat{\Phi}(z, s, a) \propto \hat{\Phi}(z, s_0, a)\,(1+O(s-s_0))$$

Expansions at $a == a_0$

For $\Phi(z, s, a)$

10.06.06.0034.01

$$\Phi(z, s, a) \propto \Phi(z, s, a_0) + s\,(\zeta(s+1, a_0) - 2\,\zeta(s+1, \max(\lfloor -\operatorname{Re}(a_0)\rfloor + 1, 0) + a_0))\,(a-a_0) + \frac{s\,(s+1)}{2} \left(z^{\max(\lfloor -\operatorname{Re}(a_0)\rfloor + 1, 0)}\,\Phi(z, s+2, \max(\lfloor -\operatorname{Re}(a_0)\rfloor + 1, 0) + a_0) + \sum_{k=0}^{\lfloor -\operatorname{Re}(a_0)\rfloor} \frac{z^k}{(k+a_0)^2\left((k+a_0)^2\right)^{s/2}} \right) (a-a_0)^2 + \ldots \;/;\; (a \to a_0)$$

10.06.06.0035.01

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{\Phi^{(0,0,k)}(z, s, a_0)}{k!}\,(a-a_0)^k$$

10.06.06.0036.01

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{(1-k-s)_k}{k!}\left(z^{\max(\lfloor-\operatorname{Re}(a_0)\rfloor+1,0)}\,\Phi(z,k+s,\max(\lfloor-\operatorname{Re}(a_0)\rfloor+1,0)+a_0)+\sum_{j=0}^{\lfloor-\operatorname{Re}(a_0)\rfloor}\frac{z^j}{(j+a_0)^k\left((j+a_0)^2\right)^{s/2}}\right)(a-a_0)^k$$

10.06.06.0037.01

$$\Phi(z,s,a)\propto\Phi(z,s,a_0)\,(1+O(a-a_0))$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0038.01

$$\hat{\Phi}(z,s,a)\propto\hat{\Phi}(z,s,a_0)-s\,\hat{\Phi}(z,s+1,a_0)\,(a-a_0)+\frac{1}{2}\,s\,(s+1)\,\hat{\Phi}(z,s+2,a_0)\,(a-a_0)^2+\ldots\;/;\;(a\to a_0)$$

10.06.06.0039.01

$$\hat{\Phi}(z,s,a)=\sum_{k=0}^{\infty}\frac{\hat{\Phi}^{(0,0,k)}(z,s,a_0)}{k!}\,(a-a_0)^k$$

10.06.06.0040.01

$$\hat{\Phi}(z,s,a)\propto\hat{\Phi}(z,s,a_0)\,(1+O(a-a_0))$$

Expansions at $a == -n$

For $\hat{\Phi}(z, s, a)$

10.06.06.0010.01

$$\hat{\Phi}(z,s,a)==\frac{z^n}{(a+n)^s}+\left(\sum_{k=0}^{n-1}\frac{z^k}{(k-n+(a+n))^s}+z^n\operatorname{Li}_s(z)\right)+z^n\sum_{j=1}^{\infty}\frac{(1-j-s)_j\operatorname{Li}_{j+s}(z)}{j!}\,(a+n)^j\;/;\;n\in\mathbb{N}$$

10.06.06.0011.01

$$\hat{\Phi}(z,s,a)\propto\frac{z^n}{(a+n)^s}+\left(\sum_{k=0}^{n-1}\frac{z^k}{(k-n+(a+n))^s}+z^n\operatorname{Li}_s(z)\right)(1+O(a+n))\;/;\;(a\to-n)\land n\in\mathbb{N}$$

Expansions on branch cuts

For $\Phi(z, s, a)$

10.06.06.0041.01

$$\Phi(z,s,a)=i^{-s}\,(-i\,(n+a_0))^{-s}\,e^{-i\pi s\left\lfloor-\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i\,(n+a_0))\right)\right\rfloor}\,z^n\sum_{k=0}^{\infty}\frac{(-n-a_0)^{-k}\,(s)_k}{k!}\,(a-a_0)^k+$$
$$\sum_{k=0}^{n-1}\frac{z^k}{(-a_0-k)^s}\sum_{j=0}^{\infty}\frac{(k+a_0)^{-j}\,(s)_j}{j!}\,(a-a_0)^j+\Phi(z,s,a+n+1)\,z^{n+1}\;/;\;(a\to a_0)\bigwedge\operatorname{Re}(a_0)=-n\bigwedge n\in\mathbb{N}$$

10.06.06.0042.01

$$\Phi(z,\,s,\,a) = i^{-s}\,(-i\,(n+a_0))^{-s}\,e^{-i\,\pi\,s\left\lfloor -\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i\,(n+a_0))\right)\right\rfloor}\,z^n\sum_{k=0}^{\infty}\frac{(-n-a_0)^{-k}\,(s)_k}{k!}\,(a-a_0)^k + \sum_{k=0}^{n-1}\frac{z^k}{(-a-k)^s} + \Phi(z,\,s,\,a+n+1)\,z^{n+1}\;/;\;(a\to a_0)\bigwedge \operatorname{Re}(a_0) = -n\bigwedge n\in\mathbb{N}$$

10.06.06.0043.01

$$\Phi(z,\,s,\,a) \propto i^{-s}\,(-i\,(n+a_0))^{-s}\,e^{-i\,\pi\,s\left\lfloor -\frac{1}{\pi}\left(\arg\left(\frac{a+n}{n+a_0}\right)+\arg(-i\,(n+a_0))\right)\right\rfloor}\,z^n(1+O(a-a_0)) + \left(\Phi(z,\,s,\,n+a_0+1)\,z^{n+1}+\sum_{k=0}^{n-1}\frac{z^k}{(-k-a_0)^s}\right)(1+O(a-a_0))\;/;\;\operatorname{Re}(a_0) = -n\bigwedge n\in\mathbb{N}$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0044.01

$$\hat{\Phi}(z,\,s,\,a) = z^{n+1}\,\hat{\Phi}(z,\,s,\,a+n+1)\;+z^n(n+a_0)^{-s}\sum_{k=0}^{\infty}\frac{(n+a_0)^{-k}\,(s)_k}{k!}\,(a-a_0)^k + \sum_{k=0}^{n-1}z^k\,e^{-2\,i\,\pi\,s\left\lfloor -\frac{1}{2\pi}\left(\arg(-k-a_0)+\arg\left(\frac{a+k}{k+a_0}\right)\right)\right\rfloor}\,(k+a_0)^{-s}\sum_{j=0}^{\infty}\frac{(k+a_0)^{-j}\,(s)_j}{j!}\,(a-a_0)^j\;/;\;(a\to a_0)\bigwedge -n< a_0 <1-n\bigwedge n\in\mathbb{N}^+$$

