WeierstrassSigma

Notations

Traditional name

Weierstrass sigma function

Traditional notation

$\sigma(z; g_2, g_3)$

Mathematica StandardForm notation

`WeierstrassSigma[z, {g2, g3}]`

Primary definition

09.15.02.0001.01

$$\sigma(z; g_2, g_3) = z \prod_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \left(1 - \frac{z}{2 m \omega_1 + 2 n \omega_3}\right) \exp\left(\frac{z^2}{2 (2 m \omega_1 + 2 n \omega_3)^2} + \frac{z}{2 m \omega_1 + 2 n \omega_3}\right) /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Special notations for this file:

09.15.02.0002.01

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.15.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.15.02.0004.01

$$e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.15.02.0005.01

$$\eta_n = \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.15.02.0006.01

$$q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed z

Degenerate cases:

09.15.03.0001.01

$\sigma(z;\,0,\,0) = z$

09.15.03.0002.01

$$\sigma(z;\,3,\,1) = \sqrt{\frac{2}{3}}\; e^{\frac{z^2}{4}} \sin\left(\sqrt{\frac{3}{2}}\; z\right)$$

For fixed $\{g_2,\,g_3\}$

09.15.03.0003.01

$\sigma(0;\,g_2,\,g_3) = 0$

09.15.03.0004.01

$\sigma(2\,m\,\omega_1 + 2\,n\,\omega_3;\,g_2,\,g_3) = 0 \;/;\; \{m,\,n\} \in \mathbb{Z}$

Half-period values:

09.15.03.0005.01

$$\frac{\sigma(\omega_k;\,g_2,\,g_3)^2}{\sigma(\omega_i;\,g_2,\,g_3)^2\,\sigma(\omega_j;\,g_2,\,g_3)^2} = e^{2\,\eta_j\,\omega_i}\left(e_i - e_j\right) /;\; \{i,\,j,\,k\} \in \{1,\,2,\,3\} \bigwedge i \neq j \neq k$$

09.15.03.0006.01

$$\frac{\sigma(\omega_k;\,g_2,\,g_3)}{\sigma(\omega_j;\,g_2,\,g_3)\,\sigma_j(\omega_i;\,g_2,\,g_3)} = e^{\eta_j\,\omega_i} /;\; \{i,\,j,\,k\} \in \{1,\,2,\,3\} \bigwedge i \neq j \neq k$$

09.15.03.0007.01

$$\frac{\sigma_j(\omega_i;\,g_2,\,g_3)^2}{\sigma(\omega_i;\,g_2,\,g_3)^2} = e_i - e_j /;\; \{i,\,j\} \in \{1,\,2,\,3\} \bigwedge i \neq j$$

One-third period values:

09.15.03.0008.01

$$\sigma\left(\frac{2\,\omega_i}{3};\,g_2,\,g_3\right)^3 = -\frac{\exp\left(\frac{2\,\eta_i\,\omega_i}{3}\right)}{\wp'\left(\frac{2\,\omega_i}{3};\,g_2,\,g_3\right)} /;\; i \in \{1,\,2,\,3\}$$

Values at fixed points

Equianharmonic case ($g_2 = 0$, $g_3 = 1$):

09.15.03.0009.01

$$\sigma(\omega_1;\,0,\,1) = e^{\frac{\pi}{4\sqrt{3}}}\,\frac{\sqrt[3]{2}}{\sqrt[4]{3}}$$

09.15.03.0010.01

$$\sigma(\omega_2;\,0,\,1) = e^{\frac{\pi}{4\sqrt{3}}}\,\frac{\sqrt[3]{2}}{\sqrt[4]{3}}\,e^{\frac{4\pi i}{3}}$$

09.15.03.0011.01

$$\sigma(\omega_3; 0, 1) == e^{\frac{\pi}{4\sqrt{3}}} \frac{\sqrt[3]{2}}{\sqrt[4]{3}} e^{\frac{2\pi i}{3}}$$

Lemniscatic case ($g_2 = 1, g_3 = 0$):

09.15.03.0012.01

$$\sigma(\omega_1; 1, 0) == e^{\pi/8} \sqrt[4]{2}$$

09.15.03.0013.01

$$\sigma(\omega_2; 1, 0) == -e^{\pi/4} \sqrt{2} e^{\frac{i\pi}{4}}$$

09.15.03.0014.01

$$\sigma(\omega_3; 1, 0) == e^{\pi/8} \sqrt[4]{2} i$$

Values at infinities

09.15.03.0015.01

$$\sigma(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) == \frac{2\omega_1}{\pi} \exp\left(\frac{1}{6}\left(\frac{\pi z}{2\omega_1}\right)^2\right) \sin\left(\frac{\pi z}{2\omega_1}\right)$$

09.15.03.0016.01

$$\sigma(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == z$$

General characteristics

Domain and analyticity

$\sigma(z; g_2, g_3)$ is an entire analytical function of z, g_2, and g_3, which is defined in $\mathbb{C}^3$.

