SpheroidalPS

Notations

Traditional name

Angular spheroidal function of the first kind

Traditional notation

$PS_{\nu,\mu}(\gamma, z)$

Mathematica StandardForm notation

```
SpheroidalPS[ν, μ, γ, z]
```

Primary definition

11.08.02.0001.01

$PS_{\nu,\mu}(\gamma, z)$

$PS_{\nu,\mu}(\gamma, z)$ is the angular spheroidal function of the first kind with variable z and parameters ν, μ, γ. It is defined as the normalizable solution $w(z) == PS_{\nu,\mu}(\gamma, z)$ of the wave differential equation $(1-z^2)\, w''(z) - 2\, z\, w'(z) + \left(\lambda + \gamma^2 (1-z^2) - \mu^2 / (1-z^2)\right) w(z) == 0$ with parameter λ equal to spheroidal eigenvalue $\lambda == \lambda_{\nu,\mu}(\gamma)$. The parameter ν enumerates the spheroidal eigenvalues in such a manner that in the spherical limit ($\gamma \to 0$), the eigenvalues are $\lambda_{\nu,\mu}(0) == \nu(\nu+1)$ and $PS_{\nu,\mu}(0, z) = P_\nu^\mu(z)$, where $P_\nu^\mu(z)$ is the Legendre function of the first kind. The angular spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme, meaning $\int_{-1}^{1} PS_{n_1,m}(\gamma, \eta)\, PS_{n_2,m}(\gamma, \eta)\, d\eta = 2\,(m+n_1)!\, / ((2\, n_1 + 1)\, (n_1 - m)!)\, \delta_{n_1,n_2} /; n_1 \in \mathbb{N} \bigwedge n_2 \in \mathbb{N} \bigwedge m \in \mathbb{N} \bigwedge n_1 \geq m \bigwedge n_2 \geq m$. $PS_{\nu,\mu}(\gamma, z)$ is an analytical function in the variables ν, μ, γ and z.

Specific values

Specialized values

For fixed ν, μ, z

11.08.03.0001.01

$PS_{\nu,\mu}(0, z) = P_\nu^\mu(z)$

11.08.03.0002.01

$$PS_{\nu,\mu}(0, z) = \frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}} P_\nu^\mu(z)$$

For fixed ν, γ, z

11.08.03.0003.01

$$PS_{\nu,\frac{1}{2}}(\gamma, z) = \frac{\sqrt{2}}{\sqrt{\pi}\,\sqrt[4]{1-z^2}} \operatorname{Ce}\left(a_{\nu+\frac{1}{2}}\left(\frac{\gamma^2}{4}\right), \frac{\gamma^2}{4}, \cos^{-1}(z)\right)$$

For fixed ν, z

11.08.03.0004.01

$$PS_{\nu,0}(0, z) = P_\nu(z)$$

11.08.03.0005.01

$$PS_{\nu,\frac{1}{2}}(0, z) = \frac{\sqrt{\pi}}{\sqrt{2}\,\sqrt[4]{1-z^2}} \cos\left(\left(\nu + \frac{1}{2}\right)\cos^{-1}(z)\right)$$

11.08.03.0006.01

$$PS_{\nu,\frac{1}{2}}(0, z) = \frac{\sqrt{\pi}}{\sqrt{2}\,\sqrt[4]{1-z^2}} T_{\nu+\frac{1}{2}}(z)$$

General characteristics

Domain and analyticity

$PS_{\nu,\mu}(\gamma, z)$ is an analytical function of ν, μ, γ, z which is defined in $\mathbb{C}^4$.

11.08.04.0001.01

$$(\nu * \mu * \gamma * z) \longrightarrow PS_{\nu,\mu}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$PS_{\nu,\mu}(\gamma, z)$ is an even function with respect to γ.