10.06.06.0045.01

$$\hat{\Phi}(z,\,s,\,a) \propto \left(z^n(n+a_0)^{-s}\;+z^{n+1}\,\hat{\Phi}(z,\,s,\,a_0+n+1)\right)(1+O(a-a_0)) + \sum_{k=0}^{n-1}z^k\,e^{-2\,i\,\pi\,s\left\lfloor -\frac{1}{2\pi}\left(\arg(-k-a_0)+\arg\left(\frac{a+k}{k+a_0}\right)\right)\right\rfloor}\,(k+a_0)^{-s}\,(1+O(a-a_0))\;/;\;-n< a_0 <1-n\bigwedge n\in\mathbb{N}^+$$

Residue representations

Residue representations

For $\hat{\Phi}(z, s, a)$

10.06.06.0012.01

$$\hat{\Phi}(z,\,s,\,a) = \sum_{j=0}^{\infty}\operatorname{res}_t\left(\left(\Gamma(1-t)\left(\frac{\Gamma(a-t)}{\Gamma(a-t+1)}\right)^s(-z)^{-t}\right)\Gamma(t)\right)(-j)\;/;\;(|z|<1\vee(|z|=1\wedge\operatorname{Re}(s)>1))\bigwedge -a\notin\mathbb{N}$$

10.06.06.0016.01

$$\hat{\Phi}(z,\,s,\,a) = \sum_{j=0}^{\infty}\operatorname{res}_t\left(\pi\,(a-t)^{-s}\,(-z)^{-t}\csc(\pi\,t)\right)(-j)\;/;\;(|z|<1\vee(|z|=1\wedge\operatorname{Re}(s)>1))\bigwedge -a\notin\mathbb{N}$$

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Other series representations

Other series representations

For $\Phi(z, s, a)$

10.06.06.0013.01

$$\Phi(z, s, a) = z^{-a}\,\Gamma(1-s)\sum_{k=-\infty}^{\infty}(2\pi i k - \log(z))^{s-1}\,e^{2\pi k i a} \;/;\; \operatorname{Re} s < 0 \land 0 < a \leq 1$$

10.06.06.0014.01

$$\Phi\left(e^{2\pi i t}, s, \frac{p}{q}\right) = (2 q \pi)^{s-1}\,\Gamma(1-s)\left(\sum_{k=1}^{q}\zeta\left(1-s, \frac{k-t}{q}\right)\exp\left(\frac{2(k-t)\pi i p}{q} - \frac{1-s}{2}\pi i\right) + \sum_{k=1}^{q}\zeta\left(1-s, \frac{k+t-1}{q}\right)\exp\left(\frac{1-s}{2}\pi i - \frac{2(k+t-1)\pi i p}{q}\right)\right) \;/;\; p \in \mathbb{N} \land q \in \mathbb{N}^{+} \land p \leq q$$

For $\hat{\Phi}(z, s, a)$

10.06.06.0015.01

$$\hat{\Phi}\left(\exp\left(\frac{2\pi i p}{q}\right), s, a\right) = q^{-s}\sum_{k=1}^{q}\hat{\zeta}\left(s, \frac{a+k-1}{q}\right)\exp\left(\frac{2(k-1)\pi i p}{q}\right) \;/;\; p \in \mathbb{N} \land q \in \mathbb{N}^{+} \land p \leq q$$

Integral representations

On the real axis

Of the direct function

For $\Phi(z, s, a)$

10.06.07.0001.01

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}\,e^{-a t}}{1 - z\,e^{-t}}\,dt \;/;\; \operatorname{Re}(a) > 0 \land \operatorname{Re}(s) > 0 \land |z| \leq 1 \land z \neq 1 \lor \operatorname{Re}(s) > 1 \land z = 1$$

10.06.07.0002.01

$$\Phi(z, s, a) = \int_0^{\infty}(a+t)^{-s}\,z^t\,dt - 2\int_0^{\infty}\frac{\sin\left(t\log(z) - s\tan^{-1}\left(\frac{t}{a}\right)\right)}{\left(a^2+t^2\right)^{s/2}\left(e^{2\pi t}-1\right)}\,dt + \frac{1}{2 a^s} \;/;\; \operatorname{Re}(a) > 0$$

10.06.07.0006.01

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)}\int_0^{1}\frac{(-\log(t))^{s-1}\,t^{a-1}}{1 - z t}\,dt \;/;\; \operatorname{Re}(a) > 0 \land \operatorname{Re}(s) > 0 \land |z| \leq 1 \land z \neq 1 \lor \operatorname{Re}(s) > 1 \land z = 1$$

Contour integral representations

Contour integral representations

For $\hat{\Phi}(z, s, a)$

10.06.07.0003.01

$$\hat{\Phi}(z, s, a) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(t)\,\Gamma(1-t)\,\Gamma(a-t)^s\,(-z)^{-t}}{\Gamma(1+a-t)^s}\,dt \,/;\, s \in \mathbb{N}^+$$

10.06.07.0004.02

$$\hat{\Phi}(z, s, a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{i\infty+\gamma} \Gamma(t)\,\Gamma(1-t)\left(\frac{\Gamma(a-t)}{\Gamma(a-t+1)}\right)^s (-z)^{-t}\,dt \,/;\, 0 < \gamma < 1 \bigwedge \mathrm{Re}(a) > \gamma \bigwedge z \neq 0 \bigwedge (\arg(-z) < \pi \bigvee \mathrm{Re}(s) > 1)$$

