StieltjesGamma

Notations

Traditional name

Stieltjes constant

Traditional notation

γ_n

Mathematica StandardForm notation

`StieltjesGamma[n]`

Primary definition

10.05.02.0001.01

$$\gamma_n = (-1)^n\, n!\left([(s-1)^n]\left(\zeta(s) - \frac{1}{s-1}\right)\right) \;/;\; n \in \mathbb{N}$$

Specific values

Values at fixed points

10.05.03.0001.01

$$\gamma_0 = \gamma$$

10.05.03.0002.01

$$\gamma_n = (-1)^n \left([(s-1)^n]\, \zeta(s)\right) \;/;\; n \in \mathbb{N}$$

General characteristics

Domain and analyticity

γ_n is a nonanalytical function which is defined only for nonnegative integer n.

10.05.04.0001.01

$$n \longrightarrow \gamma_n :: \mathbb{Z} \longrightarrow \mathbb{R}$$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Other series representations

10.05.06.0001.01

$$\gamma_n == \frac{\log^n(2)}{n+1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} B_{n+1}\left(\frac{\log(k)}{\log(2)}\right)$$

10.05.06.0002.01

$$\gamma_n == n! \log^n(2) \sum_{m=1}^{n+1} \frac{(-1)^{m-1}}{m!} \sum_{k=1}^{\infty} \frac{(-1)^k \lfloor \log_2(k) \rfloor^m}{k} B_{n-m+1}\left(\frac{\log(k)}{\log(2)}\right)$$

10.05.06.0003.01

$$\gamma_1 == \frac{\log(2)}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (2 \log_2(k) - \lfloor \log_2(2\,k) \rfloor) \lfloor \log_2(k) \rfloor$$

10.05.06.0004.01

$$\gamma_n == (-1)^n\, n! \sum_{k_0=1}^{n+1} \sum_{k_1=1}^{n+1} \ldots \sum_{k_{n+1}=1}^{n+1} \delta_{n+1, \sum_{j=0}^{n+1} (j+1)\, k_j} \prod_{j=0}^{n+1} \frac{\left(-\frac{\eta_j}{j+1}\right)^{k_j}}{k_j!} /; n \in \mathbb{N}^+ \bigwedge \eta_n == [s^n]\left(-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)$$

```
(StieltjesGamma[n_] /; n ∈ Integers ∧ n ≥ 0) :>
 (Module[{k}, Expand[(-1)^n (n)!
     Sum[KroneckerDelta[n + 1, Sum[(1 + j) k_j, {j, 0, n+1}]] Product[1/(k_j !) (-η_j/(j + 1))^(k_j), {j, 0, n+1}],
      Evaluate[Sequence @@ Table[{k_j, 0, n + 1}, {j, 0, n + 1}]]]]] /.
   η_k_ :> SeriesTerm[-Zeta'[1 + s] / Zeta[1 + s], {s, 0, k}])
```

K. Maslanka: An Explicit Formula Relating Stieltjes Numbers and Li's Numbers math.NT/0406312 ({}[[1,1]]) {}[[1,1]]

Integral representations

Contour integral representations

10.05.07.0001.01

$$\gamma_n == \frac{(-1)^n\, n!}{2\,\pi\, i} \int_{|s-1|==1} \frac{1}{(s-1)^{n+1}} \left(\zeta(s) - \frac{1}{s-1}\right) ds \;/;\; n \in \mathbb{N}$$

Limit representations

10.05.09.0001.01

$$\gamma_n == \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{\log^n(k)}{k} - \frac{\log^{n+1}(m)}{n+1}\right)$$

10.05.09.0002.01

$$\gamma_n == \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{\log^n(k)}{k} - \int_1^m \frac{\log^n(t)}{t}\, dt\right)$$

Generating functions

10.05.11.0001.01

$$\gamma_n == (-1)^n\, n! \left([z^n]\left(\zeta(s) - \frac{1}{s-1}\right)\right) \;/;\; n \in \mathbb{N}$$

Identities

Relation to Li's numbers

10.05.17.0001.01

$$\eta_n == (n+1) \sum_{k_0=1}^{n+1} \sum_{k_1=1}^{n+1} \ldots \sum_{k_{n+1}=1}^{n+1} \left(\Gamma\left(\sum_{j=0}^{n+1} k_j\right) \delta_{n+1, \sum_{j=0}^{n+1} (j+1)\, k_j}\right) \prod_{j=0}^{n+1} \frac{\left(-\frac{(-1)^{k_j}\, \gamma_j}{j!}\right)^{k_j}}{k_j!} \;/;\; n \in \mathbb{Z} \bigwedge \nu \geq 0 \bigwedge \eta_n == [s^n]\left(-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)$$

```
(SeriesTerm[-Zeta'[1 + s_] / Zeta[1 + s_], {s_, 0, n_}] /;
   n ∈ Integers ⋀ n ≥ 0) :>
Module[{k}, Expand[(n + 1) Sum[If[n + 1 == Sum[(1 + j) k_j, {j, 0, n+1}], Gamma[Sum[k_j, {j, 0, n+1}]], 0]
   Product[(-(-1)^(k_j) StieltjesGamma[j] / j!)^(k_j) / k_j!, {j, 0, n+1}],
   Evaluate[Sequence @@ Table[{k_j, 0, n + 1}, {j, 0, n + 1}]]]]]
```

K. Maslanka: An Explicit Formula Relating Stieltjes Numbers and Li's Numbers math.NT/0406312 ({}[[1,1]]) {}[[1,1]]

Inequalities

10.05.29.0001.01

$$\left|\gamma_n\right| < \frac{2\,(n-1)!}{\pi^n}$$

Theorems

Riemann hypothesis

The Riemann hypothesis is equivalent to (Li 1997) $\lambda_n > 0$ for all integer $n > 0$. The

$$\lambda_n = \frac{1}{(n-1)!}\,\frac{\partial^n}{\partial s^n}\left(s^{n-1}\log(\zeta(s))\right)\Big|_{s==1}$$

contain derivatives of the Riemann zeta function at $s == 1$ and can be expressed through Stieltjes constants.

History

– E. Cahen (1894)

Copyright