PrimePi

Notations

Traditional name

Prime counting function

Traditional notation

$\pi(x)$

Mathematica StandardForm notation

`PrimePi[x]`

Primary definition

13.04.02.0001.01

$$\pi(x) == \sum_{k=1}^{\lfloor x \rfloor} \theta(x - \text{prime}(k)) \;/;\; x \in \mathbb{R} \land x \geq 0$$

For real x, the prime counting function $\pi(x)$ is the number of primes less than or equal to x.

13.04.02.0002.01

$$\pi(x) == 0 \;/;\; x < 2$$

Examples: The primes under 25 are 2, 3, 5, 7, 11, 13, 17, 19 and 23, so $\pi(3) == 2$, $\pi(10) == 4$, $\pi(25) == 9$.

Specific values

Specialized values

13.04.03.0001.01

$$\pi(x) == 0 \;/;\; x < 2$$

Values at fixed points

13.04.03.0002.01

$$\pi(1) == 0$$

13.04.03.0003.01

$$\pi(2) == 1$$

13.04.03.0004.01

$$\pi(3) == 2$$

13.04.03.0005.01

$\pi(4) == 2$

13.04.03.0006.01

$\pi(5) == 3$

13.04.03.0007.01

$\pi(6) == 3$

13.04.03.0008.01

$\pi(7) == 4$

13.04.03.0009.01

$\pi(8) == 4$

13.04.03.0010.01

$\pi(9) == 4$

13.04.03.0011.01

$\pi(10) == 4$

13.04.03.0012.01

$\pi(11) == 5$

13.04.03.0013.01

$\pi(12) == 5$

13.04.03.0014.01

$\pi(13) == 6$

13.04.03.0015.01

$\pi(14) == 6$

13.04.03.0016.01

$\pi(15) == 6$

13.04.03.0017.01

$\pi(16) == 6$

13.04.03.0018.01

$\pi(17) == 7$

13.04.03.0019.01

$\pi(18) == 7$

13.04.03.0020.01

$\pi(19) == 8$

13.04.03.0021.01

$\pi(20) == 8$

13.04.03.0022.01

$\pi(21) == 8$

13.04.03.0023.01

$\pi(22) == 8$

13.04.03.0024.01

$\pi(23) == 9$

13.04.03.0025.01

$\pi(24) == 9$

13.04.03.0026.01

$\pi(25) == 9$

13.04.03.0027.01

$\pi(26) == 9$

13.04.03.0028.01

$\pi(27) == 9$

13.04.03.0029.01

$\pi(28) == 9$

13.04.03.0030.01

$\pi(29) == 10$

13.04.03.0031.01

$\pi(30) == 10$

13.04.03.0032.01

$\pi(31) == 11$

13.04.03.0033.01

$\pi(32) == 11$

13.04.03.0034.01

$\pi(33) == 11$

13.04.03.0035.01

$\pi(34) == 11$

13.04.03.0036.01

$\pi(35) == 11$

13.04.03.0037.01

$\pi(36) == 11$

13.04.03.0038.01

$\pi(37) == 12$

13.04.03.0039.01

$\pi(38) == 12$

13.04.03.0040.01

$\pi(39) == 12$

13.04.03.0041.01

$\pi(40) == 12$

13.04.03.0042.01

$\pi(41) == 13$

13.04.03.0043.01

$\pi(42) == 13$

13.04.03.0044.01

$\pi(43) == 14$

13.04.03.0045.01
$\pi(44) == 14$

13.04.03.0046.01
$\pi(45) == 14$

13.04.03.0047.01
$\pi(46) == 14$

13.04.03.0048.01
$\pi(47) == 15$

13.04.03.0049.01
$\pi(48) == 15$

13.04.03.0050.01
$\pi(49) == 15$

13.04.03.0051.01
$\pi(50) == 15$

13.04.03.0052.01
$\pi(100) == 25$

13.04.03.0053.01
$\pi(1000) == 168$

13.04.03.0056.01
$\pi(10\,000) == 1229$

13.04.03.0054.01
$\pi(1\,000\,000) == 78\,498$

13.04.03.0055.01
$\pi(1\,000\,000\,000) == 50\,847\,534$

13.04.03.0057.01
$\pi(-100) == 0$

General characteristics

Domain and analyticity

$\pi(x)$ is a nonanalytical function which is defined over the reals $\mathbb{R}$.

