

ProductLog2

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Notations

Traditional name

Generalized Lambert function

Traditional notation

$W_k(z)$

Mathematica StandardForm notation

ProductLog[k, z]

LambertW[k, z]

Primary definition

01.32.02.0001.01

$z = w e^w$; $w = W_k(z) \wedge k \in \mathbb{Z}$

Specific values

Specialized values

01.32.03.0001.01

$W_0(z) = W(z)$

01.32.03.0002.01

$W_k(0) = -\infty$; $k \neq 0$

01.32.03.0003.01

$W_{-1}\left(-\frac{\pi}{2}\right) = -\frac{i\pi}{2}$

01.32.03.0004.01

$W_{-1}\left(-\frac{1}{e}\right) = -1$

General characteristics

Domain and analyticity

$W_k(z)$ is an analytical function of z which is defined over \mathbb{C} .

01.32.04.0001.01

$$(k * z) \rightarrow W_k(z) :: (\mathbb{Z} \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

Symmetries and periodicities

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $W_k(z)$ does not have poles and essential singularities.

01.32.04.0002.01

$$\text{Sing}_z(W_k(z)) = \{\}$$

Branch points

With respect to z

The function $W_k(z)$, $k \neq 0$, $k \neq 1$, has two branch points: $z = 0$, $z = \infty$.

01.32.04.0003.01

$$\mathcal{BP}_z(W_k(z)) = \{0, \infty\} /; k \notin \{0, 1\}$$

The function $W_0(z)$ has two branch points: $z = -\frac{1}{e}$, $z = \infty$.

01.32.04.0004.01

$$\mathcal{BP}_z(W_0(z)) = \left\{-\frac{1}{e}, \infty\right\}$$

The function $W_1(z)$ has three branch points: $z = 0$, $z = -\frac{1}{e}$, $z = \infty$.

01.32.04.0005.01

$$\mathcal{BP}_z(W_1(z)) = \left\{0, -\frac{1}{e}, \infty\right\}$$

01.32.04.0006.01

$$\mathcal{R}_z(W_k(z), 0) = \log$$

01.32.04.0007.01

$$\mathcal{R}_z\left(W_1(z), -\frac{1}{e}\right) = 2$$

01.32.04.0008.01

$$\mathcal{R}_z(W_k(z), \infty) = \log$$

Branch cuts

With respect to z

The function $W_k(z)$, $k \neq 0$, is a single-valued function on the z -plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

01.32.04.0009.01

$$\mathcal{BC}_z(W_k(z)) = \{(-\infty, 0), -i\} /; k \notin \{0, 1\}$$

01.32.04.0010.01

$$\lim_{\epsilon \rightarrow +0} W_k(x + i\epsilon) = W_k(x) /; k \neq 0 \wedge k \neq 1 \wedge x < 0$$

01.32.04.0011.01

$$\lim_{\epsilon \rightarrow +0} W_k(x - i\epsilon) = W_{k-1}(x) /; k \neq 0 \wedge k \neq 1 \wedge x < 0$$

01.32.04.0012.01

$$\mathcal{BC}_z(W_0(z)) = \left\{ \left(\left(-\infty, -\frac{1}{e} \right), -i \right) \right\}$$

01.32.04.0013.01

$$\lim_{\epsilon \rightarrow +0} W_0(x + i\epsilon) = W_0(x) /; x < -\frac{1}{e}$$

01.32.04.0014.01

$$\lim_{\epsilon \rightarrow +0} W_0(x - i\epsilon) = W_{-1}(x) /; x < -\frac{1}{e}$$

01.32.04.0015.01

$$\mathcal{BC}_z(W_1(z)) = \left\{ \left(\left(-\infty, -\frac{1}{e} \right), -i \right), \left(\left(-\frac{1}{e}, 0 \right), -i \right) \right\}$$

01.32.04.0016.01

$$\lim_{\epsilon \rightarrow +0} W_1(x + i\epsilon) = W_1(x) /; x < 0$$

01.32.04.0017.01

$$\lim_{\epsilon \rightarrow +0} W_1(x - i\epsilon) = W_0(x) /; x < -\frac{1}{e}$$

01.32.04.0018.01

$$\lim_{\epsilon \rightarrow +0} W_1(x - i\epsilon) = W_{-1}(x) /; -\frac{1}{e} < x < 0$$

Series representations

Asymptotic series expansions

01.32.06.0001.02

$$W_k(z) \propto w - \zeta + \frac{\zeta}{w} - \frac{2\zeta - \zeta^2}{2w^2} + \frac{\zeta(6 - 9\zeta + 2\zeta^2)}{6w^3} + \dots /;$$

$$(z \rightarrow 0 \mid \infty) \wedge w = 2i\pi k + \log(z) \wedge \zeta = \log(w) \wedge \neg(k = 0 \wedge |z| < e^2) \wedge \neg(k = -1 \wedge z \in \left(-\frac{1}{e}, 0\right))$$

