WeierstrassZetaHalfPeriodValues

Notations

Traditional name

Weierstrass zeta function values at half periods

Traditional notation

$\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$

Mathematica StandardForm notation

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WeierstrassZetaHalfPeriodValues[{g2, g3}]
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Primary definition

09.21.02.0001.01

$\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ /;
$\{\eta_1, \eta_2, \eta_3\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \omega_2 = -\omega_1 - \omega_3$

Special notations for this file:

09.21.02.0002.01

$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), \omega_2(g_2, g_3), \omega_3(g_2, g_3)\}$ /; $\omega_2 = \omega_2(g_2, g_3) = -\omega_1(g_2, g_3) - \omega_3(g_2, g_3)$

09.21.02.0003.01

$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$

09.21.02.0004.01

$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$

09.21.02.0008.01

$\{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$

09.21.02.0009.01

$\{e_1', e_2', e_3'\} = \{e_1'(g_2, g_3), e_2'(g_2, g_3), e_3'(g_2, g_3)\}$

09.21.02.0005.01

$e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$

09.21.02.0010.01

$e_n' = \wp'(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$

09.21.02.0006.01

$\eta_n = \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$

09.21.02.0007.01

$$q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

Specific values

Values at fixed points

Equianharmonic case $\{g_2, g_3\} == \{0, 1\}$

09.21.03.0001.01

$$\{\eta_1(0, 1), \eta_2(0, 1), \eta_3(0, 1)\} == \left\{\frac{\pi}{2\omega_1\sqrt{3}}, \frac{\pi}{2\omega_1\sqrt{3}} e^{2\pi i/3}, \frac{\pi}{2\omega_1\sqrt{3}} e^{4\pi i/3}\right\} /; \{g_2, g_3\} == \{0, 1\}$$

Lemniscatic case $\{g_2, g_3\} == \{1, 0\}$

09.21.03.0002.01

$$\{\eta_1(1, 0), \eta_2(1, 0), \eta_3(1, 0)\} == \left\{\frac{\pi}{4\omega_1}, \frac{\pi}{4\omega_1}(1 - i), -\frac{\pi i}{4\omega_1}\right\} /; \{g_2, g_3\} == \{1, 0\}$$

Values at infinities

09.21.03.0003.01

$$\eta_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) == \frac{\pi^2}{12\omega_1}$$

09.21.03.0004.01

$$\eta_3(g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) == \frac{\pi^2}{12\omega_3}$$

General characteristics

Domain and analyticity

$\{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$ is an vector-valued function of g_2 and g_3 that is analytic in each component, and it is defined over $\mathbb{C}^2$.

09.21.04.0001.01

$$\{g_2 * g_3\} \longrightarrow \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \longrightarrow \{\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}\}$$

Symmetries and periodicities

Mirror symmetry

09.21.04.0002.01

$$\eta_1(\overline{g_2}, \overline{g_3}) == \overline{\eta_1(g_2, g_3)}$$

09.21.04.0003.01

$$\eta_3(\overline{g_2}, \overline{g_3}) == -\overline{\eta_3(g_2, g_3)}$$

Transformation of half-periods

09.21.04.0005.01

$$\{\{\eta_1(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3)),\,\eta_2(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3)),\,\eta_3(g_2(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3),\,g_3(a\,\omega_1+b\,\omega_3,\,c\,\omega_1+d\,\omega_3))\}\}=\{\{\eta_1(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)),\,\eta_2(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3)),\,\eta_3(g_2(\omega_1,\omega_3),\,g_3(\omega_1,\omega_3))\}\}\;/;\;\{a,b,c,d\}\in\mathbb{Z}\land a\,d-b\,c=\pm 1$$

Series representations

q-series

09.21.06.0001.01

$$\eta_1 = \frac{\pi^2}{12\,\omega_1} - \frac{2\,\pi^2}{\omega_1}\sum_{k=1}^{\infty}\frac{k\,q^{2k}}{1-q^{2k}}$$

09.21.06.0002.01

$$\eta_1 = \frac{\pi^2}{2\,\omega_1}\left(4\sum_{k=1}^{\infty}\frac{q^{2k}}{\left(q^{2k}+1\right)^2}+\frac{1}{2}\right)-e_1\,\omega_1$$

09.21.06.0003.01

$$\eta_1 = \frac{2\,\pi^2}{\omega_1}\sum_{k=1}^{\infty}\frac{q^{2k-1}}{\left(q^{2k-1}+1\right)^2}-e_2\,\omega_1$$

09.21.06.0004.01

$$\eta_1 = -e_3\,\omega_1-\frac{2\,\pi^2}{\omega_1}\sum_{k=1}^{\infty}\frac{q^{2k-1}}{\left(1-q^{2k-1}\right)^2}$$

09.21.06.0005.01

$$\eta_1 = \frac{\pi^2}{2\,\omega_1}\left(\frac{1}{6}-4\sum_{k=1}^{\infty}\frac{q^{2k}}{\left(1-q^{2k}\right)^2}\right)$$