10.06.07.0005.01

$$\hat{\Phi}(z, s, a-n) = \frac{z^n}{2 i}\int_{\gamma-i\infty}^{i\infty+\gamma} (a-t)^{-s}\,(-z)^{-t}\,\csc(\pi t)\,dt \,/;\, n \in \mathbb{Z} \bigwedge n < \gamma < n+1 \bigwedge \mathrm{Re}(a) > \gamma \bigwedge z \neq 0 \bigwedge (\arg(-z) < \pi \bigvee \mathrm{Re}(s) > 1)$$

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Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

For $\Phi(z, s, a)$

10.06.16.0001.01

$$\Phi(z, s, a+1) = \frac{1}{z}\left(\Phi(z, s, a) - \frac{1}{(a^2)^{s/2}}\right)$$

10.06.16.0002.01

$$\Phi(z, s, a-1) = z\,\Phi(z, s, a) + \frac{1}{\left((a-1)^2\right)^{s/2}}$$

10.06.16.0003.01

$$\Phi(z, s, a+n) = z^{-n}\left(\Phi(z, s, a) - \sum_{k=0}^{n-1} \frac{z^k}{\left((a+k)^2\right)^{s/2}}\right) /;\, n \in \mathbb{N}$$

10.06.16.0004.01

$$\Phi(z, s, a-n) = z^n\,\Phi(z, s, a) + \sum_{k=0}^{n-1} \frac{z^k}{\left((a+k-n)^2\right)^{s/2}} \,/;\, n \in \mathbb{N}$$

For $\hat{\Phi}(z, s, a)$

10.06.16.0005.01

$$\hat{\Phi}(z, s, a+1) = \frac{1}{z}\left(\hat{\Phi}(z, s, a) - \frac{1}{a^s}\right)$$

10.06.16.0006.01

$$\hat{\Phi}(z, s, a-1) = z\,\hat{\Phi}(z, s, a) + \frac{1}{(a-1)^s}$$

10.06.16.0007.01

$$\hat{\Phi}(z, s, a+n) = z^{-n}\left(\hat{\Phi}(z, s, a) - \sum_{k=0}^{n-1}\frac{z^k}{(a+k)^s}\right) /; n \in \mathbb{N}$$

10.06.16.0008.01

$$\hat{\Phi}(z, s, a-n) = z^n\,\hat{\Phi}(z, s, a) + \sum_{k=0}^{n-1}\frac{z^k}{(a+k-n)^s} /; n \in \mathbb{N}$$

Multiple arguments

Argument involving numeric multiples of variable

For $\hat{\Phi}(z, s, a)$

10.06.16.0009.01

$$\hat{\Phi}(z, s, 2\,a) = 2^{-s}\left(\hat{\Phi}\left(z^2, s, a\right) + z\,\hat{\Phi}\left(z^2, s, a+\frac{1}{2}\right)\right)$$

10.06.16.0010.01

$$\hat{\Phi}(z, s, 3\,a) = 3^{-s}\left(\hat{\Phi}\left(z^3, s, a\right) + z\,\hat{\Phi}\left(z^3, s, a+\frac{1}{3}\right) + z^2\,\hat{\Phi}\left(z^3, s, a+\frac{2}{3}\right)\right)$$

Argument involving symbolic multiples of variable

For $\hat{\Phi}(z, s, a)$

10.06.16.0011.01

$$\hat{\Phi}(z, s, m\,a) = m^{-s}\sum_{k=0}^{m-1}\hat{\Phi}\left(z^m, s, a+\frac{k}{m}\right)z^k /; m \in \mathbb{Z} \land m > 1$$

Identities

Recurrence identities

Consecutive neighbors

For $\Phi(z, s, a)$

10.06.17.0001.01

$$\Phi(z, s, a) = z\,\Phi(z, s, a+1) + \frac{1}{(a^2)^{s/2}}$$

10.06.17.0002.01

$$\Phi(z, s, a) = \frac{1}{z}\left(\Phi(z, s, a-1) - \frac{1}{((a-1)^2)^{s/2}}\right)$$

For $\hat{\Phi}(z, s, a)$

10.06.17.0003.01

$$\hat{\Phi}(z, s, a) = z\,\hat{\Phi}(z, s, a+1) + \frac{1}{a^s}$$

10.06.17.0004.01

$$\hat{\Phi}(z, s, a) = \frac{1}{z}\left(\hat{\Phi}(z, s, a-1) - \frac{1}{(a-1)^s}\right)$$

Distant neighbors

For $\Phi(z, s, a)$

10.06.17.0005.01

$$\Phi(z, s, a) = z^n\,\Phi(z, s, a+n) + \sum_{k=0}^{n-1} \frac{z^k}{((a+k)^2)^{s/2}} \;/;\; n \in \mathbb{N}$$

10.06.17.0006.01

$$\Phi(z, s, a) = z^{-n}\left(\Phi(z, s, a-n) - \sum_{k=0}^{n-1} \frac{z^k}{((a+k-n)^2)^{s/2}}\right) /;\; n \in \mathbb{N}$$

10.06.17.0012.01

$$\Phi(z, s, a) = z^{-a+b-\min(0,\operatorname{Re}(b-a))}\,\Phi(z, s, b-\min(0, \operatorname{Re}(b-a))) + \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{z^k}{((a+k)^2)^{s/2}} \;/;\; \operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0$$

For $\hat{\Phi}(z, s, a)$

10.06.17.0007.01

$$\hat{\Phi}(z, s, a) == z^n \hat{\Phi}(z, s, a+n) + \sum_{k=0}^{n-1} \frac{z^k}{(a+k)^s} /; n \in \mathbb{N}$$

10.06.17.0008.01

$$\hat{\Phi}(z, s, a) == z^{-n} \left(\hat{\Phi}(z, s, a-n) - \sum_{k=0}^{n-1} \frac{z^k}{(a+k-n)^s} \right) /; n \in \mathbb{N}$$