01.32.06.0002.01

$$W_k(z) = w - \zeta - \sum_{n=1}^{\infty} w^{-n} \sum_{m=1}^n \frac{(-1)^n}{m!} S_n^{(-m+n+1)} \zeta^m /;$$

$$(z \rightarrow 0 | \infty) \wedge w = 2 i \pi k + \log(z) \wedge \zeta = \log(w) \wedge \neg(k = 0 \wedge |z| < e^2) \wedge \neg(k = -1 \wedge z \in \left(-\frac{1}{e}, 0\right))$$

01.32.06.0003.01

$$W_k(z) \propto \log(z) + 2 i \pi k - \log(2 i \pi k + \log(z)) \left(1 + O\left(\frac{1}{\log(z)}\right)\right) /; (|z| \rightarrow 0 | \infty) \wedge k \neq 0 \wedge k \neq 1$$

01.32.06.0004.01

$$W_k(z) = (2 i \pi k + \log(z)) - \log(2 i \pi k + \log(z)) - \sum_{p=0}^{\infty} \frac{(-1)^p}{(2 i \pi k + \log(z))^p} \sum_{j=1}^p \frac{S_p^{(p-j+1)} \log^j(2 i \pi k + \log(z))}{j!} /; (|z| \rightarrow 0 | \infty)$$

Integral representations

On the real axis

Of the direct function

01.32.07.0001.01

$$W_k(z) = 1 + (\log(z) + 2 i \pi k - 1) \exp\left(\frac{i}{2 \pi} \int_0^{\infty} \frac{1}{t+1} \log\left(\frac{\log(z) + t - \log(t) + (2k-1)i\pi}{\log(z) + t - \log(t) + (2k+1)i\pi}\right) dt\right) /; k \in \mathbb{Z} \wedge z \notin \left(-\frac{1}{e}, 0\right)$$

Limit representations

01.32.09.0001.01

$$W_{-1}(-\log(x)) = -\log(x) \lim_{n \rightarrow \infty} \underbrace{\log_x(\log_x(\dots \log_x(e)))}_{n\text{-times}} /; x > 1$$

Differential equations

Ordinary nonlinear differential equations

01.32.13.0001.01

$$w'(z) z(w(z) + 1) - w(z) = 0 /; w(z) = W_k(c_1 z)$$

Identities

Functional identities

01.32.17.0001.01

$$e^{n w(z)} = z^n w(z)^{-n} /; w(z) = W_k(z) \wedge n \in \mathbb{Z}$$

Differentiation

Low-order differentiation

01.32.20.0001.01

$$\frac{\partial W_k(z)}{\partial z} = \frac{W_k(z)}{z(W_k(z) + 1)}$$

01.32.20.0002.01

$$\frac{\partial^2 W_k(z)}{\partial z^2} = -\frac{W_k(z)^2(W_k(z) + 2)}{z^2(W_k(z) + 1)^3}$$

Symbolic differentiation

01.32.20.0003.01

$$\frac{\partial^n W_k(z)}{\partial z^n} = \text{boole}(n = 0, W_k(z)) + \frac{n! W_k(z)^n}{z^n (W_k(z) + 1)^{2n-1}} \sum_{q=0}^n \sum_{m=0}^q \sum_{l=0}^{q-m} \sum_{j=0}^{q-l-m} \frac{(-1)^{q-j+n-1} (l+n)^{j+l+n-1} \binom{l+n}{n} (-2n)_{q-j-l-m}}{j!(l+n)!(q-j-l-m)!} W_k(z)^q /;$$

$n \in \mathbb{N}^+$

Eric Weisstein and Oleg Marichev

Integration

Indefinite integration

Involving only one direct function

01.32.21.0001.01

$$\int W_k(z) dz = \frac{z(W_k(z)^2 - W_k(z) + 1)}{W_k(z)}$$

Involving one direct function and elementary functions

Involving power function

01.32.21.0002.01

$$\int z^{\alpha-1} W_k(z) dz = \frac{1}{\alpha^2} (e^{-(\alpha-1)W_k(z)-W_k(z)} z^\alpha (\alpha \Gamma(\alpha + 1, -\alpha W_k(z)) - \Gamma(\alpha + 2, -\alpha W_k(z))) (-\alpha W_k(z))^{-\alpha})$$

Representations through equivalent functions

With inverse function

01.32.27.0001.01

$$e^{W_k(z)} W_k(z) = z$$

With related functions

01.32.27.0002.01

$$e^n W_k(z) = z^n W_k(z)^{-n} /; n \in \mathbb{Z}$$

01.32.27.0003.01

$$\log(W_k(z)) = \log(z) - W_k(z) + 2i\pi k$$

01.32.27.0004.01

$$\log(W_{-1}(z)) = \log(z) - W_{-1}(z) - 2i\pi /; z \notin \left(-\infty, -\frac{1}{e}\right) \wedge z \notin (0, \infty)$$

01.32.27.0005.01

$$\log(W_{-1}(x)) = \log(x) - W_{-1}(x) /; -\frac{1}{e} < x < 0$$

History

- J. H. Lambert (1758)
- L. Euler (1764, 1779)
- R. Corless, D. Knuth (1993)

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