

SpheroidalQS

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Notations

Traditional name

Angular spheroidal function of the second kind

Traditional notation

$$QS_{\nu,\mu}(\gamma, z)$$

Mathematica StandardForm notation

SpheroidalQS[ν, μ, γ, z]

Primary definition

11.09.02.0001.01

$$QS_{\nu,\mu}(\gamma, z)$$

$QS_{\nu,\mu}(\gamma, z)$ is the angular spheroidal function of the first kind with variable z and parameters ν, μ, γ . It is defined as the normalizable solution $w(z) = QS_{\nu,\mu}(\gamma, z)$ of the wave differential equation $(1 - z^2)w''(z) - 2zw'(z) + (\lambda + \gamma^2(1 - z^2) - \mu^2/(1 - z^2))w(z) = 0$ with parameter λ equal to spheroidal eigenvalue $\lambda = \lambda_{\nu,\mu}(\gamma)$. The parameter ν enumerates the spheroidal eigenvalues in such a manner that in the limit ($\gamma \rightarrow 0$), the eigenvalues are $\lambda_{\nu,\mu}(0) = \nu(\nu + 1)$ and $QS_{\nu,\mu}(0, z) = Q_{\nu}^{\mu}(z)$, where $Q_{\nu}^{\mu}(z)$ is the Legendre function of the second kind. The angular spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme. $QS_{\nu,\mu}(\gamma, z)$ is an analytical function in the variables ν, μ, γ and z .

Specific values

Specialized values

For fixed ν, μ, z

11.09.03.0001.01

$$QS_{\nu,\mu}(0, z) = Q_{\nu}^{\mu}(z)$$

For fixed ν, γ, z

11.09.03.0002.01

$$QS_{\nu, \frac{1}{2}}(\gamma, z) = -\frac{\sqrt{\pi}}{\sqrt{2} \sqrt[4]{1-z^2}} \operatorname{Se}\left(b_{\nu+\frac{1}{2}}\left(\frac{\gamma^2}{4}\right), \frac{\gamma^2}{4}, \cos^{-1}(z)\right)$$

For fixed ν, z

11.09.03.0003.01

$$QS_{\nu, 0}(0, z) = Q_{\nu}(z)$$

11.09.03.0004.01

$$QS_{\nu, \frac{1}{2}}(0, z) = -\frac{\sqrt{\pi}}{\sqrt{2} \sqrt[4]{1-z^2}} \sin\left(\left(\nu + \frac{1}{2}\right) \cos^{-1}(z)\right)$$

11.09.03.0005.01

$$QS_{\nu, \frac{1}{2}}(0, z) = -\frac{\sqrt{\pi} \sqrt[4]{1-z^2}}{\sqrt{2}} U_{\nu-\frac{1}{2}}(z)$$

General characteristics

Domain and analyticity

$QS_{\nu, \mu}(\gamma, z)$ is an analytical function of ν, μ, γ, z which is defined in \mathbb{C}^4 .

11.09.04.0001.01

$$(\nu * \mu * \gamma * z) \rightarrow QS_{\nu, \mu}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$QS_{\nu, \mu}(\gamma, z)$ is an even function with respect to γ .

11.09.04.0002.01

$$QS_{\nu, \mu}(-\gamma, z) = QS_{\nu, \mu}(\gamma, z)$$

Mirror symmetry

11.09.04.0003.01

$$QS_{\bar{\nu}, \bar{\mu}}(\bar{\gamma}, \bar{z}) = \overline{QS_{\nu, \mu}(\gamma, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at $\gamma = 0$

11.09.06.0001.01

$$\begin{aligned}
 QS_{v,\mu}(\gamma, z) \propto & Q_v^\mu(z) + \frac{1}{2(2v+1)} \left(\frac{(-\mu+v+1)(-\mu+v+2)}{(2v+3)^2} Q_{v+2}^\mu(z) - \frac{(\mu+v-1)(\mu+v)}{(1-2v)^2} Q_{v-2}^\mu(z) \right) \gamma^2 + \\
 & \frac{1}{8} \left(\frac{(\mu+v-3)(\mu+v-2)(\mu+v-1)(\mu+v)}{(4v^2-8v-5)(4v^2-8v+3)^2} Q_{v-4}^\mu(z) + \frac{8(4\mu^2-1)(\mu^2+(2v-1)\mu+(v-1)v)}{(1-2v)^4(8v^3-4v^2-34v-15)} Q_{v-2}^\mu(z) - \right. \\
 & \frac{1}{2v+1} \left(\frac{(\mu-v)(\mu-v+1)(\mu+v-1)(\mu+v)}{(1-2v)^4(2v-3)} + \frac{(-\mu+v+1)(-\mu+v+2)(\mu+v+1)(\mu+v+2)}{(2v+3)^4(2v+5)} \right) Q_v^\mu(z) - \\
 & \frac{8(4\mu^2-1)(\mu^2-(2v+3)\mu+v^2+3v+2)}{(2v+3)^4(8v^3+28v^2-2v-7)} Q_{v+2}^\mu(z) + \\
 & \left. \frac{(-\mu+v+1)(-\mu+v+2)(-\mu+v+3)(-\mu+v+4)}{(2v+1)(2v+3)^2(2v+5)^2(2v+7)} Q_{v+4}^\mu(z) \right) \gamma^4 + \dots /; (\gamma \rightarrow 0)
 \end{aligned}$$