13.04.04.0001.01
$x \longrightarrow \pi(x) :: \mathbb{R} \longrightarrow \mathbb{Z}$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Other series representations

13.04.06.0001.01

$$\pi(n) == \sum_{k=1}^{n} \left\lfloor \cos^2\left(\frac{\pi\,((k-1)!+1)}{k}\right) \right\rfloor - 1 \;/;\; n \in \mathbb{N}^+$$

13.04.06.0002.01

$$\pi(n) == \sum_{k=2}^{n} \frac{\sin^2\left(\frac{\pi\,(k-1)!^2}{k}\right)}{\sin^2\left(\frac{\pi}{k}\right)} \;/;\; n \in \mathbb{N}$$

13.04.06.0003.01

$$\pi(n) == \sum_{k=2}^{n} \left\lfloor \frac{(k-1)!+1}{k} - \left\lfloor \frac{(k-1)!}{k} \right\rfloor \right\rfloor \;/;\; n \in \mathbb{N}$$

13.04.06.0004.01

$$\pi(n) == \sum_{k=3}^{n} \left((k-2)! - k \left\lfloor \frac{(k-2)!}{k} \right\rfloor \right) - 1 \;/;\; n - 3 \in \mathbb{N}^+$$

13.04.06.0005.01

$$\pi(n) == \sum_{k=2}^{n} \left\lfloor \frac{1}{\sum_{i=1}^{k-1} \left\lfloor \frac{i \left\lfloor \frac{k}{i} \right\rfloor}{k} \right\rfloor} \right\rfloor \;/;\; n \in \mathbb{N}$$

13.04.06.0013.01

$$\pi(x) == \sum_{j=2}^{\lfloor x \rfloor} \left(1 + \left\lfloor \frac{2 - \sum_{i=1}^{j} \left(\left\lfloor \frac{j}{i} \right\rfloor - \left\lfloor \frac{j-1}{i} \right\rfloor\right)}{j} \right\rfloor\right)$$

13.04.06.0006.01

$$\pi(n) == \sum_{n=2}^{n} \delta_{\frac{e^{\frac{2\pi i (n-1)!}{n}}-1}{e^{-\frac{2\pi i}{n}}-1},1} \;/;\; n \in \mathbb{N}$$

13.04.06.0007.01

$$\pi(x) == \sum_{k=1}^{\lfloor x \rfloor} \delta_{g(k),n_{g(k)},1} \;/;\; \text{factors}(k) == \left\{\{p_1, n_1\}, \{p_2, n_2\}, \ldots, \left\{p_{g(k)}, n_{g(k)}\right\}\right\} \bigwedge p_j \in \mathbb{P} \bigwedge n_j \in \mathbb{N}^+$$

13.04.06.0008.01

$$\pi(x) == \sum_{i=1}^{n} 1 \;/;\; \text{prime}(n) \leq x < \text{prime}(n+1)$$

13.04.06.0009.01

$$\pi(x) == \sum_{k=1}^{\lfloor x \rfloor} \theta(x - p_k) \;/;\; p_k \in \mathbb{P} \bigwedge x \in \mathbb{R} \bigwedge x \geq 0$$

13.04.06.0010.01

$$\pi(x) = \sum_{k=2}^{\lfloor x \rfloor} \left\lfloor \frac{\phi(k)}{k-1} \right\rfloor$$

13.04.06.0011.01

$$\pi(x) = -\sum_{k=1}^{\lfloor \log_2(x) \rfloor} \mu(k) \sum_{n=2}^{\lfloor x^{1/k} \rfloor} \mu(n)\, \Omega(n) \left\lfloor \frac{x^{1/k}}{n} \right\rfloor /;\ n = \prod_{k=1}^{l} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+ \bigwedge \Omega(n) = \sum_{k=1}^{l} n_k$$

