# ProductLog2



## Notations

### Traditional name

Generalized Lambert function

### Traditional notation

$W_k(z)$

### *Mathematica* StandardForm notation

ProductLog[$k$, $z$]

LambertW[$k$, $z$]

## Primary definition

01.32.02.0001.01

$z = w\, e^w \;/;\; w = W_k(z) \bigwedge k \in \mathbb{Z}$

## Specific values

### Specialized values

01.32.03.0001.01

$W_0(z) = W(z)$

01.32.03.0002.01

$W_k(0) = -\infty \;/;\; k \neq 0$

01.32.03.0003.01

$$W_{-1}\left(-\frac{\pi}{2}\right) = -\frac{i\,\pi}{2}$$

01.32.03.0004.01

$$W_{-1}\left(-\frac{1}{e}\right) = -1$$

## General characteristics

### Domain and analyticity

$W_k(z)$ is an analytical function of $z$ which is defined over $\mathbb{C}$.

01.32.04.0001.01

$$(k * z) \longrightarrow W_k(z) :: (\mathbb{Z} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

## Symmetries and periodicities

### Periodicity

No periodicity

## Poles and essential singularities

### With respect to z

The function $W_k(z)$ does not have poles and essential singularities.

01.32.04.0002.01

$$\mathcal{S}ing_z(W_k(z)) == \{\}$$

## Branch points

### With respect to z

The function $W_k(z)$, $k \neq 0$, $k \neq 1$, has two branch points: $z == 0$, $z == \tilde{\infty}$.

01.32.04.0003.01

$$\mathcal{BP}_z(W_k(z)) == \{0, \tilde{\infty}\} /; k \notin \{0, 1\}$$

The function $W_0(z)$ has two branch points: $z == -\frac{1}{e}$, $z == \tilde{\infty}$.

01.32.04.0004.01

$$\mathcal{BP}_z(W_0(z)) == \left\{-\frac{1}{e}, \tilde{\infty}\right\}$$

The function $W_1(z)$ has three branch points: $z == 0$, $z == -\frac{1}{e}$, $z == \tilde{\infty}$.

01.32.04.0005.01

$$\mathcal{BP}_z(W_1(z)) == \left\{0, -\frac{1}{e}, \tilde{\infty}\right\}$$

01.32.04.0006.01

$$\mathcal{R}_z(W_k(z), 0) == \log$$

01.32.04.0007.01

$$\mathcal{R}_z\left(W_1(z), -\frac{1}{e}\right) == 2$$

01.32.04.0008.01

$$\mathcal{R}_z(W_k(z), \tilde{\infty}) == \log$$

## Branch cuts

### With respect to z

The function $W_k(z)$, $k \neq 0$, is a single-valued function on the $z$-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

01.32.04.0009.01

$$\mathcal{BC}_z(W_k(z)) = \{\{(-\infty, 0), -i\}\} /; k \notin \{0, 1\}$$

01.32.04.0010.01

$$\lim_{\epsilon \to +0} W_k(x + i\,\epsilon) = W_k(x) /; k \neq 0 \bigwedge k \neq 1 \bigwedge x < 0$$

01.32.04.0011.01

$$\lim_{\epsilon \to +0} W_k(x - i\,\epsilon) = W_{k-1}(x) /; k \neq 0 \bigwedge k \neq 1 \bigwedge x < 0$$

01.32.04.0012.01

$$\mathcal{BC}_z(W_0(z)) = \left\{\left\{\left(-\infty, -\frac{1}{e}\right), -i\right\}\right\}$$

01.32.04.0013.01

$$\lim_{\epsilon \to +0} W_0(x + i\,\epsilon) = W_0(x) /; x < -\frac{1}{e}$$

01.32.04.0014.01

$$\lim_{\epsilon \to +0} W_0(x - i\,\epsilon) = W_{-1}(x) /; x < -\frac{1}{e}$$

01.32.04.0015.01

$$\mathcal{BC}_z(W_1(z)) = \left\{\left\{\left(-\infty, -\frac{1}{e}\right), -i\right\}, \left\{\left(-\frac{1}{e}, 0\right), -i\right\}\right\}$$

01.32.04.0016.01

$$\lim_{\epsilon \to +0} W_1(x + i\,\epsilon) = W_1(x) /; x < 0$$

01.32.04.0017.01

$$\lim_{\epsilon \to +0} W_1(x - i\,\epsilon) = W_0(x) /; x < -\frac{1}{e}$$

01.32.04.0018.01

$$\lim_{\epsilon \to +0} W_1(x - i\,\epsilon) = W_{-1}(x) /; -\frac{1}{e} < x < 0$$

# Series representations

## Asymptotic series expansions

01.32.06.0001.02

$$W_k(z) \propto w - \zeta + \frac{\zeta}{w} - \frac{2\,\zeta - \zeta^2}{2\,w^2} + \frac{\zeta\left(6 - 9\,\zeta + 2\,\zeta^2\right)}{6\,w^3} + \ldots /;$$

$$(z \to 0 \mid \infty) \bigwedge w = 2\,i\,\pi\,k + \log(z) \bigwedge \zeta = \log(w) \bigwedge \neg\left(k = 0 \bigwedge |z| < e^2\right) \bigwedge \neg\left(k = -1 \bigwedge z \in \left(-\frac{1}{e}, 0\right)\right)$$