11.08.04.0002.01

$$PS_{\nu,\mu}(-\gamma, z) = PS_{\nu,\mu}(\gamma, z)$$

11.08.04.0003.01

$$PS_{-\nu,\mu}(\gamma, z) = PS_{\nu-1,\mu}(\gamma, z)$$

Mirror symmetry

11.08.04.0004.01

$$PS_{\overline{\nu},\overline{\mu}}(\overline{\gamma}, \overline{z}) == \overline{PS_{\nu,\mu}(\gamma, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at $\gamma = 0$

11.08.06.0001.01

$$PS_{\nu,\mu}(\gamma, z) \propto P_\nu^\mu(z) + \frac{1}{2(2\nu+1)}\left(\frac{(-\mu+\nu+1)(-\mu+\nu+2)}{(2\nu+3)^2} P_{\nu+2}^\mu(z) - \frac{(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^2} P_{\nu-2}^\mu(z)\right)\gamma^2 + \frac{1}{8}\left(\frac{(\mu+\nu-3)(\mu+\nu-2)(\mu+\nu-1)(\mu+\nu)}{(4\nu^2-8\nu-5)(4\nu^2-8\nu+3)^2} P_{\nu-4}^\mu(z) + \frac{8(4\mu^2-1)(\mu^2+(2\nu-1)\mu+(\nu-1)\nu)}{(1-2\nu)^4(8\nu^3-4\nu^2-34\nu-15)} P_{\nu-2}^\mu(z) - \frac{1}{2\nu+1}\left(\frac{(\mu-\nu)(\mu-\nu+1)(\mu+\nu-1)(\mu+\nu)}{(1-2\nu)^4(2\nu-3)} + \frac{(-\mu+\nu+1)(-\mu+\nu+2)(\mu+\nu+1)(\mu+\nu+2)}{(2\nu+3)^4(2\nu+5)}\right) P_\nu^\mu(z) + \frac{(-\mu+\nu+1)(-\mu+\nu+2)(-\mu+\nu+3)(-\mu+\nu+4)}{(2\nu+1)(2\nu+3)^2(2\nu+5)^2(2\nu+7)} P_{\nu+4}^\mu(z) - \frac{8(4\mu^2-1)(\mu^2-(2\nu+3)\mu+\nu^2+3\nu+2)}{(2\nu+3)^4(8\nu^3+28\nu^2-2\nu-7)} P_{\nu+2}^\mu(z)\right)\gamma^4 + \ldots /; (\gamma \to 0)$$

11.08.06.0002.01

$$PS_{\nu,\mu}(\gamma, z) \propto P_\nu^\mu(z)\left(1 + O(\gamma^2)\right)$$

Expansions at generic point $z = z_0$

11.08.06.0003.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, z_0) + PS_{\nu,\mu}'(\gamma, z_0)(z-z_0) - \frac{1}{2(z_0^2-1)^2}\left(2 PS_{\nu,\mu}'(\gamma, z_0) z_0 (z_0^2-1) + PS_{\nu,\mu}(\gamma, z_0)\left(-\mu^2 + \gamma^2 (z_0^2-1)^2 - \lambda_{\nu,\mu}(\gamma)(z_0^2-1)\right)\right)(z-z_0)^2 - \frac{1}{6(z_0^2-1)^3}\left(PS_{\nu,\mu}'(\gamma, z_0)(z_0^2-1)\left(\gamma^2 z_0^4 - 2(\gamma^2+3) z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma)(z_0^2-1) - 2\right) - 2 PS_{\nu,\mu}(\gamma, z_0) z_0 \left(-3\mu^2 + \gamma^2 (z_0^2-1)^2 - 2\lambda_{\nu,\mu}(\gamma)(z_0^2-1)\right)\right)(z-z_0)^3 + \ldots /; (z \to z_0)$$

11.08.06.0004.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, z_0)(1 + O(z-z_0))$$

Expansions at $z = 0$

11.08.06.0005.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, 0) + PS_{\nu,\mu}'(\gamma, 0) z - \frac{1}{2}\left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) PS_{\nu,\mu}(\gamma, 0) z^2 + \frac{1}{6}\left(2 PS_{\nu,\mu}'(\gamma, 0) - \left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) PS_{\nu,\mu}'(\gamma, 0)\right) z^3 + \ldots /; (z \to 0)$$