09.15.04.0001.01

$$(z * \{g_2 * g_3\}) \longrightarrow \sigma(z; g_2, g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\sigma(z; g_2, g_3)$ is an odd function with respect to z.

09.15.04.0002.01

$$\sigma(-z; g_2, g_3) == -\sigma(z; g_2, g_3)$$

Mirror symmetry

09.15.04.0003.01

$$\sigma(\bar{z}; \overline{g_2}, \overline{g_3}) == \overline{\sigma(z; g_2, g_3)}$$

Periodicity

$\sigma(z; g_2, g_3)$ is a quasi-periodic function with respect to z.

09.15.04.0004.01

$$\sigma(z + 2\omega_n; g_2, g_3) == -e^{2\eta_n (z+\omega_n)} \sigma(z; g_2, g_3) /; n \in \{1, 2, 3\}$$

09.15.04.0005.01

$$\sigma(z + 2\,m\,\omega_1 + 2\,n\,\omega_2 + 2\,r\,\omega_3;\, g_2,\, g_3) = (-1)^{n\,r + r\,m + m\,n + m + n + r}\, e^{2\,(m\,\eta_1 + n\,\eta_2 + r\,\eta_3)\,(z + m\,\omega_1 + n\,\omega_2 + r\,\omega_3)}\, \sigma(z;\, g_2,\, g_3)\; /;\; \{m,\, n,\, r\} \in \mathbb{Z}$$

Transformation of half-periods

09.15.04.0006.01

$$\sigma(z;\, g_2(a\,\omega_1 + b\,\omega_3,\, c\,\omega_1 + d\,\omega_3),\, g_3(a\,\omega_1 + b\,\omega_3,\, c\,\omega_1 + d\,\omega_3)) = \sigma(z;\, g_2(\omega_1,\, \omega_3),\, g_3(\omega_1,\, \omega_3))\; /;\; \{a,\, b,\, c,\, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c = \pm 1$$

Homogeneity

09.15.04.0007.01

$$\sigma(z\,t;\, g_2,\, g_3) = t\,\sigma\left(z;\, g_2\,t^4,\, g_3\,t^6\right)\; /;\; t \in \mathbb{R}$$

09.15.04.0008.01

$$\sigma(\lambda\,z;\, g_2(\lambda\,\omega_1,\, \lambda\,\omega_3),\, g_3(\lambda\,\omega_1,\, \lambda\,\omega_3)) = \lambda\,\sigma(z;\, g_2(\omega_1,\, \omega_3),\, g_3(\omega_1,\, \omega_3))$$

09.15.04.0009.01

$$\sigma(\lambda\,z;\, g_2(\lambda\,\omega_1,\, \lambda\,\omega_3),\, g_3(\lambda\,\omega_1,\, \lambda\,\omega_3)) = \sigma\left(\lambda\,z;\, \frac{g_2(\omega_1,\, \omega_3)}{\lambda^4},\, \frac{g_3(\omega_1,\, \omega_3)}{\lambda^6}\right)$$

Poles and essential singularities

With respect to z

For fixed g_2, g_3, the function $\sigma(z;\, g_2,\, g_3)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

09.15.04.0010.01

$$\mathcal{S}ing_z(\sigma(z;\, g_2,\, g_3)) = \{\{\tilde{\infty},\, \infty\}\}$$

Branch points

With respect to z

For fixed g_2, g_3, the function $\sigma(z;\, g_2,\, g_3)$ does not have branch points.

09.15.04.0011.01

$$\mathcal{BP}_z(\sigma(z;\, g_2,\, g_3)) = \{\}$$

Branch cuts

With respect to z

For fixed g_2, g_3, the function $\sigma(z;\, g_2,\, g_3)$ does not have branch cuts.

09.15.04.0012.01

$$\mathcal{BC}_z(\sigma(z;\, g_2,\, g_3)) = \{\}$$

Series representations

Generalized power series

Expansions at z == 0

09.15.06.0005.01

$$\sigma(z; g_2, g_3) \propto z - \frac{g_2}{240} z^5 - \frac{g_3}{840} z^7 - \frac{g_2^2}{161\,280} z^9 + \ldots \;/;\; (z \to 0)$$

09.15.06.0006.01

$$\sigma(z; g_2, g_3) \propto z - \frac{g_2}{240} z^5 - \frac{g_3}{840} z^7 - \frac{g_2^2}{161\,280} z^9 + O\left(z^{11}\right)$$