01.32.06.0002.01

$$W_k(z) == w - \zeta - \sum_{n=1}^{\infty} w^{-n} \sum_{m=1}^{n} \frac{(-1)^n}{m!} S_n^{(-m+n+1)} \zeta^m \; /;$$

$$(z \to 0 \,|\, \infty) \bigwedge w == 2\,i\,\pi\,k + \log(z) \bigwedge \zeta == \log(w) \bigwedge \neg\left(k == 0 \bigwedge |z| < e^2\right) \bigwedge \neg\left(k == -1 \bigwedge z \in \left(-\frac{1}{e}, 0\right)\right)$$

01.32.06.0003.01

$$W_k(z) \propto \log(z) + 2\,i\,\pi\,k - \log(2\,i\,\pi\,k + \log(z)) \left(1 + O\left(\frac{1}{\log(z)}\right)\right) /; (|z| \to 0 \,|\, \infty) \land k \neq 0 \land k \neq 1$$

01.32.06.0004.01

$$W_k(z) == (2\,i\,\pi\,k + \log(z)) - \log(2\,i\,\pi\,k + \log(z)) - \sum_{p=0}^{\infty} \frac{(-1)^p}{(2\,i\,\pi\,k + \log(z))^p} \sum_{j=1}^{p} \frac{S_p^{(p-j+1)} \log^j(2\,i\,\pi\,k + \log(z))}{j!} \; /; (|z| \to 0 \,|\, \infty)$$

# Integral representations

## On the real axis

### Of the direct function

01.32.07.0001.01

$$W_k(z) == 1 + (\log(z) + 2\,i\,\pi\,k - 1) \exp\left(\frac{i}{2\,\pi} \int_0^{\infty} \frac{1}{t+1} \log\left(\frac{\log(z) + t - \log(t) + (2\,k-1)\,i\,\pi}{\log(z) + t - \log(t) + (2\,k+1)\,i\,\pi}\right) dt\right) /; k \in \mathbb{Z} \bigwedge z \notin \left(-\frac{1}{e}, 0\right)$$

# Limit representations

01.32.09.0001.01

$$W_{-1}(-\log(x)) == -\log(x) \lim_{n \to \infty} \underbrace{\log_x(\log_x(\ldots \log_x(e)))}_{n-\text{times}} \; /; x > 1$$

# Differential equations

## Ordinary nonlinear differential equations

01.32.13.0001.01

$$w'(z)\, z\, (w(z) + 1) - w(z) == 0 \; /; w(z) == W_k(c_1\, z)$$

# Identities

## Functional identities

01.32.17.0001.01

$$e^{n\, w(z)} == z^n\, w(z)^{-n} \; /; w(z) == W_k(z) \land n \in \mathbb{Z}$$

# Differentiation

## Low-order differentiation

01.32.20.0001.01

$$\frac{\partial W_k(z)}{\partial z} == \frac{W_k(z)}{z\,(W_k(z)+1)}$$

01.32.20.0002.01

$$\frac{\partial^2 W_k(z)}{\partial z^2} == -\frac{W_k(z)^2\,(W_k(z)+2)}{z^2\,(W_k(z)+1)^3}$$

### Symbolic differentiation

01.32.20.0003.01

$$\frac{\partial^n W_k(z)}{\partial z^n} == \text{boole}(n == 0,\ W_k(z)) + \frac{n!\,W_k(z)^n}{z^n\,(W_k(z)+1)^{2n-1}} \sum_{q=0}^{n} \sum_{m=0}^{q} \sum_{l=0}^{q-m} \sum_{j=0}^{q-l-m} \frac{(-1)^{q-j+n-1}\,(l+n)^{j+l+n-1} \binom{l+n}{n} (-2n)_{q-j-l-m}}{j!\,(l+n)!\,(q-j-l-m)!} W_k(z)^q \text{ /; } n \in \mathbb{N}^+$$

Eric Weisstein and Oleg Marichev

## Integration

### Indefinite integration

#### Involving only one direct function

01.32.21.0001.01

$$\int W_k(z)\,dz == \frac{z\left(W_k(z)^2 - W_k(z) + 1\right)}{W_k(z)}$$

#### Involving one direct function and elementary functions

### Involving power function

01.32.21.0002.01

$$\int z^{\alpha-1}\,W_k(z)\,dz == \frac{1}{\alpha^2}\left(e^{-(\alpha-1)\,W_k(z)-W_k(z)}\,z^{\alpha}\,(\alpha\,\Gamma(\alpha+1,\,-\alpha\,W_k(z)) - \Gamma(\alpha+2,\,-\alpha\,W_k(z)))\,(-\alpha\,W_k(z))^{-\alpha}\right)$$

## Representations through equivalent functions

### With inverse function

01.32.27.0001.01

$$e^{W_k(z)}\,W_k(z) == z$$

### With related functions

01.32.27.0002.01

$$e^{n\,W_k(z)} == z^n\,W_k(z)^{-n} \text{ /; } n \in \mathbb{Z}$$

01.32.27.0003.01

$$\log(W_k(z)) == \log(z) - W_k(z) + 2\,i\,\pi\,k$$

01.32.27.0004.01

$$\log(W_{-1}(z)) == \log(z) - W_{-1}(z) - 2\,i\,\pi \;/;\; z \notin \left(-\infty, -\frac{1}{e}\right) \bigwedge z \notin (0, \infty)$$

01.32.27.0005.01

$$\log(W_{-1}(x)) == \log(x) - W_{-1}(x) \;/;\; -\frac{1}{e} < x < 0$$

## History

– J. H. Lambert (1758)

– L. Euler (1764, 1779)

- R. Corless, D. Knuth (1993)

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