10.06.17.0013.01

$$\hat{\Phi}(z, s, a) = z^{-a+b-\min(0,\mathrm{Re}(b-a))} \hat{\Phi}(z, s, b-\min(0, \mathrm{Re}(b-a))) + \sum_{k=0}^{\mathrm{Re}(b-a)-1} \frac{z^k}{\left((a+k)^2\right)^{s/2}} /; \mathrm{Re}(b-a) \in \mathbb{Z} \bigwedge \mathrm{Im}(b-a) = 0$$

Functional identities

Major general cases

For $\Phi(z, s, a)$

10.06.17.0009.01

$$\Phi\left(e^{2 i \pi x}, 1-s, a\right) == (2 \pi)^{-s} \Gamma(s) \left(e^{i \pi \left(\frac{s}{2} - 2 a x\right)} \Phi\left(e^{-2 i a \pi}, s, x\right) + e^{i \pi \left(2 a (1-x) - \frac{s}{2}\right)} \Phi\left(e^{2 i a \pi}, s, 1-x\right) \right) /; 0 < x < 1$$

For $\hat{\Phi}(z, s, a)$

10.06.17.0010.01

$$\hat{\Phi}(z, s, a) == 2^{-s} \left(\hat{\Phi}\left(z^2, s, \frac{a}{2}\right) + z \hat{\Phi}\left(z^2, s, \frac{a+1}{2}\right) \right)$$

10.06.17.0011.01

$$\hat{\Phi}(z, s, a) == q^{-s} \sum_{k=1}^{q} \hat{\Phi}\left(z^q, s, \frac{a+k-1}{q}\right) z^{k-1} /; q \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to *z*

For $\Phi(z, s, a)$

10.06.20.0015.01

$$\frac{\partial \Phi(z, s, a)}{\partial z} = z^{\max(\lfloor -\mathrm{Re}(a) \rfloor, 0)} (\Phi(z, s-1, a+\max(\lfloor -\mathrm{Re}(a) \rfloor, 0)+1) - a\,\Phi(z, s, a+\max(\lfloor -\mathrm{Re}(a) \rfloor, 0)+1)) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor -1} \frac{(k+1)\, z^k}{\left((a+k+1)^2\right)^{s/2}}$$

10.06.20.0016.01

$$\frac{\partial \Phi(z, s, a)}{\partial z} = \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor -1} \frac{(k+1)\, z^k}{\left((a+k+1)^2\right)^{s/2}} - \sum_{k=0}^{\max(\lfloor -\mathrm{Re}(a) \rfloor, 0)-1} \frac{(k+1)\, z^k}{(a+k+1)^s} + \Gamma(a+1)^s\, {}_{s+1}\tilde{F}_s(a_1, a_2, \ldots, a_s, 2; a_1+1, a_2+1, \ldots, a_s+1; z) \,/;\, a_1 = a_2 = \ldots = a_s = a+1 \land s \in \mathbb{N}^+$$

10.06.20.0017.01

$$\frac{\partial^2 \Phi(z, s, a)}{\partial z^2} = z^{\max(0, \lfloor -\mathrm{Re}(a) \rfloor)-1} (\Phi(z, s-2, a+\max(0, \lfloor -\mathrm{Re}(a) \rfloor)+1) - (2a+1)\,\Phi(z, s-1, a+\max(0, \lfloor -\mathrm{Re}(a) \rfloor)+1) + (a+1)\, a\, \Phi(z, s, a+\max(0, \lfloor -\mathrm{Re}(a) \rfloor)+1)) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor -2} \frac{(k+1)(k+2)\, z^k}{\left((a+k+2)^2\right)^{s/2}}$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0001.01

$$\frac{\partial \hat{\Phi}(z, s, a)}{\partial z} = \frac{\hat{\Phi}(z, s-1, a) - a\,\hat{\Phi}(z, s, a)}{z}$$

10.06.20.0002.01

$$\frac{\partial^2 \hat{\Phi}(z, s, a)}{\partial z^2} = \frac{1}{z^2}\left(\hat{\Phi}(z, s-2, a) - (2a+1)\,\hat{\Phi}(z, s-1, a) + a\,(a+1)\,\hat{\Phi}(z, s, a)\right)$$

With respect to *s*

For $\Phi(z, s, a)$

10.06.20.0018.01

$$\frac{\partial \Phi(z, s, a)}{\partial s} = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{\log\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} \,/;\, -a \notin \mathbb{N} \land |z| < 1 \land \mathrm{Re}(s) > 1$$

10.06.20.0019.01

$$\frac{\partial^2 \Phi(z, s, a)}{\partial s^2} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{\log^2\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} \text{ /; } -a \notin \mathbb{N} \bigwedge |z| < 1 \bigwedge \operatorname{Re}(s) > 1$$

10.06.20.0020.01

$$\frac{\partial \Phi(z, s, a+1)}{\partial s} = \frac{1}{z} \frac{\partial \Phi(z, s, a)}{\partial s} + \frac{\log\left(a^2\right)}{2 z \left(a^2\right)^{s/2}}$$

10.06.20.0021.01

$$\frac{\partial \Phi(z, s, a+m)}{\partial s} = z^{-m} \frac{\partial \Phi(z, s, a)}{\partial s} + \frac{z^{-m}}{2^n} \sum_{k=0}^{m-1} \frac{\log\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} \text{ /; } m \in \mathbb{N}^+$$

10.06.20.0022.01

$$\frac{\partial \Phi(z, s, m)}{\partial s} = \frac{\partial \operatorname{Li}_s(z)}{\partial s} z^{-m} + \frac{z^{1-m}}{2^n} \sum_{k=0}^{m-2} \frac{\log\left((k+1)^2\right) z^k}{\left((k+1)^2\right)^{s/2}} \text{ /; } m \in \mathbb{N}$$