09.15.06.0001.01

$$\sigma(z; g_2, g_3) = \sum_{k=0}^{\infty} d_k z^{2k+1} \;/;$$

$$d_0 = 1 \bigwedge d_1 = 0 \bigwedge d_2 = -\frac{g_2}{240} \bigwedge d_3 = -\frac{g_3}{840} \bigwedge d_4 = -\frac{g_2^2}{161\,280} \bigwedge d_n = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_n=0}^{n} \delta_{n-\sum_{j=1}^{n} j\,k_j,0} \prod_{j=1}^{n} \frac{c_j^{k_j}}{k_j!} \bigwedge$$

$$c_j = -\frac{a_j}{2j(2j-1)} \bigwedge a_1 = 0 \bigwedge a_2 = \frac{g_2}{20} \bigwedge a_3 = \frac{g_3}{28} \bigwedge a_k = \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l\, a_{k-l}$$

09.15.06.0002.01

$$\sigma(z; g_2, g_3) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n} \left(\frac{g_2}{2}\right)^m (2 g_3)^n}{(4m+6n+1)!} z^{4m+6n+1} \;/;\; \Big(a_{0,0} = 1 \land a_{m,n} = 0 \;/;$$

$$m < 0 \bigvee n < 0 \bigwedge a_{m,n} = \frac{16}{3}(n+1)\, a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n-1)\, a_{m-1,n} + 3(m+1)\, a_{m+1,n-1}\Big)$$

09.15.06.0007.01

$$\sigma(z; g_2, g_3) \propto z\left(1 + O\left(z^4\right)\right)$$

q-series

q -series for logarithms:

09.15.06.0003.01

$$\log(\sigma(z; g_2, g_3)) = \log\left(\frac{2\omega_1}{\pi}\right) + \frac{\eta_1 z^2}{2\omega_1} + \log\left(\sin\left(\frac{\pi z}{2\omega_1}\right)\right) + 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{k\left(1-q^{2k}\right)} \sin^2\left(\frac{k\pi z}{2\omega_1}\right)$$

Other series representations

09.15.06.0004.01

$$\sigma(z; g_2, g_3) = z \exp\left(-\sum_{j=2}^{\infty} \frac{z^{2j}}{2j} \sum_{\substack{m,n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2j}}\right)$$

Integral representations

On the real axis

Of the direct function

09.15.07.0001.01

$$\sigma(z; g_2, g_3) = z \exp\left(\int_0^z \left(\zeta(t; g_2, g_3) - \frac{1}{t}\right) dt\right)$$

Product representations

Infinite products involving trigonometric functions

09.15.08.0001.01

$$\sigma(z; g_2, g_3) = \frac{2\,\omega_i}{\pi} \exp\left(\frac{\eta_i\, z^2}{2\,\omega_i}\right) \sin\left(\frac{\pi\, z}{2\,\omega_i}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\sin^2\left(\frac{\pi z}{2\,\omega_i}\right)}{\sin^2\left(\frac{n\,\pi\,\omega_j}{\omega_i}\right)}\right) /;\ \{i,\ j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

Infinite products involving exponentials

09.15.08.0002.01

$$\sigma(z; g_2, g_3) = z \prod_{k=1}^{\infty} \exp\left(-\frac{G_{2k+2}(\omega_1, \omega_3)\, z^{2k+2}}{2\,k+2}\right) /;\ G_p(\omega_1, \omega_3) = \sum_{\substack{m,\, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_3)^p}$$

Infinite products involving q, trigonometrics and exponentials

09.15.08.0003.01

$$\sigma(z; g_2, g_3) = \frac{2\,\omega_1}{\pi} \sin\left(\frac{\pi\, z}{2\,\omega_1}\right) \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n} \exp\left(-\frac{i\,\pi\, z}{\omega_1}\right)}{1 - q^{2n}}\right) \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n} \exp\left(\frac{i\,\pi\, z}{\omega_1}\right)}{1 - q^{2n}}\right)$$

09.15.08.0004.01

$$\sigma(z; g_2, g_3) = \frac{2\,\omega_1}{\pi} \sin\left(\frac{\pi\, z}{2\,\omega_1}\right) \exp\left(\frac{\eta_1\, z^2}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \frac{1 - 2\, q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}{\left(1 - q^{2n}\right)^2}$$

q products for half-period values

09.15.08.0005.01

$$\sigma(\omega_1; g_2, g_3) = \frac{2\,\omega_1}{\pi} \exp\left(\frac{\eta_1\,\omega_1}{2}\right) \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n}}{1 - q^{2n}}\right)^2$$

09.15.08.0006.01

$$\sigma(\omega_2; g_2, g_3) = -\sqrt{i}\, \frac{\omega_1}{\pi} \exp\left(\frac{\eta_2\,\omega_2}{2}\right) \frac{1}{\sqrt[4]{q}} \left(\prod_{n=1}^{\infty} \frac{1 + q^{2n-1}}{1 - q^{2n}}\right)^2$$

