SpheroidalS1

Notations

Traditional name

Radial spheroidal function of the first kind

Traditional notation

$S_{\nu,\mu}^{(1)}(\gamma, z)$

Mathematica StandardForm notation

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SpheroidalS1[ν, μ, γ, z]
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Primary definition

11.10.02.0001.01

$S_{\nu,\mu}^{(1)}(\gamma, z)$

$S_{\nu,\mu}^{(1)}(\gamma, z)$ is the radial spheroidal function of the first kind with variable z and parameters ν, μ, γ. It is defined as the normalizable solution $w(z) == S_{\nu,\mu}^{(1)}(\gamma, z)$ of the wave differential equation $\left(1-z^2\right) w''(z) - 2 z\, w'(z) + \left(\lambda + \gamma^2\left(1-z^2\right) - \mu^2 / \left(1-z^2\right)\right) w(z) == 0$ with parameter λ equal to spheroidal eigenvalue $\lambda == \lambda_{\nu,\mu}(\gamma)$. The parameter ν enumerates the spheroidal eigenvalues in such a manner that in the limit $(\gamma \to 0)$, the eigenvalues are $\lambda_{\nu,\mu}(0) == \nu\,(\nu+1)$ and $\lim_{z\to 0} S_{\nu,\mu}^{(1)}(\gamma, z/\gamma) = j_\nu(z)$, where $j_\nu(z)$ is the spherical Bessel function of the first kind. The radial spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme, meaning $\lim_{z\to\infty} S_{\nu,\mu}^{(1)}(\gamma, z/\gamma) / j_\nu(z) = 1$. $S_{\nu,\mu}^{(1)}(\gamma, z)$ is an analytical function in the variables ν, μ, γ and z.

Specific values

General characteristics

Domain and analyticity

$S_{\nu,\mu}^{(1)}(\gamma, z)$ is an analytical function of ν, μ, γ, z which is defined in $\mathbb{C}^4$.

11.10.04.0001.01

$(\nu * \mu * \gamma * z) \longrightarrow S_{\nu,\mu}^{(1)}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$S_{\nu,\mu}^{(1)}(\gamma, z)$ is an even function with respect to μ.

11.10.04.0002.01

$$S_{\nu,-\mu}^{(1)}(\gamma, z) = S_{\nu,\mu}^{(1)}(\gamma, z)$$

Mirror symmetry

11.10.04.0003.01

$$S_{\overline{\nu},\overline{\mu}}^{(1)}(\overline{\gamma}, \overline{z}) = \overline{S_{\nu,\mu}^{(1)}(\gamma, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at generic point $z = z_0$

11.10.06.0001.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, z_0) + S_{\nu,\mu}^{(1)\prime}(\gamma, z_0)(z - z_0) + \frac{1}{2\left(1 - z_0^2\right)}\left(2 S_{\nu,\mu}^{(1)\prime}(\gamma, z_0) z_0 + S_{\nu,\mu}^{(1)}(\gamma, z_0)\left(\left(z_0^2 - 1\right)\gamma^2 - \lambda_{\nu,\mu}(\gamma) + \frac{\mu^2}{1 - z_0^2}\right)\right)(z - z_0)^2 - \frac{1}{6\left(z_0^2 - 1\right)^3}\left(S_{\nu,\mu}^{(1)\prime}(\gamma, z_0)\left(z_0^2 - 1\right)\left(\gamma^2 z_0^4 - 2\left(\gamma^2 + 3\right) z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right) - 2\right) - 2 S_{\nu,\mu}^{(1)}(\gamma, z_0) z_0\left(-3\mu^2 + \gamma^2\left(z_0^2 - 1\right)^2 - 2\lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right)\right)\right)(z - z_0)^3 + \frac{1}{24\left(z_0^2 - 1\right)^4}\left(4 S_{\nu,\mu}^{(1)\prime}(\gamma, z_0) z_0\left(z_0^2 - 1\right)\left(\gamma^2 z_0^4 - 2\left(\gamma^2 + 3\right) z_0^2 + \gamma^2 - 3\mu^2 - 2\lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right) - 6\right) + S_{\nu,\mu}^{(1)}(\gamma, z_0)\left(\gamma^4\left(z_0^2 - 1\right)^4 + \lambda_{\nu,\mu}(\gamma)^2\left(z_0^2 - 1\right)^2 - 2\gamma^2\left(\mu^2 + 4 z_0^2 + 2\right)\left(z_0^2 - 1\right)^2 - 2\lambda_{\nu,\mu}(\gamma)\left(\gamma^2 z_0^4 - \left(2\gamma^2 + 9\right) z_0^2 + \gamma^2 - \mu^2 - 3\right)\left(z_0^2 - 1\right) + \mu^2\left(\mu^2 + 36 z_0^2 + 8\right)\right)\right)(z - z_0)^4 + \ldots /; (z \to z_0)$$

