PolyLog3

Notations

Traditional name

Nielsen polylogarithm

Traditional notation

$S_{\nu,p}(z)$

Mathematica StandardForm notation

PolyLog[ν, p, z]

Primary definition

10.09.02.0001.01

$$S_{\nu,p}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+p} S_k^{(p)} z^k}{k^\nu\, k!} \,/;\, |z| < 1 \bigwedge p \in \mathbb{N}^+$$

$S_\nu^p(z)$ is the Nielsen generalized polylogarithm function or hyperlogarithm.

Specific values

Specialized values

For fixed ν, p

10.09.03.0001.01

$$S_{\nu,p}(0) = 0$$

For fixed ν, z

10.09.03.0002.01

$$S_{\nu,1}(z) = \mathrm{Li}_{\nu+1}(z)$$

For fixed p, z

10.09.03.0003.01

$$S_{0,p}(z) = \frac{(-\log(1-z))^p}{\Gamma(p+1)}$$

10.09.03.0004.01

$$S_{1,p}(z) = \frac{(-1)^p \log(z) \log^p(1-z)}{p!} + (-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k \log^{-k+p-1}(1-z)}{(p-k-1)!} \operatorname{Li}_{k+2}(1-z) + \zeta(p+1) /; p \in \mathbb{N}^+$$

For fixed ν

10.09.03.0005.01

$$S_{\nu,1}(1) = \zeta(\nu+1)$$

For fixed z

10.09.03.0007.01

$$S_{0,1}(z) = -\log(1-z)$$

Values at fixed points

10.09.03.0006.01

$$S_{-1,2}\left(\frac{1}{2}\right) = \log(2)$$

General characteristics

Domain and analyticity

$S_{\nu,p}(z)$ is an analytical function of ν, p, z which is defined in $\mathbb{C}^3$.

10.09.04.0001.01

$$(\nu * p * z) \longrightarrow S_{\nu,p}(z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

10.09.04.0002.02

$$S_{\overline{\nu},\overline{p}}(\overline{z}) = \overline{S_{\nu,p}(z)} /; z \notin (1, \infty)$$

Permutation symmetry

Permutation symmetry for $z = 1$:

10.09.04.0003.01

$$S_{\nu,p}(1) = S_{p,1}(1) /; \nu \in \mathbb{N}^+ \land p \in \mathbb{N}^+$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, p, the function $S_{\nu,p}(z)$ does not have poles and essential singularities.

10.09.04.0004.01

$\mathcal{S}ing_z(S_{\nu,p}(z)) = \{\}$

With respect to ν

For fixed p, z, the function $S_{\nu,p}(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

10.09.04.0005.01

$\mathcal{S}ing_\nu(S_{\nu,p}(z)) = \{\{\tilde{\infty}, \infty\}\}$

Branch points

With respect to z

For fixed ν, p the function $S_{\nu,p}(z)$ has two branch points: $z = 1$, $z = \tilde{\infty}$.

10.09.04.0006.01

$\mathcal{BP}_z(S_{\nu,p}(z)) = \{1, \tilde{\infty}\}$

With respect to ν

For fixed p, z, the function $S_{\nu,p}(z)$ does not have branch points.

10.09.04.0007.01

$\mathcal{BP}_\nu(S_{\nu,p}(z)) = \{\}$

Branch cuts

With respect to z

For fixed ν, p, the function $S_{\nu,p}(z)$ is a single-valued function on the z-plane cut along the interval $\{1, \infty\}$, where it is continuous from below.

10.09.04.0008.01

$\mathcal{BC}_z(S_{\nu,p}(z)) = \{\{\{1, \infty\}, i\}\} /; \nu \in \mathbb{N}^+ \wedge p \in \mathbb{N}^+$

10.09.04.0009.01

$$\lim_{\epsilon \to +0} S_{n,2}(x - i\,\epsilon) = S_{n,2}(x) /; n \in \mathbb{N}^+ \wedge x > 1$$

10.09.04.0010.01

$$\lim_{\epsilon \to +0} S_{n,2}(x + i\,\epsilon) = S_{n,2}(x) - \frac{2\,\pi^2 \log^n(x)}{n!} + 2\,i\,\pi \left(\mathrm{Li}_{n+1}(x) - \sum_{k=0}^{n-1} \frac{\log^k(x)\,\zeta(1-k+n)}{k!} \right) /; n \in \mathbb{N}^+ \wedge x > 1$$

With respect to ν

For fixed p, z, the function $S_\nu^p(z)$ does not have branch cuts.