09.15.08.0007.01

$$\sigma(\omega_3; g_2, g_3) = i\, \frac{\omega_1}{\pi} \exp\left(\frac{\eta_3\,\omega_3}{2}\right) \frac{1}{\sqrt[4]{q}} \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 - q^{2n}}\right)^2$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

09.15.13.0001.01

$$\sigma'(z; g_2, g_3) = \zeta(z; g_2, g_3)\, \sigma(z; g_2, g_3)$$

Involving related functions

09.15.13.0002.01

$$\frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z} = \zeta(z; g_2, g_3)$$

Ordinary nonlinear differential equations

09.15.13.0003.01

$$2\, \sigma(z; g_2, g_3)\, \frac{\partial^4 \sigma(z; g_2, g_3)}{\partial z^4} - 8\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}\, \frac{\partial^3 \sigma(z; g_2, g_3)}{\partial z^3} + 6 \left(\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2}\right)^2 - g_2\, \sigma(z; g_2, g_3)^2 = 0$$

Partial differential equations

09.15.13.0004.01

$$\left(g_2^3 - 27\, g_3^2\right) \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} = -\frac{9}{4}\, g_3\, \frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} - \frac{1}{4}\, g_2^2\, \sigma(z; g_2, g_3) - \frac{3}{16}\, g_2\, g_3\, z^2\, \sigma(z; g_2, g_3) + \frac{1}{4}\, g_2^2\, z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}$$

09.15.13.0005.01

$$\left(g_2^3 - 27\, g_3^2\right) \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} = \frac{3}{2}\, g_2\, \frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} + \frac{9}{2}\, g_3\, \sigma(z; g_2, g_3) + \frac{1}{8}\, g_2^2\, z^2\, \sigma(z; g_2, g_3) - \frac{9}{2}\, g_2\, z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z}$$

09.15.13.0006.01

$$z\, \frac{\partial \sigma(z; g_2, g_3)}{\partial z} - 4\, g_2\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} - 6\, g_3\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} - \sigma(z; g_2, g_3) = 0$$

09.15.13.0007.01

$$\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} - 12\, g_3\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} - \frac{2}{3}\, g_2^2\, \frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} + \frac{1}{12}\, g_2\, \sigma(z; g_2, g_3)\, z^2 = 0$$

Transformations

Addition formulas

Translation by half-periods

09.15.16.0001.01

$$\sigma(z \pm \omega_i; g_2, g_3) = \pm e^{\pm \eta_i z}\, \sigma(\omega_i; g_2, g_3)\, \sigma_i(\omega_i; g_2, g_3) \;/;\; i \in \{1, 2, 3\}$$

09.15.16.0002.01

$$\sigma(z_2 + z_1; g_2, g_3) = \exp\left(z_1\, \zeta(z_2; g_2, g_3) - \frac{z_1^2}{2}\, \wp(z_2; g_2, g_3)\right) \sigma(z_2; g_2, g_3)$$
$$\prod_{m,n=-\infty}^{\infty} \left(1 - \frac{z_1}{2\, m\, \omega_1 + 2\, n\, \omega_3 - z_2}\right) \exp\left(\frac{z_1}{2\, m\, \omega_1 + 2\, n\, \omega_3 - z_2} + \frac{z_1^2}{2\, (2\, m\, \omega_1 + 2\, n\, \omega_3 - z_2)^2}\right)$$

09.15.16.0003.01

$$\sigma(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)=-\sigma(z_1;g_2,g_3)^2\,\sigma(z_2;g_2,g_3)^2\,(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))$$

09.15.16.0004.01

$$\sigma(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)=\sigma(z_1;g_2,g_3)^2\,\sigma_i(z_2;g_2,g_3)^2-\sigma(z_2;g_2,g_3)^2\,\sigma_i(z_1;g_2,g_3)^2\;/;\;i\in\{1,2,3\}$$

09.15.16.0005.01

$$\sigma(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)\,\sigma(b+c;g_2,g_3)\,\sigma(b-c;g_2,g_3)+\sigma(z_1+b;g_2,g_3)\,\sigma(z_1-b;g_2,g_3)\,\sigma(c+z_2;g_2,g_3)\,\sigma(c-z_2;g_2,g_3)+\sigma(z_1+c;g_2,g_3)\,\sigma(z_1-c;g_2,g_3)\,\sigma(z_2+b;g_2,g_3)\,\sigma(z_2-b;g_2,g_3)=0$$

09.15.16.0006.01

$$\sigma_i(z_1+z_2;g_2,g_3)\,\sigma_i(z_1-z_2;g_2,g_3)=\sigma_i(z_1;g_2,g_3)^2\,\sigma_i(z_2;g_2,g_3)^2-(e_i-e_j)(e_i-e_k)\,\sigma(z_1;g_2,g_3)^2\,\sigma(z_2;g_2,g_3)^2\;/;\;\{i,j,k\}\in\{1,2,3\}\bigwedge i\neq j\neq k$$