11.08.06.0006.01

$$PS_{\nu,\mu}(\gamma, z) \propto PS_{\nu,\mu}(\gamma, 0)(1 + O(z))$$

Other series representations

11.08.06.0007.01

$$M_{\nu,\mu}(0) = 0 \;/;\; \operatorname{Re}(\mu) > -\frac{1}{2}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.08.13.0001.01

$$\left(1 - z^2\right) w''(z) - 2\, z\, w'(z) + \left(\left(1 - z^2\right) \gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1 - z^2}\right) w(z) = 0 \;/;\; w(z) = c_1\, PS_{\nu,\mu}(\gamma, z) + c_2\, QS_{\nu,\mu}(\gamma, z)$$

11.08.13.0002.01

$$W_z\left(PS_{\nu,\mu}(\gamma, z), QS_{\nu,\mu}(\gamma, z)\right) = \frac{1}{1 - z^2} \left(PS_{\nu,\mu}(\gamma, 0)\, QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0)\, QS_{\nu,\mu}(\gamma, 0)\right)$$

11.08.13.0003.01

$$\left(1 - g(z)^2\right) w''(z) + \left(\left(g(z)^2 - 1\right) \frac{g''(z)}{g'(z)} - 2\, g(z)\, g'(z)\right) w'(z) + \left(\frac{\left(\gamma\, g(z)^2 + \mu - \gamma\right)\left(-\gamma\, g(z)^2 + \mu + \gamma\right) g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu,\mu}(\gamma)\, g'(z)^2\right) w(z) = 0 \;/;$$
$$w(z) = c_1\, PS_{\nu,\mu}(\gamma, g(z)) + c_2\, QS_{\nu,\mu}(\gamma, g(z))$$

11.08.13.0004.01

$$W_z\left(PS_{\nu,\mu}(\gamma, g(z)), QS_{\nu,\mu}(\gamma, g(z))\right) = \frac{g'(z)}{1 - g[z]^2} \left(PS_{\nu,\mu}(\gamma, 0)\, QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0)\, QS_{\nu,\mu}(\gamma, 0)\right)$$

11.08.13.0005.01

$$\left(1 - g(z)^2\right) w''(z) + \left(\left(g(z)^2 - 1\right)\left(\frac{2\, h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) - 2\, g(z)\, g'(z)\right) w'(z) +$$
$$\left(\frac{\left(\gamma\, g(z)^2 + \mu - \gamma\right)\left(-\gamma\, g(z)^2 + \mu + \gamma\right) g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu,\mu}(\gamma)\, g'(z)^2 + \frac{2\, g(z)\, h'(z)\, g'(z)}{h(z)} + \frac{\left(g(z)^2 - 1\right)\left(h(z)\, h''(z) - 2\, h'(z)^2\right)}{h(z)^2} - \right.$$
$$\left. \frac{\left(g(z)^2 - 1\right) h'(z)\, g''(z)}{h(z)\, g'(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\, h(z)\, PS_{\nu,\mu}(\gamma, g(z)) + c_2\, h(z)\, QS_{\nu,\mu}(\gamma, g(z))$$

11.08.13.0006.01

$$W_z\left(h(z)\, PS_{\nu,\mu}(\gamma, g(z)), h(z)\, QS_{\nu,\mu}(\gamma, g(z))\right) = \frac{h(z)^2\, g'(z)}{1 - g[z]^2} \left(PS_{\nu,\mu}(\gamma, 0)\, QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0)\, QS_{\nu,\mu}(\gamma, 0)\right)$$