11.10.06.0002.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, z_0)\left(1 + O(z - z_0)\right)$$

Expansions at $z = 0$

11.10.06.0003.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, 0) + S_{\nu,\mu}^{(1)\prime}(\gamma, 0) z - \frac{1}{2}\left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) S_{\nu,\mu}^{(1)}(\gamma, 0) z^2 + \frac{1}{6}\left(2 S_{\nu,\mu}^{(1)\prime}(\gamma, 0) - \left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) S_{\nu,\mu}^{(1)\prime}(\gamma, 0)\right) z^3 + \ldots /; (z \to 0)$$

11.10.06.0004.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, 0)\left(1 + O(z)\right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.10.13.0001.01

$$\left(1-z^2\right) w''(z) - 2\, z\, w'(z) + \left(\left(1-z^2\right) \gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1-z^2}\right) w(z) = 0 \;/;\; w(z) = c_1\, S_{\nu,\mu}^{(1)}(\gamma,\, z) \;+ c_2\, S_{\nu,\mu}^{(2)}(\gamma,\, z)$$

11.10.13.0002.01

$$W_z\left(S_{\nu,\mu}^{(1)}(\gamma,\, z),\, S_{\nu,\mu}^{(2)}(\gamma,\, z)\right) = \frac{1}{1-z^2} \left(S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0) - S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)\right)$$

11.10.13.0003.01

$$\left(1-g(z)^2\right) w''(z) + \left(\left(g(z)^2-1\right) \frac{g''(z)}{g'(z)} - 2\, g(z)\, g'(z)\right) w'(z) + \left(\frac{\left(\gamma\, g(z)^2 + \mu - \gamma\right)\left(-\gamma\, g(z)^2 + \mu + \gamma\right) g'(z)^2}{g(z)^2-1} + \lambda_{\nu,\mu}(\gamma)\, g'(z)^2\right) w(z) = 0 \;/;$$
$$w(z) = c_1\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z)) \;+ c_2\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))$$

11.10.13.0004.01

$$W_z\left(S_{\nu,\mu}^{(1)}(\gamma,\, g(z)),\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))\right) = \frac{g'(z)}{1-g[z]^2} \left(S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0) - S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)\right)$$

11.10.13.0005.01

$$\left(1-g(z)^2\right) w''(z) + \left(\left(g(z)^2-1\right)\left(\frac{2\, h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) - 2\, g(z)\, g'(z)\right) w'(z) +$$
$$\left(\frac{\left(\gamma\, g(z)^2 + \mu - \gamma\right)\left(-\gamma\, g(z)^2 + \mu + \gamma\right) g'(z)^2}{g(z)^2-1} + \lambda_{\nu,\mu}(\gamma)\, g'(z)^2 + \frac{2\, g(z)\, h'(z)\, g'(z)}{h(z)} + \frac{\left(g(z)^2-1\right)\left(h(z)\, h''(z) - 2\, h'(z)^2\right)}{h(z)^2} - \right.$$
$$\left.\frac{\left(g(z)^2-1\right) h'(z)\, g''(z)}{h(z)\, g'(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\, h(z)\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z)) \;+ c_2\, h(z)\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))$$