10.09.04.0011.01

$\mathcal{BC}_\nu(S_{\nu,p}(z)) = \{\}$

Series representations

Generalized power series

Expansions at $z == 0$

For the function itself

10.09.06.0001.02

$$S_{\nu,p}(z) \propto \frac{p^{-\nu} z^p}{p!} \left(1 + \frac{1}{2} p^{\nu+1} (p+1)^{-\nu} z + \frac{p^\nu (p+2)^{-\nu-1}}{p+1} S_{p+2}^{(p)} z^2 + \ldots\right) /; (z \to 0) \bigwedge p \in \mathbb{N}^+$$

10.09.06.0005.01

$$S_{\nu,p}(z) \propto \frac{p^{-\nu} z^p}{p!} \left(1 + \frac{1}{2} p^{\nu+1} (p+1)^{-\nu} z + \frac{p^\nu (p+2)^{-\nu-1}}{p+1} S_{p+2}^{(p)} z^2 + O(z^3)\right)$$

10.09.06.0002.01

$$S_{\nu,p}(z) == \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+p}^{(p)} z^{k+p}}{(k+p)! (k+p)^\nu} /; |z| \leq 1 \bigwedge p \in \mathbb{N}^+$$

10.09.06.0003.02

$$S_{\nu,p}(z) \propto \frac{p^{-\nu} z^p}{p!} (1 + O(z))$$

Other series representations

10.09.06.0004.01

$$S_{\nu,1}(z) \propto \Gamma(-\nu) \sum_{k=-\infty}^{\infty} (2 \pi i k - \log(z))^\nu e^{2 \pi k i} /; \operatorname{Re} \nu < -1$$

Integral representations

On the real axis

Of the direct function

10.09.07.0001.01

$$S_{\nu,p}(z) == \frac{1}{\Gamma(\nu) \Gamma(p+1)} \int_0^1 \frac{(-\log(t))^{\nu-1} (-\log(1-z t))^p}{t} \, dt /; \operatorname{Re}(\nu) > 0 \bigwedge |z| \leq 1 \bigwedge p \in \mathbb{N}^+$$

10.09.07.0002.01

$$S_{\nu,p}(z) == \frac{(-1)^{n+p-1}}{(n-1)! \, p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1-z t)}{t} \, dt /; n \in \mathbb{N}^+ \bigwedge p \in \mathbb{N}^+$$

Differential equations

Ordinary nonlinear differential equations

10.09.13.0001.01

$$\left(\left(-2\,z\,p+5\,p+9\,z-9\right)w''(z)^2+z\left(\left(p+6\,z-6\right)w^{(3)}(z)+p\,(z-1)\,z\,w^{(4)}(z)\right)w''(z)+z^2\left(-z\,p+p+z-1\right)w^{(3)}(z)^2\right)z^2+ w'(z)\left(2\left(z\,p+p+3\,z-3\right)w''(z)+z\left(\left(4\,z\,p-3\,p+2\,z-2\right)w^{(3)}(z)+p\,(z-1)\,z\,w^{(4)}(z)\right)\right)z+ \left(p+z-1\right)w'(z)^2=0\;/;\;w(z)=S_{2,p}(z)$$

10.09.13.0002.01

$$\left(\left(\left(36\,(z-1)+p\,(17-11\,z)\right)w^{(3)}(z)^2+ z\left(\left(12\,(z-1)+p\,(3-2\,z)\right)w^{(4)}(z)+p\,(z-1)\,z\,w^{(5)}(z)\right)w^{(3)}(z)+z^2\left(-z\,p+p+z-1\right)w^{(4)}(z)^2\right)z^2+ w''(z)\left(\left(84\,(z-1)+p\,(6\,z+19)\right)w^{(3)}(z)+z\left(\left(16\,z\,p-13\,p+14\,z-14\right)w^{(4)}(z)+3\,p\,(z-1)\,z\,w^{(5)}(z)\right)\right)z+ \left(49\,(z-1)+p\,(25-4\,z)\right)w''(z)^2\right)z^2+w'(z)\left(2\left(2\,z\,p+3\,p+7\,z-7\right)w''(z)+ z\left(\left(12\,(z-1)+7\,p\,(2\,z-1)\right)w^{(3)}(z)+z\left(\left(8\,z\,p-7\,p+2\,z-2\right)w^{(4)}(z)+p\,(z-1)\,z\,w^{(5)}(z)\right)\right)\right) z+\left(p+z-1\right)w'(z)^2=0\;/;\;w(z)=S_{3,p}(z)$$