10.06.20.0023.01

$$\frac{\partial \Phi(z, s, a)}{\partial s} = z^{-a+b-\min(0,\operatorname{Re}(b-a))} \frac{\partial \Phi(z, s, b-\min(0, \operatorname{Re}(b-a)))}{\partial s} - \frac{1}{2} \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{\log\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} \text{ /; } \operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0$$

10.06.20.0024.01

$$\frac{\partial \Phi(z, s, a)}{\partial s} = z^{-\lfloor \operatorname{Re}(a) \rfloor} \frac{\partial \Phi(z, s, a - \lfloor \operatorname{Re}(a) \rfloor)}{\partial s} + \frac{\operatorname{sgn}(\operatorname{Re}(a))}{2} z^{-\theta(\operatorname{Re}(a))} \sum_{k=0}^{|\lfloor \operatorname{Re}(a) \rfloor|-1} \frac{\log\left(\left(k + \frac{1}{2}(1-2a)\operatorname{sgn}(\operatorname{Re}(a)) + \frac{1}{2}\right)^2\right) z^{-\operatorname{sgn}(\operatorname{Re}(a))\, k}}{\left(\left(k + \frac{1}{2}(1-2a)\operatorname{sgn}(\operatorname{Re}(a)) + \frac{1}{2}\right)^2\right)^{s/2}} \text{ /; }$$

$$\neg (a \in \mathbb{Z} \land a > 0)$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0003.01

$$\frac{\partial \hat{\Phi}(z, s, a)}{\partial s} == -\sum_{k=0}^{\infty} \frac{\log(a+k) z^k}{(a+k)^s} \text{ /; } |z| < 1 \land \operatorname{Re}(s) > 1$$

10.06.20.0004.01

$$\frac{\partial^2 \hat{\Phi}(z, s, a)}{\partial s^2} == \sum_{k=0}^{\infty} \frac{\log^2(a+k) z^k}{(a+k)^s} \text{ /; } |z| < 1 \land \operatorname{Re}(s) > 1$$

10.06.20.0025.01

$$\frac{\partial \hat{\Phi}(z, s, a)}{\partial s} = z^{-a+b-\min(0,\operatorname{Re}(b-a))} \frac{\partial \hat{\Phi}(z, s, b-\min(0, \operatorname{Re}(b-a)))}{\partial s} - \frac{1}{2} \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{\log\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} \text{ /; } \operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0$$

With respect to *a*

For $\Phi(z, s, a)$

10.06.20.0026.01

$$\Phi^{(0,0,1)}(z, s, a) = -s \sum_{k=0}^{\infty} \frac{(a+k)\, z^k}{\left((a+k)^2\right)^{\frac{s}{2}+1}} \;/;\; -a \notin \mathbb{N}$$

10.06.20.0027.01

$$\Phi^{(0,0,1)}(z, s, a) = -s \left(z^{\max(\lfloor -\mathrm{Re}(a) \rfloor + 1, 0)}\, \Phi(z, s+1, a + \max(\lfloor -\mathrm{Re}(a) \rfloor + 1, 0)) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{z^k}{(a+k)\left((a+k)^2\right)^{s/2}} \right) /;\; -a \notin \mathbb{N}$$

10.06.20.0028.01

$$\Phi^{(0,0,2)}(z, s, a) = s\,(s+1) \sum_{k=0}^{\infty} \frac{z^k}{\left((a+k)^2\right)^{\frac{s}{2}+1}} \;/;\; -a \notin \mathbb{N}$$

10.06.20.0029.01

$$\Phi^{(0,0,2)}(z, s, a) = s\,(s+1) \left(z^{\max(\lfloor -\mathrm{Re}(a) \rfloor + 1, 0)}\, \Phi(z, s+2, a + \max(\lfloor -\mathrm{Re}(a) \rfloor + 1, 0)) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor} \frac{z^k}{(a+k)^2 \left((a+k)^2\right)^{s/2}} \right) /;\; -a \notin \mathbb{N}$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0005.01

$$\frac{\partial \hat{\Phi}(z, s, a)}{\partial a} == -s\, \hat{\Phi}(z, s+1, a)$$

10.06.20.0006.01

$$\frac{\partial^2 \hat{\Phi}(z, s, a)}{\partial a^2} == s\,(s+1)\, \hat{\Phi}(z, s+2, a)$$

Symbolic differentiation

With respect to *z*

For $\Phi(z, s, a)$

10.06.20.0030.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial z^n} = \sum_{k=0}^{\infty} \frac{(k+1)_n\, z^k}{\left((a+k+n)^2\right)^{s/2}} \;/;\; |z| < 1 \land n \in \mathbb{N}$$

10.06.20.0031.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial z^n} == z^{\max(-n + \lfloor -\mathrm{Re}(a) \rfloor + 1, 0)} \sum_{k=0}^{\infty} \frac{(k + \max(-n + \lfloor -\mathrm{Re}(a) \rfloor + 1, 0) + 1)_n\, z^k}{(a + k + n + \max(-n + \lfloor -\mathrm{Re}(a) \rfloor + 1, 0))^s} + \sum_{k=0}^{\lfloor -\mathrm{Re}(a) \rfloor - n} \frac{(k+1)_n\, z^k}{\left((a+k+n)^2\right)^{s/2}} \;/;\; |z| < 1 \land n \in \mathbb{N}$$

10.06.20.0032.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial z^n} = \sum_{k=0}^{\lfloor -\text{Re}(a)\rfloor - n} \frac{(k+1)_n z^k}{\left((a+k+n)^2\right)^{s/2}} - \sum_{k=0}^{\max(-n+\lfloor -\text{Re}(a)\rfloor+1,0)-1} \frac{(k+1)_n z^k}{(a+k+n)^s} + n!\,\Gamma(a+n)^s\, {}_{s+1}\tilde{F}_s(a_1, a_2, \ldots, a_s, n+1; a_1+1, a_2+1, \ldots, a_s+1; z) /; a_1 = a_2 = \ldots = a_s = a+n \bigwedge n \in \mathbb{N} \bigwedge s \in \mathbb{N}^+$$