11.08.13.0007.01

$$\left(1 - a^2\, z^{2r}\right) w''(z) - \frac{a^2\, (r - 2\, s + 1)\, z^{2r} + r + 2\, s - 1}{z}\, w'(z) +$$
$$\left(a^2\, r^2\, \lambda_{\nu,\mu}(\gamma)\, z^{2r-2} + \frac{-a^2\, r^2 \left(\left(a^2\, z^{2r} - 1\right)^2 \gamma^2 - \mu^2\right) z^{2r} - s^2 \left(a^2\, z^{2r} - 1\right)^2 + r\, s \left(a^4\, z^{4r} - 1\right)}{z^2 \left(a^2\, z^{2r} - 1\right)}\right) w(z) =$$
$$0 \;/;\; w(z) = c_1\, z^s\, PS_{\nu,\mu}(\gamma, a\, z^r) + c_2\, z^s\, QS_{\nu,\mu}(\gamma, a\, z^r)$$

11.08.13.0008.01

$$W_z\left(z^s PS_{\nu,\mu}(\gamma, a z^r), z^s QS_{\nu,\mu}(\gamma, a z^r)\right) = \frac{a r z^{r+2s-1}}{1 - a^2 z^{2r}} \left(PS_{\nu,\mu}(\gamma, 0) QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0) QS_{\nu,\mu}(\gamma, 0)\right)$$

11.08.13.0009.01

$$\left(1 - a^2 r^{2z}\right) w''(z) + \left(-a^2 \log(r) r^{2z} + 2 a^2 \log(s) r^{2z} - \log(r) - 2\log(s)\right) w'(z) +$$
$$\left(a^2 \log^2(r) \lambda_{\nu,\mu}(\gamma) r^{2z} + \frac{1}{a^2 r^{2z} - 1} \left(-a^2 \left(\left(a^2 r^{2z} - 1\right)^2 \gamma^2 - \mu^2\right) \log^2(r) r^{2z} - \left(a^2 r^{2z} - 1\right)^2 \log^2(s) + \left(a^4 r^{4z} - 1\right) \log(r) \log(s)\right)\right)$$
$$w(z) = 0 /; w(z) = c_1 s^z PS_{\nu,\mu}(\gamma, a r^z) + c_2 s^z QS_{\nu,\mu}(\gamma, a r^z)$$

11.08.13.0010.01

$$W_z\left(s^z PS_{\nu,\mu}(\gamma, a r^z), s^z QS_{\nu,\mu}(\gamma, a r^z)\right) = \frac{a r^z s^{2z} \log(r)}{1 - a^2 r^{2z}} \left(PS_{\nu,\mu}(\gamma, 0) QS_{\nu,\mu}'(\gamma, 0) - PS_{\nu,\mu}'(\gamma, 0) QS_{\nu,\mu}(\gamma, 0)\right)$$

Differentiation

Low-order differentiation

With respect to z

11.08.20.0001.01

$$\frac{\partial PS_{\nu,\mu}(\gamma, z)}{\partial z} = PS_{\nu,\mu}'(\gamma, z)$$

11.08.20.0002.01

$$\frac{\partial^2 PS_{\nu,\mu}(\gamma, z)}{\partial z^2} = \frac{1}{1 - z^2} \left(2 z PS_{\nu,\mu}'(\gamma, z) - \left(\left(1 - z^2\right) \gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1 - z^2}\right) PS_{\nu,\mu}(\gamma, z)\right)$$

Integration

Definite integration

Involving the direct function

11.08.21.0001.01

$$\int_{-1}^{1} PS_{n,m}(\gamma, t)^2 \, dt = \frac{2 (m+n)!}{(2n+1)(n-m)!} /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \le n$$

11.08.21.0002.01

$$\int_{-1}^{1} PS_{n_1,m}(\gamma, t) PS_{n_2,m}(\gamma, t) \, dt = \frac{(2 (m+n_1)!) \delta_{n_1,n_2}}{(2 n_1 + 1)(n_1 - m)!} /; n_1 \in \mathbb{Z} \bigwedge n_2 \in \mathbb{Z} \bigwedge m \in \mathbb{Z} \bigwedge n_1 \ge m \bigwedge n_2 \ge m$$

Representations through equivalent functions

Theorems

History

Copyright