09.15.16.0007.01

$$\sigma_i(z_1+z_2;g_2,g_3)\,\sigma(z_1-z_2;g_2,g_3)=\sigma(z_1;g_2,g_3)\,\sigma_i(z_1;g_2,g_3)\,\sigma_j(z_2;g_2,g_3)\,\sigma_k(z_2;g_2,g_3)-\sigma(z_2;g_2,g_3)\,\sigma_i(z_2;g_2,g_3)\,\sigma_j(z_1;g_2,g_3)\,\sigma_k(z_1;g_2,g_3)\;/;\;\{i,j,k\}\in\{1,2,3\}\bigwedge i\neq j\neq k$$

Multiple arguments

Argument involving numeric multiples of variable

Double angle formulas

09.15.16.0008.01

$$\sigma(2z;g_2,g_3)=\frac{2\,\sigma(z;g_2,g_3)\,\sigma(\omega_1-z;g_2,g_3)\,\sigma(\omega_2-z;g_2,g_3)\,\sigma(\omega_3-z;g_2,g_3)}{\sigma(\omega_1;g_2,g_3)\,\sigma(\omega_2;g_2,g_3)\,\sigma(\omega_3;g_2,g_3)}$$

09.15.16.0009.01

$$\sigma(2z;g_2,g_3)=2\,\sigma(z;g_2,g_3)\,\sigma_1(z;g_2,g_3)\,\sigma_2(z;g_2,g_3)\,\sigma_3(z;g_2,g_3)$$

09.15.16.0010.01

$$\sigma(2z;g_2,g_3)=\sigma(z;g_2,g_3)\left(2\left(\frac{\partial\sigma(z;g_2,g_3)}{\partial z}\right)^3-3\,\sigma(z;g_2,g_3)\,\frac{\partial^2\sigma(z;g_2,g_3)}{\partial z^2}\,\frac{\partial\sigma(z;g_2,g_3)}{\partial z}+\frac{\partial^3\sigma(z;g_2,g_3)}{\partial z^3}\,\sigma(z;g_2,g_3)^2\right)$$

09.15.16.0011.01

$$\sigma(2z;g_2,g_3)=-\wp'(z;g_2,g_3)\,\sigma(z;g_2,g_3)^4$$

Triple angle formulas

09.15.16.0012.01

$$\sigma(3z;g_2,g_3)=-\wp'(z;g_2,g_3)^2\,\sigma(z;g_2,g_3)^9\,(\wp(2z;g_2,g_3)-\wp(z;g_2,g_3))$$

09.15.16.0013.01

$$\sigma(3z;g_2,g_3)=\sigma(z;g_2,g_3)^9\left(3\,\wp(z;g_2,g_3)^4-\frac{3}{2}\,g_2\,\wp(z;g_2,g_3)^2-3\,g_3\,\wp(z;g_2,g_3)-\frac{g_2^2}{16}\right)$$

Argument involving symbolic multiples of variable

Multiple angle formula:

09.15.16.0014.01

$$\sigma(n\,z;\,g_2,\,g_3) = n\,(-1)^{n^2-1}\,e^{-n\,(n-1)\,\eta_2\,z}\,\sigma(z;\,g_2,\,g_3)\prod_{\substack{j,k=0\\ \{j,k\}\neq\{0,0\}}}^{n-1}\frac{\sigma\left(z-\frac{2\,j\,\omega_1+2\,k\,\omega_3}{n};\,g_2,\,g_3\right)}{\sigma\left(\frac{2\,j\,\omega_1+2\,k\,\omega_3}{n};\,g_2,\,g_3\right)}\;/;\;n-1\in\mathbb{N}^+$$

Related transformations

Halving half-period

09.15.16.0015.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{2},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{2},\,\omega_3\right)\right) = \exp\left(\frac{e_1\,z^2}{2}\right)\sigma(z;\,g_2,\,g_3)\,\sigma_1(z;\,g_2,\,g_3)$$

Third of half-period

09.15.16.0016.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{3},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{3},\,\omega_3\right)\right) = \exp\left(z^2\,\wp\left(\frac{2\,\omega_1}{3};\,g_2,\,g_3\right)-2\,z\,\eta_1\right)\frac{\sigma(z;\,g_2,\,g_3)\,\sigma\left(z+\frac{2\,\omega_1}{3};\,g_2,\,g_3\right)\sigma\left(z+\frac{4\,\omega_1}{3};\,g_2,\,g_3\right)}{\sigma\left(\frac{2\,\omega_1}{3};\,g_2,\,g_3\right)\sigma\left(\frac{4\,\omega_1}{3};\,g_2,\,g_3\right)}$$