10.06.20.0033.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial z^n} = z^{1-n+\max(\lfloor -\text{Re}(a)\rfloor,0)} \sum_{j=1}^{n} \sum_{p=0}^{j} S_n^{(j)} \binom{j}{p} \Phi(z, s-p, a+\max(\lfloor -\text{Re}(a)\rfloor, 0)+1)\,(-a)^{j-p} + \sum_{k=0}^{\lfloor -\text{Re}(a)\rfloor - n} \frac{(k+1)_n z^k}{\left((a+k+n)^2\right)^{s/2}} /; n \in \mathbb{N}$$

10.06.20.0034.01

$$\frac{\partial^n \Phi(z, s, n)}{\partial z^n} = \sum_{j=0}^{n} \sum_{p=0}^{j} (-1)^{j+n} S_n^{(j)} \binom{j}{p} (1-a-n)^p \left(\Phi(z, s-j+p, a+n+\max(\lfloor -\text{Re}(a+n)\rfloor, 0)+1)\, z^{\max(\lfloor -\text{Re}(a+n)\rfloor,0)+1} + \sum_{k=0}^{\max(\lfloor -\text{Re}(a+n)\rfloor,0)} \frac{z^k}{(a+k+n)^{s-j+p}}\right) + \sum_{k=0}^{\lfloor -\text{Re}(a)\rfloor - n} \frac{(k+1)_n z^k}{\left((a+k+n)^2\right)^{s/2}} - \sum_{k=0}^{\max(\lfloor -\text{Re}(a)\rfloor - n, -1)} \frac{(k+1)_n z^k}{(a+k+n)^s} /; n \in \mathbb{N}$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0007.02

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial z^n} = \sum_{k=0}^{\infty} \frac{(k+1)_n z^k}{(a+k+n)^s} /; |z| < 1 \land n \in \mathbb{N}$$

10.06.20.0008.02

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial z^n} = n!\,\Gamma(a+n)^s\, {}_{s+1}\tilde{F}_s(a_1, a_2, \ldots, a_s, n+1; a_1+1, a_2+1, \ldots, a_s+1; z) /; a_1 = a_2 = \ldots = a_s = a+n \bigwedge n \in \mathbb{N} \bigwedge s \in \mathbb{N}^+$$

10.06.20.0009.02

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial z^n} = z^{-n} \sum_{j=1}^{n} \sum_{p=0}^{j} S_n^{(j)} \binom{j}{p} \left(z^{\max(\lfloor -\text{Re}(a)\rfloor,0)+1}\, \Phi(z, s-p, a+\max(\lfloor -\text{Re}(a)\rfloor, 0)+1) + \sum_{k=0}^{\lfloor -\text{Re}(a)\rfloor} \frac{z^k}{(a+k)^{s-p}}\right) (-a)^{j-p} /; n \in \mathbb{N}$$

10.06.20.0035.01

$$\frac{\partial^n \hat{\Phi}(z, s, n)}{\partial z^n} = \sum_{j=0}^{n} \sum_{p=0}^{j} (-1)^{j+n} S_n^{(j)} \binom{j}{p} (1-a-n)^p \left(\Phi(z, s-j+p, a+n+\max(\lfloor -\text{Re}(a+n)\rfloor, 0)+1)\, z^{\max(\lfloor -\text{Re}(a+n)\rfloor,0)+1} + \sum_{k=0}^{\max(\lfloor -\text{Re}(a+n)\rfloor,0)} \frac{z^k}{(a+k+n)^{s-j+p}}\right) /; n \in \mathbb{N}$$

With respect to s

For $\Phi(z, s, a)$

10.06.20.0036.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial s^n} == \frac{(-1)^n}{2^n} \sum_{k=0}^{\infty} \frac{\log^n\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} /; -a \notin \mathbb{N} \bigwedge |z| < 1 \bigwedge \operatorname{Re}(s) > 1 \bigwedge n \in \mathbb{N}$$

10.06.20.0037.01

$$\frac{\partial^n \Phi(z, s, a+1)}{\partial s^n} = \frac{1}{z} \frac{\partial^n \Phi(z, s, a)}{\partial s^n} + \frac{(-1)^{n-1} \log^n\left(a^2\right)}{2^n z \left(a^2\right)^{s/2}} /; n \in \mathbb{N}$$

10.06.20.0038.01

$$\frac{\partial^n \Phi(z, s, a+m)}{\partial s^n} = \frac{\partial^n \Phi(z, s, a)}{\partial s^n} z^{-m} + \frac{(-1)^{n-1} z^{-m}}{2^n} \sum_{k=0}^{m-1} \frac{\log^n\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

10.06.20.0039.01

$$\frac{\partial^n \Phi(z, s, m)}{\partial s^n} = \frac{\partial^n \operatorname{Li}_s(z)}{\partial s^n} z^{-m} + \frac{(-1)^{n-1} z^{1-m}}{2^n} \sum_{k=0}^{m-2} \frac{\log^n\left((k+1)^2\right) z^k}{\left((k+1)^2\right)^{s/2}} /; m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

10.06.20.0040.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial s^n} = z^{-a+b-\min(0,\operatorname{Re}(b-a))} \frac{\partial^n \Phi(z, s, b-\min(0, \operatorname{Re}(b-a)))}{\partial s^n} + \frac{(-1)^n}{2^n} \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{\log^n\left((a+k)^2\right) z^k}{\left((a+k)^2\right)^{s/2}} /;$$

$$\operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0 \bigwedge n \in \mathbb{N}^+$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0010.02

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial s^n} == (-1)^n \sum_{k=0}^{\infty} \frac{\log^n(a+k) z^k}{(a+k)^s} /; |z| < 1 \wedge \operatorname{Re}(s) > 1 \wedge n \in \mathbb{N}$$

10.06.20.0041.01

$$\frac{\partial^n \hat{\Phi}(z, s, a+1)}{\partial s^n} = \frac{1}{z} \frac{\partial^n \hat{\Phi}(z, s, a)}{\partial s^n} + \frac{(-1)^{n-1} \log^n(a)}{z a^s} /; n \in \mathbb{N}$$