11.10.13.0006.01

$$W_z\left(h(z)\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z)),\, h(z)\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))\right) = \frac{h(z)^2\, g'(z)}{1-g[z]^2} \left(S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0) - S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)\right)$$

11.10.13.0007.01

$$\left(1-a^2\, z^{2r}\right) w''(z) - \frac{a^2\,(r-2\,s+1)\, z^{2r} + r + 2\,s - 1}{z}\, w'(z) +$$
$$\left(a^2\, r^2\, \lambda_{\nu,\mu}(\gamma)\, z^{2r-2} + \frac{-a^2\, r^2\left(\left(a^2\, z^{2r}-1\right)^2 \gamma^2 - \mu^2\right) z^{2r} - s^2\left(a^2\, z^{2r}-1\right)^2 + r\, s\left(a^4\, z^{4r}-1\right)}{z^2\left(a^2\, z^{2r}-1\right)}\right) w(z) =$$
$$0 \;/;\; w(z) = c_1\, z^s\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, z^r) \;+ c_2\, z^s\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, z^r)$$

11.10.13.0008.01

$$W_z\left(z^s\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, z^r),\, z^s\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, z^r)\right) = \frac{a\, r\, z^{r+2s-1}}{1-a^2\, z^{2r}} \left(S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0) - S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)\right)$$

11.10.13.0009.01

$$\left(1-a^2 r^{2z}\right) w''(z) + \left(-a^2 \log(r) r^{2z} + 2 a^2 \log(s) r^{2z} - \log(r) - 2\log(s)\right) w'(z) + \left(a^2 \log^2(r)\, \lambda_{\nu,\mu}(\gamma)\, r^{2z} + \frac{1}{a^2 r^{2z}-1}\left(-a^2\left(\left(a^2 r^{2z}-1\right)^2 \gamma^2 - \mu^2\right)\log^2(r)\, r^{2z} - \left(a^2 r^{2z}-1\right)^2 \log^2(s) + \left(a^4 r^{4z}-1\right)\log(r)\log(s)\right)\right)$$

$$w(z) = 0 \;/;\; w(z) = c_1 s^z S_{\nu,\mu}^{(1)}(\gamma, a r^z) + c_2 s^z S_{\nu,\mu}^{(2)}(\gamma, a r^z)$$

11.10.13.0010.01

$$W_z\left(s^z S_{\nu,\mu}^{(1)}(\gamma, a r^z), s^z S_{\nu,\mu}^{(2)}(\gamma, a r^z)\right) = \frac{a r^z s^{2z}\log(r)}{1-a^2 r^{2z}}\left(S_{\nu,\mu}^{(1)}(\gamma,0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,0) - S_{\nu,\mu}^{(1)\prime}(\gamma,0)\, S_{\nu,\mu}^{(2)}(\gamma,0)\right)$$

Differentiation

Low-order differentiation

With respect to *z*

11.10.20.0001.01

$$\text{SpheroidalS1}^{(0,0,0,1)}(\nu,\mu,\gamma,z) = S_{\nu,\mu}^{(1)\prime}(\gamma,z)$$

11.10.20.0002.01

$$\text{SpheroidalS1}^{(0,0,0,2)}(\nu,\mu,\gamma,z) = \frac{1}{1-z^2}\left(2 z\, S_{\nu,\mu}^{(1)\prime}(\gamma,z) - \left(\left(1-z^2\right)\gamma^2 + \lambda_{\nu,\mu}(\gamma) - \frac{\mu^2}{1-z^2}\right) S_{\nu,\mu}^{(1)}(\gamma,z)\right)$$

Integration

Operations

Limit operation

11.10.25.0001.01

$$\lim_{z\to\infty}\frac{S_{\nu,\mu}^{(1)}\left(\gamma,\frac{z}{\gamma}\right)}{j_\nu(z)} = 1$$

Representations through equivalent functions

Theorems

History

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