11.09.06.0002.01

$$QS_{v,\mu}(\gamma, z) \propto Q_v^\mu(z) (1 + O(\gamma^2))$$

Expansions at generic point $z = z_0$

11.09.06.0003.01

$$\begin{aligned}
 QS_{v,\mu}(\gamma, z) \propto & QS_{v,\mu}(\gamma, z_0) + QS_{v,\mu}'(\gamma, z_0) (z - z_0) - \\
 & \frac{1}{2(z_0^2 - 1)^2} \left(2 QS_{v,\mu}'(\gamma, z_0) z_0 (z_0^2 - 1) + QS_{v,\mu}(\gamma, z_0) \left(-\mu^2 + \gamma^2 (z_0^2 - 1)^2 - \lambda_{v,\mu}(\gamma) (z_0^2 - 1) \right) \right) (z - z_0)^2 + \\
 & \frac{1}{6(z_0^2 - 1)^3} \left(2 QS_{v,\mu}(\gamma, z_0) z_0 \left(-3\mu^2 + \gamma^2 (z_0^2 - 1)^2 - 2\lambda_{v,\mu}(\gamma) (z_0^2 - 1) \right) - \right. \\
 & \left. QS_{v,\mu}'(\gamma, z_0) (z_0^2 - 1) \left(\gamma^2 z_0^4 - 2(\gamma^2 + 3) z_0^2 + \gamma^2 - \mu^2 - \lambda_{v,\mu}(\gamma) (z_0^2 - 1) - 2 \right) \right) (z - z_0)^3 + \dots /; (z \rightarrow z_0)
 \end{aligned}$$

11.09.06.0004.01

$$QS_{v,\mu}(\gamma, z) \propto QS_{v,\mu}(\gamma, z_0) (1 + O(z - z_0))$$

Expansions at $z = 0$

11.09.06.0005.01

$$\begin{aligned}
 QS_{v,\mu}(\gamma, z) \propto & QS_{v,\mu}(\gamma, 0) + QS_{v,\mu}'(\gamma, 0) z - \frac{1}{2} (\gamma^2 - \mu^2 + \lambda_{v,\mu}(\gamma)) QS_{v,\mu}(\gamma, 0) z^2 + \\
 & \frac{1}{6} (2 QS_{v,\mu}'(\gamma, 0) - (\gamma^2 - \mu^2 + \lambda_{v,\mu}(\gamma)) QS_{v,\mu}''(\gamma, 0)) z^3 + \dots /; (z \rightarrow 0)
 \end{aligned}$$

11.09.06.0006.01

$$QS_{v,\mu}(\gamma, z) \propto QS_{v,\mu}(\gamma, 0) (1 + O(z))$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.09.13.0001.01

$$(1 - z^2) w''(z) - 2z w'(z) + \left((1 - z^2) \gamma^2 + \lambda_{\nu, \mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) w(z) = 0 /; w(z) = c_1 Q S_{\nu, \mu}(\gamma, z) + c_2 P S_{\nu, \mu}(\gamma, z)$$

11.09.13.0002.01

$$W_z(Q S_{\nu, \mu}(\gamma, z), P S_{\nu, \mu}(\gamma, z)) = \frac{1}{1 - z^2} (P S_{\nu, \mu}'(\gamma, 0) Q S_{\nu, \mu}(\gamma, 0) - P S_{\nu, \mu}(\gamma, 0) Q S_{\nu, \mu}'(\gamma, 0))$$

11.09.13.0003.01

$$(1 - g(z)^2) w''(z) + \left((g(z)^2 - 1) \frac{g''(z)}{g'(z)} - 2g(z)g'(z) \right) w'(z) + \left(\frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu, \mu}(\gamma)g'(z)^2 \right) w(z) = 0 /;$$

$$w(z) = c_1 Q S_{\nu, \mu}(\gamma, g(z)) + c_2 P S_{\nu, \mu}(\gamma, g(z))$$

11.09.13.0004.01

$$W_z(Q S_{\nu, \mu}(\gamma, g(z)), P S_{\nu, \mu}(\gamma, g(z))) = \frac{g'(z)}{1 - g(z)^2} (P S_{\nu, \mu}'(\gamma, 0) Q S_{\nu, \mu}(\gamma, 0) - P S_{\nu, \mu}(\gamma, 0) Q S_{\nu, \mu}'(\gamma, 0))$$