10.06.20.0042.01

$$\frac{\partial^n \hat{\Phi}(z, s, a+m)}{\partial s^n} = z^{-m} \frac{\partial^n \hat{\Phi}(z, s, a)}{\partial s^n} + (-1)^{n-1} z^{-m} \sum_{k=0}^{m-1} \frac{\log^n(a+k) z^k}{(a+k)^s} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

10.06.20.0043.01

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial s^n} = z^{-a+b-\min(0,\operatorname{Re}(b-a))} \frac{\partial^n \hat{\Phi}(z, s, b-\min(0, \operatorname{Re}(b-a)))}{\partial s^n} + (-1)^n \sum_{k=0}^{\operatorname{Re}(b-a)-1} \frac{\log^n(a+k) z^k}{(a+k)^s} /;$$

$$\operatorname{Re}(b-a) \in \mathbb{Z} \bigwedge \operatorname{Im}(b-a) = 0 \bigwedge n \in \mathbb{N}^+$$

With respect to *a*

For $\Phi(z, s, a)$

10.06.20.0044.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial a^n} = (1-n-s)_n \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^n \left((a+k)^2\right)^{s/2}} \text{ /; } |z| < 1 \bigwedge -a \notin \mathbb{N} \bigwedge n \in \mathbb{N}$$

10.06.20.0045.01

$$\frac{\partial^n \Phi(z, s, a)}{\partial a^n} = (1-n-s)_n \left(z^{\max(\lfloor -\mathrm{Re}(a)\rfloor+1,0)} \Phi(z, n+s, a+\max(\lfloor -\mathrm{Re}(a)\rfloor + 1, 0)) + \sum_{k=0}^{\lfloor -\mathrm{Re}(a)\rfloor} \frac{z^k}{(a+k)^n \left((a+k)^2\right)^{s/2}} \right) \text{ /; } -a \notin \mathbb{N} \bigwedge n \in \mathbb{N}$$

For $\hat{\Phi}(z, s, a)$

10.06.20.0011.02

$$\frac{\partial^n \hat{\Phi}(z, s, a)}{\partial a^n} == (1-n-s)_n \hat{\Phi}(z, n+s, a) \text{ /; } n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

For $\hat{\Phi}(z, s, a)$

10.06.20.0012.01

$$\frac{\partial^\alpha \hat{\Phi}(z, s, a)}{\partial z^\alpha} == \sum_{k=0}^{\infty} \frac{k!\, z^{k-\alpha}}{\Gamma(k-\alpha+1)\,(a+k)^s} \text{ /; } (|z| < 1 \vee (|z| == 1 \wedge \mathrm{Re}(s) > 1))$$

With respect to *s*

For $\hat{\Phi}(z, s, a)$

10.06.20.0013.01

$$\frac{\partial^\alpha \hat{\Phi}(z, s, a)}{\partial s^\alpha} == s^{-\alpha} \sum_{k=0}^{\infty} \frac{(-s\log(a+k))^\alpha\, Q(-\alpha, 0, -s\log(a+k))\, z^k}{(a+k)^s} \text{ /; } (|z| < 1 \vee (|z| == 1 \wedge \mathrm{Re}(s) > 1))$$

With respect to *a*

For $\hat{\Phi}(z, s, a)$

10.06.20.0014.01

$$\frac{\partial^\alpha \hat{\Phi}(z, s, a)}{\partial a^\alpha} == \frac{\Gamma(1-s)\, a^{-s-\alpha}}{\Gamma(1-s-\alpha)} + \left(\sum_{k=1}^{\infty} z^k\, k^{-s}\, {}_2\tilde{F}_1\left(1, s; 1-\alpha; -\frac{a}{k}\right) \right) a^{-\alpha} \text{ /; } |z| < 1 \wedge \mathrm{Re}(s) > 1$$

Integration

Indefinite integration

Involving only one direct function

For $\hat{\Phi}(z, s, a)$

10.06.21.0001.01

$$\int \hat{\Phi}(z, s, a)\, dz = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)(a+k)^s} \;/;\; |z| < 1 \lor |z| = 1 \land \mathrm{Re}(s) > 1$$

10.06.21.0002.01

$$\int z^{\alpha-1} \hat{\Phi}(z, s, a)\, dz = \sum_{k=0}^{\infty} \frac{z^{k+\alpha}}{(k+\alpha)(a+k)^s} \;/;\; |z| < 1 \bigvee |z| = 1 \land \mathrm{Re}(s) > 1$$

Involving only one direct function with respect to *s*

For $\hat{\Phi}(z, s, a)$

10.06.21.0003.01

$$\int \hat{\Phi}(z, s, a)\, ds = -\sum_{k=0}^{\infty} \frac{(a+k)^{-s} z^k}{\log(a+k)} \;/;\; |z| < 1 \lor |z| = 1 \land \mathrm{Re}(s) > 1$$

10.06.21.0004.01

$$\int s^{\alpha-1} \hat{\Phi}(z, s, a)\, ds = -s^{\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha, s\log(a+k))\, z^k}{(s\log(a+k))^{\alpha}} \;/;\; |z| < 1 \lor |z| = 1 \land \mathrm{Re}(s) > 1$$

Involving only one direct function with respect to *a*

For $\hat{\Phi}(z, s, a)$

10.06.21.0005.01

$$\int \hat{\Phi}(z, s, a)\, da = \frac{\Phi(z, s-1, a)}{1-s}$$

10.06.21.0006.01

$$\int a^{\alpha-1} \hat{\Phi}(z, s, a)\, da = \frac{a^{\alpha}}{\alpha - s} \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s \left(\frac{a+k}{a}\right)^{-s}} \, {}_2F_1\!\left(s-\alpha, s; s-\alpha+1; -\frac{k}{a}\right)$$