10.09.13.0003.01

$$\left(\left(\left(-5\left(7\,z\,p-9\,p-20\,z+20\right)w^{(4)}(z)^2+ z\left(\left(20\,(z-1)+p\,(6-5\,z)\right)w^{(5)}(z)+p\,(z-1)\,z\,w^{(6)}(z)\right)w^{(4)}(z)+z^2\left(-z\,p+p+z-1\right)w^{(5)}(z)^2\right)z^2+ w^{(3)}(z)\left(2\,z\left(\left(20\,z\,p-17\,p+25\,z-25\right)w^{(5)}(z)+3\,p\,(z-1)\,z\,w^{(6)}(z)\right)-5\left(p\,(4\,z-21)-100\,(z-1)\right)w^{(4)}(z)\right)z+ \left(625\,(z-1)+p\,(235-85\,z)\right)w^{(3)}(z)^2\right)z^2+w''(z)\left(\left(750\,(z-1)+p\,(66\,z+199)\right)w^{(3)}(z)+ z\left(\left(300\,(z-1)+p\,(186\,z-101)\right)w^{(4)}(z)+z\left(\left(75\,z\,p-68\,p+30\,z-30\right)w^{(5)}(z)+7\,p\,(z-1)\,z\,w^{(6)}(z)\right)\right)\right)z+ \left(225\,(z-1)+p\,(113-8\,z)\right)w''(z)^2\right)z^2+w'(z)\left(2\left(15\,(z-1)+p\,(4\,z+7)\right)w''(z)+ z\left(\left(50\,(z-1)+p\,(46\,z-15)\right)w^{(3)}(z)+z\left(\left(20\,(z-1)+p\,(46\,z-35)\right)w^{(4)}(z)+ z\left(\left(13\,z\,p-12\,p+2\,z-2\right)w^{(5)}(z)+p\,(z-1)\,z\,w^{(6)}(z)\right)\right)\right)\right)z+\left(p+z-1\right)w'(z)^2=0\;/;\;w(z)=S_{4,p}(z)$$

Identities

Functional identities

10.09.17.0001.01

$$S_{n,p}(z)=\frac{(-1)^p\log^p(1-z)\log^n(z)}{n!\,p!}+\sum_{k=0}^{n-1}\frac{\log^k(z)}{k!}\left(S_{n-k}^p(1)-\sum_{j=0}^{p-1}\frac{(-1)^j\log^j(1-z)}{j!}S_{p-j}^{n-k}(1-z)\right)/;\;n\in\mathbb{N}^+\land p\in\mathbb{N}^+$$

10.09.17.0002.01

$$S_{n,p}(z)=(-1)^p\left(\frac{\log^{n+p}(-z)}{(n+p)!}+\sum_{j=0}^{n-1}\frac{\log^j(-z)\,c(n-j,\,p)}{j!}\right)+(-1)^n\sum_{k=0}^{p-1}(-1)^k\sum_{j=0}^{k}\frac{\log^j(-z)}{j!}\binom{n-j+k-1}{k-j}S_{n+k-j}^{p-k}\left(\frac{1}{z}\right)/;$$

$$c(n,\,p)=\left(1-(-1)^n\right)(-1)^p\,S_n^p(-1)-(-1)^n\sum_{j=1}^{p-1}\binom{j+n-1}{j}(-1)^{p-j}\,S_{n+j}^{p-j}(-1)\bigwedge n\in\mathbb{N}^+\bigwedge p\in\mathbb{N}^+\bigwedge z\notin(0,\,1)$$

10.09.17.0003.01

$$S_{\nu,p}(z)=\int_0^z\frac{1}{t}\,S_{\nu-1}^p(t)\,dt$$

Differentiation

Low-order differentiation

With respect to ν

10.09.20.0001.01

$$\frac{\partial S_{\nu,p}(z)}{\partial \nu} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1} \log(k+p)\, S_{k+p}^{(p)}\, z^{k+p}}{(k+p)!\,(k+p)^{\nu}} \;/;\; |z| < 1 \bigwedge p \in \mathbb{N}^+$$

10.09.20.0002.01

$$\frac{\partial^2 S_{\nu,p}(z)}{\partial \nu^2} = \sum_{k=0}^{\infty} \frac{(-1)^{k} \log^2(k+p)\, S_{k+p}^{(p)}\, z^{k+p}}{(k+p)!\,(k+p)^{\nu}} \;/;\; |z| < 1 \bigwedge p \in \mathbb{N}^+$$

With respect to z

10.09.20.0003.01

$$\frac{\partial S_{\nu,p}(z)}{\partial z} = \frac{S_{\nu-1,p}(z)}{z}$$

10.09.20.0010.01

$$\frac{\partial^2 S_{\nu,p}(z)}{\partial z^2} = \frac{S_{\nu-2,p}(z) - S_{\nu-1,p}(z)}{z^2}$$