General fractions of half-periods

09.15.16.0017.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{n},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{n},\,\omega_3\right)\right) = \exp\left(\frac{z^2}{2}\sum_{k=1}^{n-1}\wp\left(\frac{2\,k\,\omega_1}{n};\,g_2,\,g_3\right)-z\sum_{k=1}^{n-1}\frac{2\,k\,\eta_1}{n}\right)\sigma(z;\,g_2,\,g_3)\prod_{k=1}^{n-1}\frac{\sigma\left(z+\frac{2\,k\,\omega_1}{n};\,g_2,\,g_3\right)}{\sigma\left(\frac{2\,k\,\omega_1}{n};\,g_2,\,g_3\right)}\;/;\;n\in\mathbb{N}^+$$

09.15.16.0018.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{n},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{n},\,\omega_3\right)\right) =$$

$$\exp\left(\frac{z^2}{2}\sum_{j=1}^{n-1}\wp\left(\frac{j\,2\,\omega_1}{n};\,g_2,\,g_3\right)-z\sum_{j=1}^{n-1}\zeta\left(\frac{j\,2\,\omega_1}{n};\,g_2,\,g_3\right)\right)\sigma(z;\,g_2,\,g_3)\prod_{j=1}^{n-1}\frac{\sigma\left(z+\frac{2\,j\,\omega_1}{n};\,g_2,\,g_3\right)}{\sigma\left(\frac{j\,2\,\omega_1}{n};\,g_2,\,g_3\right)}\;/;\;n\in\mathbb{N}^+$$

09.15.16.0019.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{2\,n+1},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{2\,n+1},\,\omega_3\right)\right) = (-1)^n\exp\left(\frac{z^2}{2}\sum_{k=1}^{2\,n}\wp\left(\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\right)$$

$$\sigma(z;\,g_2,\,g_3)\prod_{k=1}^{n}\left(\sigma\left(z+\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\sigma\left(z-\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\Big/\sigma\left(\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\right)\;/;\;n\in\mathbb{N}^+$$

09.15.16.0020.01

$$\sigma\left(z;\,g_2\left(\frac{\omega_1}{2\,n+1},\,\omega_3\right),\,g_3\left(\frac{\omega_1}{2\,n+1},\,\omega_3\right)\right) =$$

$$\exp\left(\frac{z^2}{2}\sum_{k=1}^{2\,n}\wp\left(\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\right)\sigma(z;\,g_2,\,g_3)^{2\,n+1}\prod_{k=1}^{n}\left(\wp(z;\,g_2,\,g_3)-\wp\left(\frac{2\,k\,\omega_1}{2\,n+1};\,g_2,\,g_3\right)\right)\;/;\;n\in\mathbb{N}^+$$

09.15.16.0021.01

$$\sigma\left(z; g_2\left(\frac{\omega_1}{2n+1}, \omega_3\right), g_3\left(\frac{\omega_1}{2n+1}, \omega_3\right)\right) = \exp\left(\frac{z^2}{2}\sum_{k=1}^{2n} \wp\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right)\right)\sigma(z; g_2, g_3)\prod_{k=1}^{n}\left(\sigma_i(z; g_2, g_3)^2 - \frac{\sigma_i\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right)^2}{\sigma\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right)^2}\sigma(z; g_2, g_3)^2\right) /; i \in \{1, 2, 3\} \bigwedge n \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to z

09.15.20.0001.01

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial z} = \sigma(z; g_2, g_3)\,\zeta(z; g_2, g_3)$$

09.15.20.0002.01

$$\frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z} = \frac{\sigma'(z; g_2, g_3)}{\sigma(z; g_2, g_3)}$$

09.15.20.0003.01

$$\frac{\partial \log(\sigma(z; g_2, g_3))}{\partial z} = \zeta(z; g_2, g_3)$$

09.15.20.0004.01

$$\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial z^2} = \sigma(z; g_2, g_3)\left(\zeta(z; g_2, g_3)^2 - \wp(z; g_2, g_3)\right)$$

09.15.20.0005.01

$$\frac{\partial^3 \sigma(z; g_2, g_3)}{\partial z^3} = \sigma(z; g_2, g_3)\left(\zeta(z; g_2, g_3)^3 - 3\,\wp(z; g_2, g_3)\,\zeta(z; g_2, g_3) - \wp'(z; g_2, g_3)\right)$$

09.15.20.0006.01

$$\frac{\partial^4 \sigma(z; g_2, g_3)}{\partial z^4} = \frac{1}{2}\sigma(z; g_2, g_3)\left(g_2 + 2\left(\zeta(z; g_2, g_3)^4 - 6\,\wp(z; g_2, g_3)\,\zeta(z; g_2, g_3)^2 - 4\,\wp'(z; g_2, g_3)\,\zeta(z; g_2, g_3) - 3\,\wp(z; g_2, g_3)^2\right)\right)$$