13.04.06.0012.01

$$\pi(x) = R(x) + \sum_{\rho_k=-\infty}^{\infty} R(\rho_k) /;\ R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)\, \mathrm{li}\left(x^{1/k}\right)}{k} = \sum_{k=1}^{\infty} \frac{\log^k(x)}{k\, \zeta(k+1)\, k!} + 1 \bigwedge \zeta(\rho_k) = 0$$

Integral representations

On the real axis

Of the direct function

13.04.07.0001.01

$$\pi(n) = n - \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{m=1}^{n} \cos\left(t \prod_{k=1}^{m-1} \prod_{j=1}^{m-1} (j\, k - m) \right) \right) dt - 1 /;\ n \in \mathbb{N}^+$$

Identities

Functional identities

13.04.17.0001.01

$$\pi(x) = \sum_{d_j | n} \mu(d_j) \left\lfloor \frac{x}{d_j} \right\rfloor + \pi\left(\sqrt{x}\right) - 1 /;\ n = \prod_{k=1}^{m} p_k \bigwedge p_k \in \mathbb{P} \bigwedge p_k \leq \sqrt{x}$$

Integration

Definite integration

For the direct function itself

13.04.21.0001.01

$$\int_2^{\infty} \frac{\pi(t)}{t\,(t^s - 1)}\, dt = \frac{\log(\zeta(s))}{s} /;\ \mathrm{Re}(s) > 1$$

Representations through equivalent functions

With related functions

13.04.27.0001.01

$$\pi(x) = \sum_{i=1}^{n} 1 /;\ \mathrm{prime}(n) \leq x < \mathrm{prime}(n+1)$$

13.04.27.0002.01

$$\pi(x) = \sum_{k=1}^{\lfloor x \rfloor} \theta(x - p_k) /; p_k \in \mathbb{P} \bigwedge x \in \mathbb{R} \bigwedge x \geq 0$$

13.04.27.0003.01

$$\pi(x) = \sum_{k=1}^{\lfloor x \rfloor} \delta_{g(k), n_{g(k)}, 1} /; \text{factors}(k) = \{\{p_1, n_1\}, \{p_2, n_2\}, \ldots, \{p_{g(k)}, n_{g(k)}\}\} \bigwedge p_j \in \mathbb{P} \bigwedge n_j \in \mathbb{N}^+$$

13.04.27.0004.01

$$\pi(x) = \sum_{k=2}^{\lfloor x \rfloor} \left\lfloor \frac{\phi(k)}{k-1} \right\rfloor$$

13.04.27.0005.01

$$\pi(x) = -\sum_{k=1}^{\lfloor \log_2(x) \rfloor} \mu(k) \sum_{n=2}^{\lfloor x^{1/k} \rfloor} \mu(n)\, \Omega(n) \left\lfloor \frac{x^{1/k}}{n} \right\rfloor /; n = \prod_{k=1}^{l} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+ \bigwedge \Omega(n) = \sum_{k=1}^{l} n_k$$

Inequalities

13.04.29.0001.01

$$\frac{x}{\log(x) + 2} < \pi(x) < \frac{\log(4)\, x}{\log(x)} /; x \geq 2$$

13.04.29.0002.01

$$\pi(2\, x) < 2\, \pi(x) /; x \geq 3$$

13.04.29.0003.01

$$\pi(n) < \phi(n) /; n \in \mathbb{N} \land n \geq 91$$

Zeros

13.04.30.0001.01

$$\pi(1) = 0$$

Theorems

The prime number theorem

$$\pi(x) \propto \frac{x}{\log(x)} /; \quad (x \to \infty).$$

History

– Euclid proved that $\pi(n) \to \infty$ as $n \to \infty$

– L. Euler proved that $\pi(n) / n \to 0$ as $n \to \infty$

– P.G. Lejeune–Dirichlet (1838)

– P. L. Chebychev (1849, 1851)

– J. Hadamard and Ch. Vallée de la Poisson (1896) independently proved the prime number theorem

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