Summation

Finite summation

Finite summation

For $\hat{\Phi}(z, s, a)$

10.06.23.0001.01

$$\sum_{k=1}^{q} \hat{\Phi}\left(z^q, s, \frac{a+k-1}{q}\right) z^{k-1} = q^s \hat{\Phi}(z, s, a) /; q \in \mathbb{N}^+$$

Operations

Limit operation

Limit operation

For $\hat{\Phi}(z, s, a)$

10.06.25.0001.01

$$\lim_{z \to 1} (1-z)^{1-s} \hat{\Phi}(z, s, a) = \Gamma(1-s) /; \operatorname{Re} s < 1$$

10.06.25.0002.01

$$\lim_{z \to 1} \frac{\hat{\Phi}(z, 1, a)}{\log(1-z)} = -1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

For $\hat{\Phi}(z, s, a)$

10.06.26.0001.01

$$\hat{\Phi}(z, s, a) = a^{-s} \Gamma(a+1)^s \, {}_{s+1}\tilde{F}_s(1, a_1, a_2, \ldots, a_s; a_1+1, a_2+1, \ldots, a_s+1; z) /; a_1 = a_2 = \ldots = a_s = a \land -a \notin \mathbb{N} \land s \in \mathbb{N}^+$$

Involving ${}_pF_q$

For $\hat{\Phi}(z, s, a)$

10.06.26.0002.01

$$\hat{\Phi}(z, s, a) = a^{-s} \, {}_{s+1}F_s(1, a_1, a_2, \ldots, a_s; a_1+1, a_2+1, \ldots, a_s+1; z) /; a_1 = a_2 = \ldots = a_s = a \land -a \notin \mathbb{N} \land s \in \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

For $\Phi(z, s, a)$

10.06.26.0003.01

$$\Phi(1-z, 1, a) = \frac{1}{\Gamma(a)} G_{2,2}^{2,2}\left(z \middle| \begin{matrix} 0, 1-a \\ 0, 0 \end{matrix}\right)$$

10.06.26.0004.01

$$\Phi\left(1-\frac{1}{z}, 1, a\right) = \frac{1}{\Gamma(a)} G_{2,2}^{2,2}\left(z \middle| \begin{matrix} 1, 1 \\ 1, a \end{matrix}\right) /; z \notin (-\infty, 0)$$

For $\hat{\Phi}(z, s, a)$

10.06.26.0005.01

$$\hat{\Phi}(z, s, a) = G_{s+1,s+1}^{1,s+1}\left(-z \middle| \begin{matrix} 0, 1-a_1, \ldots, 1-a_s \\ 0, -a_1, \ldots, -a_s \end{matrix}\right) /; a_1 = a_2 = \ldots = a_s = a \land s \in \mathbb{N}^+$$

Classical cases involving unit step θ

For $\Phi(z, s, a)$

10.06.26.0006.01

$$\theta(1-|z|)(1-z)^a \Phi(1-z, 1, a) = \Gamma(a) G_{2,2}^{2,0}\left(z \middle| \begin{matrix} 1, a \\ 0, 0 \end{matrix}\right) /; z \notin (-1, 0)$$

10.06.26.0007.01

$$\theta(|z|-1)(z-1)^a \Phi(1-z, 1, a) = \Gamma(a) G_{2,2}^{0,2}\left(z \middle| \begin{matrix} 1, a \\ 0, 0 \end{matrix}\right)$$

10.06.26.0008.01

$$\theta(1-|z|)(1-z)^a \Phi\left(1-\frac{1}{z}, 1, a\right) = \Gamma(a) G_{2,2}^{2,0}\left(z \middle| \begin{matrix} a+1, a+1 \\ 1, a \end{matrix}\right)$$

10.06.26.0009.01

$$\theta(|z|-1)(z-1)^a \Phi\left(1-\frac{1}{z}, 1, a\right) = \Gamma(a) G_{2,2}^{0,2}\left(z \middle| \begin{matrix} a+1, a+1 \\ 1, a \end{matrix}\right) /; z \notin (-\infty, -1)$$

Representations through equivalent functions

Interrelations

10.06.27.0001.01

$$\Phi(z, s, a) = e^{(2\theta(\mathrm{Im}(a))-1)\pi i s} \hat{\Phi}(z, s, a) + \left(1 - e^{(2\theta(\mathrm{Im}(a))-1)\pi i s}\right) z^{\lfloor -\mathrm{Re}(a) \rfloor} \left(z \hat{\Phi}(z, s, a + \lfloor -\mathrm{Re}(a) \rfloor + 1) + \frac{(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\theta(\mathrm{Im}(a))}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{s/2}}\right)$$

10.06.27.0002.01

$$\Phi(z, s, a) = \hat{\Phi}(z, s, a) \;/;\; -\frac{\pi}{2} < \arg(a) \leq \frac{\pi}{2}$$

10.06.27.0003.01

$$\Phi(z, 2n, a) = \hat{\Phi}(z, 2n, a) \;/;\; n \in \mathbb{Z}$$

10.06.27.0004.01

$$\hat{\Phi}(z, s, a) = e^{-\theta(\lfloor -\mathrm{Re}(a) \rfloor)\,(2\,\theta(\mathrm{Im}(a))-1)\,\pi\,i\,s}\,\Phi(z, s, a) + \left(1 - e^{-(2\,\theta(\mathrm{Im}(a))-1)\,\pi\,i\,s}\right)\left(z\,\Phi(z, s, a + \lfloor -\mathrm{Re}(a) \rfloor + 1) + \frac{(\lfloor -\mathrm{Re}(a) \rfloor + \lfloor \mathrm{Re}(a) \rfloor + 1)\,\theta(\mathrm{Im}(a))}{\left((a + \lfloor -\mathrm{Re}(a) \rfloor)^2\right)^{s/2}}\right)\theta(\lfloor -\mathrm{Re}(a) \rfloor)\,z^{\lfloor -\mathrm{Re}(a) \rfloor}$$

History

– C. J. Malmstén (1849)
– R. Lipschitz (1857, 1887)
– M. Lerch (1887)
– B.C. Berndt (1972)

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