RamanujanTau

Notations

Traditional name

Ramanujan tau function

Traditional notation

$\tau(n)$

Mathematica StandardForm notation

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RamanujanTau[n]
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Primary definition

13.13.02.0001.01

$$\text{RamanujanTau}(n) = \frac{1}{2}\int_{i\gamma-1}^{i\gamma+1} e^{-2\pi i n z}\,\eta(z)^{24}\,dz\;/;\; n\in\mathbb{Z}\bigwedge n\geq 0\bigwedge \operatorname{Re}(\gamma)>0$$

Specific values

Values at fixed points

13.13.03.0001.01

$\tau(0)=0$

13.13.03.0002.01

$\tau(1)=1$

13.13.03.0003.01

$\tau(2)=-24$

13.13.03.0004.01

$\tau(3)=252$

13.13.03.0005.01

$\tau(4)=-1472$

13.13.03.0006.01

$\tau(5)=4830$

13.13.03.0007.01

$\tau(6)=-6048$

13.13.03.0008.01

$\tau(7) = -16744$

13.13.03.0009.01

$\tau(8) = 84480$

13.13.03.0010.01

$\tau(9) = -113643$

13.13.03.0011.01

$\tau(10) = -115920$

General characteristics

Domain and analyticity

$\tau(n)$ is a nonanalytical function which is defined for nonnegative integer n.

13.13.04.0001.01

$n \longrightarrow \tau(n) :: \mathbb{Z} \longrightarrow \mathbb{Z}$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

13.13.06.0001.01

$$\tau(n) = n^4 \sigma_1(n) - 24 \sum_{k=1}^{n-1} \left(35 k^4 - 52 n k^3 + 18 n^2 k^2\right) \sigma_1(k) \sigma_1(n-k) /; n > 0$$

13.13.06.0002.01

$$\tau(n) = \frac{691}{756} \sigma_5(n) + \frac{65}{756} \sigma_{11}(n) - \frac{691}{3} \sum_{k=1}^{n-1} \sigma_5(k) \sigma_5(n-k) /; n > 0$$

13.13.06.0003.01

$$\tau(n) = \frac{691}{1800} \sigma_3(n) + \frac{691}{900} \sigma_7(n) - \frac{91}{600} \sigma_{11}(n) + \frac{2764}{15} \sum_{k=1}^{n-1} \sigma_3(k) \sigma_7(n-k) /; n > 0$$

Asymptotic series expansions

13.13.06.0004.01

$$\tau(n) \propto O\left(n^6\right) /; (n \to \infty)$$

Transformations

Multiple arguments

13.13.16.0001.01

$$\tau(n\,m) = \tau(n)\,\tau(m)\,/;\ \gcd(n,\,m) = 1$$

Identities

Functional identities

13.13.17.0001.01

$$(n-1)\,\tau(n) = \sum_{k=1}^{\left\lfloor \frac{1}{2}\left(\sqrt{8n+1}-1\right)\right\rfloor} (-1)^{k-1}\,(2k+1)\left(n-1-\frac{9k(k+1)}{2}\right)\tau\left(n-\frac{k(k+1)}{2}\right)$$

13.13.17.0002.01

$$\tau\left(p_j^n\right) = \sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} (-1)^k \binom{n-k}{n-2k} p_j^{11k}\,\tau\left(p_j\right)^{n-2k}\ /;\ p_j \in \mathbb{P}$$

13.13.17.0003.01

$$\tau\left(p_j^{n+1}\right) = \tau\left(p_j\right)\tau\left(p_j^n\right) - p_j^{11}\,\tau\left(p_j^{n-1}\right)\ /;\ p_j \in \mathbb{P} \land n > 0$$

13.13.17.0004.01

$$\tau\left(n\,p_j^m\right) = \tau\left(p_j\right)\tau\left(n\,p_j^{m-1}\right) - p_j^{11}\,\tau\left(n\,p_j^{m-2}\right)\ /;\ n \in \mathbb{N}^+ \land m \in \mathbb{Z} \land m \geq 2 \land \gcd\left(n,\,p_j\right) = 1 \land p_j \in \mathbb{P}$$

Summation

Infinite summation

13.13.23.0001.01

$$\sum_{k=0}^{\infty} \text{RamanujanTau}(k)\,z^k = \eta\left(-\frac{i\log(z)}{2\pi}\right)^{24}\ /;\ |z| < 1$$

13.13.23.0002.01

$$\sum_{k=1}^{\infty} \text{RamanujanTau}(k)\,z^k = z\prod_{k=1}^{\infty}\left(1-z^k\right)^{24}\ /;\ |z| < 1$$

13.13.23.0003.01

$$\sum_{k=1}^{\infty} \text{RamanujanTau}(k)\,z^k = z\left(\sum_{k=0}^{\infty}(-1)^k\,(2k+1)\,z^{\frac{1}{2}k(k+1)}\right)^8\ /;\ |z| < 1$$

13.13.23.0004.01

$$\sum_{k=1}^{\infty} \text{RamanujanTau}(k)\,e^{2\pi i k z} = \eta(z)^{24}$$

Representations through equivalent functions

With related functions

Involving other related functions

13.13.27.0001.01

$$\tau(2\,n+1) = \frac{691}{33\,152}\, r_{24}(2\,n+1) - \frac{1}{2072}\, \sigma_{11}(2\,n+1)$$

Inequalities

13.13.29.0001.01

$$\tau(n) \neq 0 \;/;\; n > 0 \land n < 214\,928\,639\,999$$

13.13.29.0002.01

$$|\tau(n)| \leq \sigma_k(n)\, n^{11/2} \;/;\; n > 0$$

13.13.29.0003.01

$$|\tau(n)| \leq \left(2\,\chi^2 + \chi\right) n^5 \log^2(\log(n)) \;/;\; \chi = e^{\gamma} + \frac{6482}{\log^2(\log(3))} \bigwedge n > 0$$

History

– S. Ramanujan (1916)
– L.J. Mordell (1917) proved that $\tau(n)$ is a multiplicative function
– Hardy (1927)
– Wilton (1929)
– G.N. Watson (1935)
– Walfisz (1938)
– Bambah and Chowla (1947)
– Sengupta (1948)
– van der Blij (1948)

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