With respect to g_2

09.15.20.0007.01

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial g_2} = \frac{1}{16\left(g_2^3 - 27\,g_3^2\right)}\sigma(z; g_2, g_3)\left(4\,g_2^2\,(z\,\zeta(z; g_2, g_3) - 1) + 36\,g_3\left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2\right) - 3\,g_2\,g_3\,z^2\right)$$

09.15.20.0008.01

$$\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial g_2^2} = \frac{\sigma(z; g_2, g_3)}{256 \left(g_2^3 - 27 g_3^2\right)^2} \left(-16 g_2^4 \left(-\zeta(z; g_2, g_3)^2 z^2 + \wp(z; g_2, g_3) z^2 + 5 \zeta(z; g_2, g_3) z - 5\right) - 24 z^2 g_3 (z \zeta(z; g_2, g_3) - 4) g_2^3 + 9 g_3 g_2^2 \left(g_3 z^4 + 32 \left((6 - z \zeta(z; g_2, g_3)) \zeta(z; g_2, g_3)^2 + z \wp'(z; g_2, g_3) + 3 \wp(z; g_2, g_3) (z \zeta(z; g_2, g_3) - 2)\right)\right) - 216 g_3^2 g_2 \left(-\zeta(z; g_2, g_3)^2 z^2 + \wp(z; g_2, g_3) z^2 + 14 \zeta(z; g_2, g_3) z - 20\right) + 1296 g_3^2 \left(\zeta(z; g_2, g_3)^4 - 6 \wp(z; g_2, g_3) \zeta(z; g_2, g_3)^2 - 4 \wp'(z; g_2, g_3) \zeta(z; g_2, g_3) - 3 \wp(z; g_2, g_3)^2 + z^2 g_3\right)\right)$$

With respect to g_3

09.15.20.0009.01

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial g_3} = \frac{1}{8 \left(g_2^3 - 27 g_3^2\right)} \sigma(z; g_2, g_3) \left(z^2 g_2^2 + 12 \zeta(z; g_2, g_3)^2 g_2 - 12 \wp(z; g_2, g_3) g_2 - 36 g_3 (z \zeta(z; g_2, g_3) - 1)\right)$$

09.15.20.0010.01

$$\frac{\partial^2 \sigma(z; g_2, g_3)}{\partial g_3^2} = \frac{\sigma(z; g_2, g_3)}{64 \left(g_2^3 - 27 g_3^2\right)^2} \left(z^4 g_2^4 - 24 \left(-\zeta(z; g_2, g_3)^2 z^2 + \wp(z; g_2, g_3) z^2 + 10 \zeta(z; g_2, g_3) z - 16\right) g_2^3 - 72 g_2^2 \left(-2 \zeta(z; g_2, g_3)^4 + 12 \wp(z; g_2, g_3) \zeta(z; g_2, g_3)^2 + 8 \wp'(z; g_2, g_3) \zeta(z; g_2, g_3) + 6 \wp(z; g_2, g_3)^2 + z^2 g_3 (z \zeta(z; g_2, g_3) - 6)\right) + 864 g_3 g_2 \left((6 - z \zeta(z; g_2, g_3)) \zeta(z; g_2, g_3)^2 + z \wp'(z; g_2, g_3) + 3 \wp(z; g_2, g_3) (z \zeta(z; g_2, g_3) - 2)\right) - 1296 g_3^2 \left(-\zeta(z; g_2, g_3)^2 z^2 + \wp(z; g_2, g_3) z^2 + 7 \zeta(z; g_2, g_3) z - 7\right)\right)$$

With respect to ω_1

09.15.20.0013.02

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_3 \left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2 - \frac{1}{12} g_2 z^2\right) + 2 \eta_3 (z \zeta(z; g_2, g_3) - 1)\right) \sigma(z; g_2, g_3)$$

With respect to ω_3

09.15.20.0014.02

$$\frac{\partial \sigma(z; g_2, g_3)}{\partial \omega_3} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_1 \left(\wp(z; g_2, g_3) - \zeta(z; g_2, g_3)^2 - \frac{1}{12} g_2 z^2\right) + 2 \eta_1 (z \zeta(z; g_2, g_3) - 1)\right) \sigma(z; g_2, g_3)$$

Symbolic differentiation

With respect to z

09.15.20.0011.01

$$\frac{\partial^n \sigma(z; g_2, g_3)}{\partial z^n} = (2 \omega_1)^{1-n} \pi^{n-\frac{1}{2}} \left(\prod_{m=1}^{\infty} \frac{1}{1 - q^{2m}}\right)^3 \sum_{j=0}^{n} {}_2\tilde{F}_2\left(\frac{1}{2}, 1; \frac{1-j}{2}, \frac{2-j}{2}; \frac{z^2 \zeta(\omega_1; g_2, g_3)}{2 \omega_1}\right) \left(\frac{4 \omega_1}{\pi z}\right)^j \binom{n}{j} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} (2k+1)^{n-j} \sin\left(\frac{\pi ((2k+1) z + (n-j) \omega_1)}{2 \omega_1}\right) /; n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