10.09.20.0004.01

$$\frac{\partial^2 S_{\nu,p}(z)}{\partial z^2} = \sum_{k=0}^{\infty} \frac{(-1)^{k} S_{k+p}^{(p)}\, z^{k+p-2}}{(k+p-2)!\,(k+p)^{\nu}} \;/;\; |z| < 1 \bigwedge p \in \mathbb{N}^+$$

Symbolic differentiation

With respect to ν

10.09.20.0005.02

$$\frac{\partial^m S_{\nu,p}(z)}{\partial \nu^m} = \sum_{k=0}^{\infty} \frac{(-1)^{k+m} \log^m(k+p)\, S_{k+p}^{(p)}\, z^{k+p}}{(k+p)!\,(k+p)^{\nu}} \;/;\; |z| < 1 \bigwedge p \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

10.09.20.0006.02

$$\frac{\partial^m S_{\nu,p}(z)}{\partial \nu^m} = z^{-m} \sum_{j=0}^{m} S_{\nu-j}^{p}(z)\, S_m^{(j)} \;/;\; m \in \mathbb{N}$$

With respect to z

10.09.20.0011.01

$$\frac{\partial^m S_{\nu,p}(z)}{\partial z^m} = z^{-m} \sum_{j=0}^{m} S_m^{(j)}\, S_{\nu-j,p}(z)$$

10.09.20.0007.02

$$\frac{\partial^m S_{\nu,p}(z)}{\partial z^m} = \sum_{k=0}^{\infty} \frac{(-1)^{k} S_{k+p}^{(p)}\, (k-m+p+1)_m\, z^{k-m+p}}{(k+p)!\,(k+p)^{\nu}} \;/;\; |z| < 1 \bigwedge p \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to ν

10.09.20.0008.01

$$\frac{\partial^{\alpha} S_{\nu,p}(z)}{\partial \nu^{\alpha}} = \nu^{-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k (-\nu \log(k+p))^{\alpha} Q(-\alpha, 0, -\nu \log(k+p)) S_{k+p}^{(p)} z^{k+p}}{(k+p)! (k+p)^{\nu}} /; |z| \leq 1 \bigwedge p \in \mathbb{N}^+$$

With respect to z

10.09.20.0009.01

$$\frac{\partial^{\alpha} S_{\nu,p}(z)}{\partial z^{\alpha}} = \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+p}^{(p)} z^{k+p-\alpha}}{\Gamma(k+p-\alpha+1) (k+p)^{\nu}} /; |z| \leq 1 \bigwedge p \in \mathbb{N}^+$$

Integration

Indefinite integration

Involving only one direct function

10.09.21.0001.01

$$\int S_{\nu,p}(z)\, dz = \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+p}^{(p)} z^{k+p+1}}{(k+p+1)! (k+p)^{\nu}} /; |z| < 1 \bigwedge p \in \mathbb{N}^+$$

Involving one direct function and elementary functions

Involving power function

10.09.21.0002.01

$$\int z^{\alpha-1} S_{\nu,p}(z)\, dz = \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+p}^{(p)} z^{k+p+\alpha}}{(k+p+\alpha) (k+p)! (k+p)^{\nu}} /; |z| < 1 \bigwedge p \in \mathbb{N}^+$$

10.09.21.0003.01

$$\int \frac{S_{\nu,p}(z)}{z}\, dz = S_{\nu+1,p}(z)$$

Involving only one direct function with respect to ν

10.09.21.0004.01

$$\int S_{\nu,p}(z)\, d\nu = \sum_{k=0}^{\infty} \frac{(-1)^{k-1} S_{k+p}^{(p)} z^{k+p}}{(k+p)! \log(k+p) (k+p)^{\nu}}$$

Involving one direct function and elementary functions with respect to ν

Involving power function

10.09.21.0005.01

$$\int \nu^{\alpha-1} S_{\nu,p}(z)\, d\nu = -\nu^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k S_{k+p}^{(p)} \Gamma(\alpha, \nu \log(k+p)) z^{k+p}}{(k+p)! (\nu \log(k+p))^{\alpha}}$$

History

– N. Nielsen (1909)
– K. S. Kölbig
– J. A. Mignaco
– E. Remiddi (1970)

Applications include electrical network problems, number theory, group theory, K-theory, geometry, quantum electrodynamics, group cohomology, mixed Hodge structures, mixed motives, evaluation of volumes of hyperbolic polytopes, celestial mechanics.

Copyright