WeierstrassPHalfPeriodValues

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Notations

Traditional name

Weierstrass function values at half periods

Traditional notation

$\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$

Mathematica StandardForm notation

`WeierstrassPHalfPeriodValues[{g2, g3}]`

Primary definition

09.20.02.0001.01

$$\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} /; \{e_1, e_2, e_3\} == \{\wp(\omega_1; g_2, g_3), \wp(\omega_2; g_2, g_3), \wp(\omega_3; g_2, g_3)\} \bigwedge \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 == -\omega_1 - \omega_3$$

Special notations for this file:

09.20.02.0002.01

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), \omega_2(g_2, g_3), \omega_3(g_2, g_3)\} /; \omega_2 == \omega_2(g_2, g_3) == -\omega_1(g_2, g_3) - \omega_3(g_2, g_3)$$

09.20.02.0003.01

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.20.02.0005.01

$$\{e_1, e_2, e_3\} == \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$$

09.20.02.0006.01

$$\{e_1', e_2', e_3'\} == \{e_1'(g_2, g_3), e_2'(g_2, g_3), e_3'(g_2, g_3)\}$$

09.20.02.0007.01

$$e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.20.02.0008.01

$$e_n' == \wp'(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.20.02.0009.01

$$\eta_n == \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.20.02.0010.01

$$q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

Specific values

Values at fixed points

Equianharmonic case $\{g_2, g_3\} == \{0, 1\}$

09.20.03.0001.01

$$\{e_1, e_2, e_3\} == \left\{\frac{1}{\sqrt[3]{4}}, \frac{1}{\sqrt[3]{4}} e^{\frac{4\pi i}{3}}, \frac{1}{\sqrt[3]{4}} e^{\frac{2\pi i}{3}}\right\} /; \{g_2, g_3\} == \{0, 1\}$$

09.20.03.0002.01

$$(e_i - e_j)(e_i - e_k) == (e_j - e_i)(e_k - e_i) /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k \bigwedge \{g_2, g_3\} == \{0, 1\}$$

Lemniscatic case $\{g_2, g_3\} == \{1, 0\}$

09.20.03.0003.01

$$\{e_1(1, 0), e_2(1, 0), e_3(1, 0)\} == \left\{\frac{1}{2}, 0, -\frac{1}{2}\right\}$$

Values at infinities

09.20.03.0004.01

$$e_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) == \frac{\pi^2}{6\,\omega_1^2}$$

09.20.03.0005.01

$$e_3(g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) == \frac{\pi^2}{6\,\omega_3^2}$$

09.20.03.0007.01

$$\{e_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})), e_2(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})), e_3(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty}))\} == \left\{\frac{3\,g_3}{g_2}, -\frac{3\,g_3}{g_2}, -\frac{3\,g_3}{g_2}\right\}$$

General characteristics

Domain and analyticity

$\{\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}\}$ is an vector-valued function of g_2 and g_3 that is analytic in each component and it is defined over $\mathbb{C}^2$.

09.20.04.0001.01

$$\{g_2 * g_3\} \longrightarrow \{\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}\} :: \{\mathbb{C} \otimes \mathbb{C}\} \longrightarrow \{\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.20.04.0002.01

$$\{e_1(\overline{g_2}, \overline{g_3}), e_2(\overline{g_2}, \overline{g_3}), e_3(\overline{g_2}, \overline{g_3})\} == \overline{\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}}$$

Transformation of half-periods

09.20.04.0003.01

$$\{\{e_1(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3)),\,e_2(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3)),\,e_3(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3))\}\}=\{\{e_1(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)),\,e_2(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)),\,e_3(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3))\}\}\;/;\;\{a,b,c,d\}\in\mathbb{Z}\land a\,d-b\,c=\pm 1$$

Series representations

q-series

09.20.06.0002.01

$$\{e_1(g_2,g_3),\,e_2(g_2,g_3),\,e_3(g_2,g_3)\}=\left\{\frac{\pi^2}{6\,\omega_1^2}+\frac{4\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}(2\,k-1)\,\frac{q^{4k-2}}{1-q^{4k-2}},\,-\frac{\pi^2}{12\,\omega_1^2}-\frac{2\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}(-1)^k\,\frac{k\,q^k}{1+(-1)^k\,q^k},\,-\frac{\pi^2}{12\,\omega_1^2}-\frac{2\,\pi^2}{\omega_1^2}\sum_{k=1}^{\infty}\frac{k\,q^k}{1+q^k}\right\}$$

Other series representations

09.20.06.0003.01

$$\eta_i+e_i\,\omega_i=\frac{\pi^2}{4\,\omega_i}\left(1+2\sum_{n=1}^{\infty}\sec^2\!\left(\frac{\pi\,n\,\omega_j}{\omega_i}\right)\right)\;/;\;\{i,j\}\in\{1,2,3\}\land i\neq j$$

09.20.06.0004.01

$$\eta_i+e_j\,\omega_i=\frac{\pi^2}{2\,\omega_i}\sum_{n=1}^{\infty}\sec^2\!\left(\pi\,\frac{2\,n-1}{2}\,\frac{\omega_k}{\omega_i}\right)\;/;\;\{i,j,k\}\in\{1,2,3\}\land i\neq j\neq k$$

09.20.06.0005.01

$$\eta_i+e_j\,\omega_i=\frac{\pi^2}{2\,\omega_i}\sum_{n=1}^{\infty}\csc^2\!\left(\pi\,\frac{2\,n-1}{2}\,\frac{\omega_j}{\omega_i}\right)\;/;\;\{i,j\}\in\{1,2,3\}\land i\neq j$$