09.15.20.0012.01

$$\frac{\partial^\alpha \sigma(z; g_2, g_3)}{\partial z^\alpha} = 2^{\alpha-1} \sqrt{\pi}\, z^{1-\alpha} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^3$$
$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j+k} q^{k(k+1)} (2k+1)^{2j+1} \left(\frac{\pi z}{4\omega_1}\right)^{2j} {}_2\tilde{F}_2\left(j+1, j+\frac{3}{2}; j-\frac{\alpha}{2}+1, j+\frac{3-\alpha}{2}; \frac{z^2 \zeta(\omega_1; g_2, g_3)}{2\omega_1}\right)$$

Integration

Indefinite integration

Involving only one direct function

09.15.21.0001.01

$$\int \sigma(z; g_2, g_3)\, dz =$$
$$\frac{\omega_1^{3/2}}{\sqrt{2\pi\eta_1}} \left(\prod_{m=1}^{\infty} \frac{1}{1-q^{2m}}\right)^3 \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \exp\left(\frac{\pi^2 (2k+1)^2}{8\eta_1\omega_1}\right)\left(\operatorname{erf}\left(\frac{\pi(2k+1)+2 i z \eta_1}{2\sqrt{2\eta_1\omega_1}}\right) + \operatorname{erf}\left(\frac{\pi(2k+1)-2 i z\eta_1}{2\sqrt{2\eta_1\omega_1}}\right)\right)$$

Representations through equivalent functions

With related functions

Involving other Weierstrass functions

09.15.27.0001.01

$$\sigma(z; g_2, g_3) = z \exp\left(\int_0^z \left(\zeta(t; g_2, g_3) - \frac{1}{t}\right) dt\right)$$

Involving theta functions

09.15.27.0002.01

$$\sigma(z; g_2, g_3) = \frac{\omega_1}{\pi} \frac{1}{\sqrt[4]{q}} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^3 \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right) \vartheta_1\left(\frac{\pi z}{2\omega_1}, q\right)$$

09.15.27.0003.01

$$\sigma(z; g_2, g_3) = \frac{2\omega_1}{\pi} \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right) \frac{\vartheta_1\left(\frac{\pi z}{2\omega_1}, q\right)}{\vartheta_1'(0, q)}$$

Zeros

09.15.30.0001.01

$$\sigma(2m\omega_1 + 2n\omega_3; g_2, g_3) = 0 \;/;\; \{m, n\} \in \mathbb{Z}$$

Theorems

The solution to the special Lamé equation

The solution to the special Lamé equation $w''(z) == (2\,\wp(z; g_2, g_3) + \alpha)\, w(z)$ is given by

$$w(z) == c_1 \exp\left(u\,\zeta\left(\wp^{-1}(\alpha; g_2, g_3); g_2, g_3\right)\right) \frac{\sigma\left(z - \wp^{-1}(\alpha; g_2, g_3); g_2, g_3\right)}{\sigma(z; g_2, g_3)} + c_2 \exp\left(u\,\zeta\left(-\wp^{-1}(\alpha; g_2, g_3); g_2, g_3\right)\right) \frac{\sigma\left(z + \wp^{-1}(\alpha; g_2, g_3); g_2, g_3\right)}{\sigma(z; g_2, g_3)}.$$

The equations of motion for a heavy symmetric top

The equations of motion for a heavy symmetric top

$$\omega_x'(t) == -c\,\omega_y(t)\,\omega_z(t) - p_y(t),\quad \omega_y'(t) == c\,\omega_x(t)\,\omega_z(t) + p_x(t),\quad \omega_z'(t) == 0,$$
$$p_x'(t) == p_y(t)\,\omega_z(t) - p_z(t)\,\omega_y(t),\quad p_y'(t) == p_x(t)\,\omega_z(t) - p_x(t)\,\omega_z(t),\quad p_z'(t) == p_x(t)\,\omega_y(t) - p_y(t)\,\omega_x(t)$$

can be solved in closed form using the Weierstrass sigma function.

Expressing the Cartesian coordinates through ellipsoidal coordinates

Using Weierstrass sigma functions it is possible to give single–valued solutions $x(\lambda, \mu, \nu)$, $y(\lambda, \mu, \nu)$, $z(\lambda, \mu, \nu)$ for expressing the Cartesian coordinates $\{x, y, z\}$ through ellipsoidal coordinates $\{\lambda, \mu, \nu\}$ connected by $x^2 == \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}$, $y^2 == \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(a^2-b^2)}$, $z^2 == \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}$ where $a > b > c$ are the semi-axes of the defining ellipsoid.

History

– F. G. Eisenstein (1847)
– K. Weierstrass (1855, 1862, 1895)

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