11.09.13.0005.01

$$(1 - g(z)^2) w''(z) + \left((g(z)^2 - 1) \left(\frac{2h'(z)}{h(z)} + \frac{g''(z)}{g'(z)} \right) - 2g(z)g'(z) \right) w'(z) + \left(\frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu, \mu}(\gamma)g'(z)^2 + \frac{2g(z)h'(z)g'(z)}{h(z)} + \frac{(g(z)^2 - 1)(h(z)h''(z) - 2h'(z)^2)}{h(z)^2} - \frac{(g(z)^2 - 1)h'(z)g''(z)}{h(z)g'(z)} \right) w(z) = 0 /; w(z) = c_1 h(z) Q S_{\nu, \mu}(\gamma, g(z)) + c_2 h(z) P S_{\nu, \mu}(\gamma, g(z))$$

11.09.13.0006.01

$$W_z(h(z) Q S_{\nu, \mu}(\gamma, g(z)), h(z) P S_{\nu, \mu}(\gamma, g(z))) = \frac{h(z)^2 g'(z)}{1 - g(z)^2} (P S_{\nu, \mu}'(\gamma, 0) Q S_{\nu, \mu}(\gamma, 0) - P S_{\nu, \mu}(\gamma, 0) Q S_{\nu, \mu}'(\gamma, 0))$$

11.09.13.0007.01

$$(1 - a^2 z^{2r}) w''(z) - \frac{a^2(r - 2s + 1)z^{2r} + r + 2s - 1}{z} w'(z) + \left(a^2 r^2 \lambda_{\nu, \mu}(\gamma) z^{2r-2} + \frac{-a^2 r^2 \left((a^2 z^{2r} - 1)^2 \gamma^2 - \mu^2 \right) z^{2r} - s^2 (a^2 z^{2r} - 1)^2 + r s (a^4 z^{4r} - 1)}{z^2 (a^2 z^{2r} - 1)} \right) w(z) =$$

$$0 /; w(z) = c_1 z^s Q S_{\nu, \mu}(\gamma, a z^r) + c_2 z^s P S_{\nu, \mu}(\gamma, a z^r)$$

11.09.13.0008.01

$$W_z(z^s Q S_{\nu, \mu}(\gamma, a z^r), z^s P S_{\nu, \mu}(\gamma, a z^r)) = \frac{a r z^{r+2s-1}}{1 - a^2 z^{2r}} (P S_{\nu, \mu}'(\gamma, 0) Q S_{\nu, \mu}(\gamma, 0) - P S_{\nu, \mu}(\gamma, 0) Q S_{\nu, \mu}'(\gamma, 0))$$

11.09.13.0009.01

$$(1 - a^2 r^{2z}) w''(z) + (-a^2 \log(r) r^{2z} + 2a^2 \log(s) r^{2z} - \log(r) - 2 \log(s)) w'(z) + \left(a^2 \log^2(r) \lambda_{\nu, \mu}(\gamma) r^{2z} + \frac{1}{a^2 r^{2z} - 1} \left(-a^2 \left((a^2 r^{2z} - 1)^2 \gamma^2 - \mu^2 \right) \log^2(r) r^{2z} - (a^2 r^{2z} - 1)^2 \log^2(s) + (a^4 r^{4z} - 1) \log(r) \log(s) \right) \right) w(z) = 0 /;$$

$$w(z) = c_1 s^z Q S_{\nu, \mu}(\gamma, a r^z) + c_2 s^z P S_{\nu, \mu}(\gamma, a r^z)$$

11.09.13.0010.01

$$W_z(s^z QS_{v,\mu}(\gamma, ar^z), s^z PS_{v,\mu}(\gamma, ar^z)) = \frac{ar^z s^{2z} \log(r)}{1 - a^2 r^{2z}} (PS_{v,\mu}'(\gamma, 0) QS_{v,\mu}(\gamma, 0) - PS_{v,\mu}(\gamma, 0) QS_{v,\mu}'(\gamma, 0))$$

Differentiation

Low-order differentiation

With respect to z

11.09.20.0001.01

$$\frac{\partial QS_{v,\mu}(\gamma, z)}{\partial z} = QS_{v,\mu}'(\gamma, z)$$

11.09.20.0002.01

$$\frac{\partial^2 QS_{v,\mu}(\gamma, z)}{\partial z^2} = \frac{1}{1 - z^2} \left(2z QS_{v,\mu}'(\gamma, z) - \left((1 - z^2) \gamma^2 + \lambda_{v,\mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) QS_{v,\mu}(\gamma, z) \right)$$

Integration

Representations through equivalent functions

Theorems

History

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