Product representations

09.20.08.0001.01

$$\{e_1(g_2,g_3),\,e_2(g_2,g_3),\,e_3(g_2,g_3)\}=\frac{\pi^2}{12\,\omega_1^2}\left(\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\right)^4\left\{\left(\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)\right)^8+\left(\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)\right)^8,\,\left(\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)\right)^8-2\left(\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)\right)^8,\,2\left(\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)\right)^8-\left(\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)\right)^8\right\}$$

09.20.08.0002.01

$$e_2-e_3=\frac{4\,\pi^2}{\omega_1^2}\,q\left(\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\right)^4\left(\prod_{n=1}^{\infty}\left(1+q^{2n}\right)\right)^8$$

09.20.08.0003.01

$$e_1-e_3=\frac{\pi^2}{4\,\omega_1^2}\left(\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\right)^4\left(\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)\right)^8$$

09.20.08.0004.01

$$e_1 - e_2 = \frac{\pi^2}{4\,\omega_1^2} \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \right)^4 \left(\prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right) \right)^8$$

09.20.08.0005.01

$$(e_2 - e_3)\,(e_1 - e_3)\,(e_1 - e_2) = \frac{\pi^6}{4\,\omega_1^6}\, q \left(\prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \right)^{12}$$

Identities

Functional identities

09.20.17.0001.01

$$e_1 + e_2 + e_3 = 0$$

Differentiation

Low-order differentiation

With respect to g_2

09.20.20.0001.01

$$\frac{\partial \{e_1, e_2, e_3\}}{\partial g_2} = \frac{1}{4\left(g_2^3 - 27\,g_3^2\right)} \left(2\,g_2^2\,\{e_1, e_2, e_3\} + 6\,g_3\,g_2 - 36\,g_3\,\{e_1, e_2, e_3\}^2 + \left\{e_1', e_2', e_3'\right\} \left(g_2^2\,\{\omega_1, \omega_2, \omega_3\} - 18\,g_3\,\{\eta_1, \eta_2, \eta_3\}\right)\right)$$

With respect to g_3

09.20.20.0002.01

$$\frac{\partial \{e_1, e_2, e_3\}}{\partial g_3} = \frac{1}{2\left(g_2^3 - 27\,g_3^2\right)} \left(12\,g_2\,\{e_1, e_2, e_3\}^2 - 18\,g_3\,\{e_1, e_2, e_3\} - 2\,g_2^2 + (6\,g_2\,\{\eta_1, \eta_2, \eta_3\} - g_3\,\{\omega_1, \omega_2, \omega_3\}) \left\{e_1', e_2', e_3'\right\}\right)$$

Representations through more general functions

Through other functions

09.20.26.0001.01

$$\{e_1(g_2, g_3),\, e_2(g_2, g_3),\, e_3(g_2, g_3)\} = \{\wp(\omega_1; g_2, g_3),\, \wp(\omega_2; g_2, g_3),\, \wp(\omega_3; g_2, g_3)\}\ /;\ \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3),\, \omega_3(g_2, g_3)\} \bigwedge \omega_2 = -\omega_1 - \omega_3$$

Representations through equivalent functions

With related functions

Involving other Weierstrass functions

09.20.27.0001.01

$$e_i - e_j = \frac{\sigma_j(\omega_i; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2}\ /;\ \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

09.20.27.0002.01

$$e_i - e_j == e^{-2\eta_j \omega_i} \frac{\sigma(\omega_k; g_2, g_3)^2}{\sigma(\omega_i; g_2, g_3)^2 \sigma(\omega_j; g_2, g_3)^2} /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j \neq k$$

Involving Weierstrass invariants

09.20.27.0003.01

$$e_1^4 + e_2^4 + e_3^4 == \frac{g_2^2}{8}$$

09.20.27.0004.01

$$4 e_i^3 - g_2 e_i - g_3 == 0 /; i \in \{1, 2, 3\}$$

09.20.27.0005.01

$$16 (e_1 - e_2)^2 (e_3 - e_1)^2 (e_3 - e_2)^2 == g_2^3 - 27 g_3^2$$

Involving theta functions

09.20.27.0006.01

$$\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} == \frac{1}{3} \left(\frac{\pi}{2 \omega_1}\right)^2 \left\{\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4, \vartheta_2(0, q)^4 - \vartheta_4(0, q)^4, -\vartheta_2(0, q)^4 - \vartheta_3(0, q)^4\right\}$$

09.20.27.0007.01

$$e_j - e_k == \left(\frac{\pi}{2 \omega_1}\right)^2 \vartheta_{i+1}(0, q)^4 /; \{i, j, k\} \in \{1, 2, 3\} \bigwedge i \neq j < k$$

Involving elliptic integrals and modular functions

09.20.27.0008.01

$$\{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\} == \left\{\frac{(2 - m) K(m)^2}{3 \omega_1^2}, \frac{(2 m - 1) K(m)^2}{3 \omega_1^2}, \frac{-(m + 1) K(m)^2}{3 \omega_1^2}\right\} /; m == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

09.20.27.0009.01

$$\frac{e_2 - e_3}{e_1 - e_3} == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

09.20.27.0010.01

$$\frac{e_1 - e_2}{e_1 - e_3} == 1 - \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

History

- N. H. Abel (1827)
- K. Weierstrass (1855,1862)
- C. Hermite (1849) first used the notation $\wp$

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