Partitions

Notations

Traditional name

Number of unrestricted partitions of an integer

Traditional notation

$p(n)$

Mathematica StandardForm notation

PartitionsP[n]

Primary definition

\[ p(n) = \left\lfloor \frac{1}{2} \left( \frac{n^{3/2}}{e} \right) \right\rfloor ; n \in \mathbb{N} \]

$p(n)$ is the number of unrestricted partitions of the positive integer $n$ into a sum of strictly positive numbers which add up to $n$ independent of order, when repetitions are allowed.

For example, $p(5) = 7$. There are 7 possibilities to express 5 as a sum of positive integers:

$5 = 1 + 4 = 2 + 3 = 1 + 1 + 3 = 1 + 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1 + 1.$

Specific values

Values at fixed points

$p(0) = 1$

$p(1) = 1$

$p(2) = 2$
\[ p(3) = 3 \]
\[ p(4) = 5 \]
\[ p(5) = 7 \]
\[ p(6) = 11 \]
\[ p(7) = 15 \]
\[ p(8) = 22 \]
\[ p(9) = 30 \]
\[ p(10) = 42 \]
\[ p(11) = 56 \]
\[ p(12) = 77 \]
\[ p(13) = 101 \]
\[ p(14) = 135 \]
\[ p(15) = 176 \]
\[ p(16) = 231 \]
\[ p(17) = 297 \]
\[ p(18) = 385 \]
\[ p(19) = 490 \]
\[ p(20) = 627 \]
\[ p(21) = 792 \]
\[ p(22) = 1002 \]
\[ p(23) = 1255 \]
\[ p(24) = 1575 \]
\[ p(25) = 1958 \]
\[ p(26) = 2436 \]
\[ p(27) = 3010 \]
\[ p(28) = 3718 \]
\[ p(29) = 4565 \]
\[ p(30) = 5604 \]
\[ p(31) = 6842 \]
\[ p(32) = 8349 \]
\[ p(33) = 10143 \]
\[ p(34) = 12310 \]
\[ p(35) = 14883 \]
\[ p(36) = 17977 \]
\[ p(37) = 21637 \]
\[ p(38) = 26015 \]
\[ p(39) = 31185 \]
\[ p(40) = 37338 \]
\[ p(41) = 44583 \]
Values at infinities

\[ p(\infty) = \infty \]

General characteristics

Domain and analyticity

The partitions \( p(n) \) is a nonanalytical function which is defined only for integers.

\[ n \to p(n) : \mathbb{N} \to \mathbb{N}^* \]

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Generalized power series
\[ p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} A(k, n) \sqrt{k} \frac{\partial \left( \sinh \left( \frac{i}{2} \sqrt{2} \sqrt{n - \frac{1}{24}} \right) \left(n - \frac{1}{24}\right)^{-1/2} \right)}{\partial n} /; \]

\[ A(k, n) = \sum_{h=1}^{k} \delta_{\gcd(h,k),1} \exp \left( \pi i \sum_{j=1}^{k-1} \frac{j}{k} \left( \frac{h j}{k} - \left\lfloor \frac{h j}{k} \right\rfloor - \frac{1}{2} \right) \right) - \frac{2 \pi i h n}{k} \]

**Asymptotic series expansions**

\[ p(n) \asymp \frac{1}{4 n \sqrt{3}} \exp \left( \sqrt{\frac{2}{3}} \sqrt{n} \left( 1 + O \left( \frac{1}{n} \right) \right) \right) /; (n \to \infty) \]

**Generating functions**

\[ p(n) = \left[ t^n \right] \prod_{k=1}^{\infty} \frac{1}{1 - t^k} /; n \in \mathbb{N} \]

\[ p(n) = \left[ t^n \right] \left( \sum_{k=1}^{\infty} (-1)^k t^k \right)^{1/2} /; n \in \mathbb{N} \]

\[ p(n) = \left[ t^n \right] \frac{2 \sqrt{t}}{\sqrt{\theta_3(0, \sqrt{t})}} /; n \in \mathbb{N} \]

**Identities**

**Functional identities**

\[ p(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma_1(k) p(n - k) \]

\[ p(n) = \sum_{k=1}^{n} (-1)^{k-1} \left[ p \left( n - \frac{1}{2} (3 k^2 - k) \right) + p \left( n - \frac{1}{2} (3 k^2 + k) \right) \right] \]
Complex characteristics

Real part

\[ \text{Re}(p(n)) = p(n) \]

Imaginary part

\[ \text{Im}(p(n)) = 0 \]

Absolute value

\[ |p(n)| = p(n) \]

Argument

\[ \text{arg}(p(n)) = 0 \]

Conjugate value

\[ \overline{p(n)} = p(n) \]

Summation

Finite summation

\[ \sum_{k=1}^{\infty} \left( \frac{1}{2} \left( \sqrt{24n+1} - 1 \right) \right) (-1)^k \frac{p(n - \frac{1}{2} k(3k+1))}{\left( \sqrt{24n+1} + 1 \right)} = 0 \]

Infinite summation

\[ \sum_{k=0}^{\infty} p(k) r^k = \prod_{k=1}^{\infty} \frac{1}{1 - r^k} \]

Representations through equivalent functions

With related functions
\[ p(n) = \sum_{k=0}^{n-2} q(n - 2k) p(k) \]

**Inequalities**

\[ p(n) \leq \frac{1}{2} (p(n - 1) + p(n + 1)); n \in \mathbb{N}^+ \]

**Other identities**

**Congruence properties**

\[ p(5n + 4) \mod 5 = 0 \]

\[ p(7n + 5) \mod 7 = 0 \]

\[ p(11n + 6) \mod 11 = 0 \]

\[ p(n) \mod (5^{k_1} 7^{k_2} 11^{k_3}) = 0; (24n) \mod (5^{k_1} 7^{k_2} 11^{k_3}) = 1 \bigcap k_1 \in \mathbb{N} \bigcap k_2 \in \mathbb{N} \bigcap k_3 \in \mathbb{N} \]

**History**

- G. W. Leibniz (1669) investigated the number of ways a given positive integer can be decomposed into smaller ones
- L. Euler (1740)
- S. Ramanujan (1917)
- G.H. Hardy (1920) introduced the notation \( p(n) \)
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