Other series representations

09.21.06.0006.01

$$\eta_j = \frac{\pi^2}{2\,\omega_j}\left(\frac{1}{6}+\sum_{n=1}^{\infty}\frac{1}{\sin^2\left(\frac{\pi\,n\,\omega_k}{\omega_j}\right)}\right)\;/;\;\{j,k\}\in\{1,2,3\}\land j\neq k$$

09.21.06.0007.01

$$\eta_i+e_i\,\omega_i = \frac{\pi^2}{4\,\omega_i}\left(1+2\sum_{n=1}^{\infty}\sec^2\left(\frac{\pi\,n\,\omega_j}{\omega_i}\right)\right)\;/;\;\{i,j\}\in\{1,2,3\}\land i\neq j$$

09.21.06.0008.01

$$\eta_i+e_j\,\omega_i = \frac{\pi^2}{2\,\omega_i}\sum_{n=1}^{\infty}\sec^2\left(\pi\,\frac{2n-1}{2}\,\frac{\omega_k}{\omega_i}\right)\;/;\;\{i,j,k\}\in\{1,2,3\}\land i\neq j\neq k$$

09.21.06.0009.01

$$\eta_i+e_j\,\omega_i = \frac{\pi^2}{2\,\omega_i}\sum_{n=1}^{\infty}\csc^2\left(\pi\,\frac{2n-1}{2}\,\frac{\omega_j}{\omega_i}\right)\;/;\;\{i,j\}\in\{1,2,3\}\land i\neq j$$

Differential equations

Partial differential equations

09.21.13.0001.01

$$72\, g_3 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_2} + 4\, g_2^2 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_3} - \omega_1\, g_2 = 0$$

09.21.13.0002.01

$$72\, g_3 \frac{\partial \zeta(\omega_2; g_2, g_3)}{\partial g_2} + 4\, g_2^2 \frac{\partial \zeta(\omega_2; g_2, g_3)}{\partial g_3} - \omega_2\, g_2 = 0$$

Identities

Functional identities

09.21.17.0001.01

$$\eta_1 + \eta_2 + \eta_3 = 0$$

Legendre's relation:

09.21.17.0002.01

$$\eta_1\, \omega_3 - \eta_3\, \omega_1 = \frac{\pi\, i}{2}$$

Legendre's relation:

09.21.17.0003.01

$$\eta_i\, \omega_j - \eta_j\, \omega_i = \operatorname{sgn}\left(\operatorname{Re}\left(-i \frac{\omega_j}{\omega_i}\right)\right) \frac{\pi\, i}{2} \;/;\; \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

Differentiation

Low-order differentiation

With respect to g_2

09.21.20.0001.01

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27\, g_3^2\right)} \left(18\, g_3 \left\{e_1', e_2', e_3'\right\} - g_2 \left(3\, g_3 + 2\, g_2 \{e_1, e_2, e_3\}\right) \{\omega_1, \omega_2, \omega_3\} + 2\left(g_2^2 + 18\, g_3 \{e_1, e_2, e_3\}\right) \{\eta_1, \eta_2, \eta_3\}\right)$$

With respect to g_3

09.21.20.0002.01

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27\, g_3^2\right)} \left(\left(g_2^2 + 18\, g_3 \{e_1, e_2, e_3\}\right) \{\omega_1, \omega_2, \omega_3\} - 6\, g_2 \left\{e_1', e_2', e_3'\right\} - 6\left(3\, g_3 + 2\, g_2 \{e_1, e_2, e_3\}\right) \{\eta_1, \eta_2, \eta_3\}\right)$$

With respect to ω_1

09.21.20.0003.01

$$\frac{\partial \eta_1}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(2 \eta_1 \eta_3 - \frac{1}{6} g_2 \omega_1 \omega_3\right)$$

09.21.20.0004.01

$$\frac{\partial \eta_3}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(2 \eta_3^2 - \frac{1}{6} g_2 \omega_3^2\right)$$

With respect to ω_3

09.21.20.0005.01

$$\frac{\partial \eta_1}{\partial \omega_3} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(2 \eta_1^2 - \frac{1}{6} g_2 \omega_1^2\right)$$

09.21.20.0006.01

$$\frac{\partial \eta_3}{\partial \omega_1} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(2 \eta_1 \eta_3 - \frac{1}{6} g_2 \omega_1 \omega_3\right)$$

Representations through more general functions

Through other functions

09.21.26.0001.01

$$\{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} /; \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \land \omega_2 = -\omega_1 - \omega_3$$

Representations through equivalent functions

With related functions

Involving theta functions

09.21.27.0001.01

$$\eta_1 = -\frac{\pi^2}{12 \omega_1} \frac{\vartheta_1^{(3)}(0, q)}{\vartheta_1'(0, q)}$$

09.21.27.0002.01

$$\eta_1 = -e_i \omega_1 - \frac{\pi^2}{4 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} /; i \in \{1, 2, 3\}$$

09.21.27.0003.01

$$\eta_1^2 = \left(\frac{g_2}{6} - e_i^2\right) \omega_1^2 - e_i - \frac{\pi^2 \eta_1}{2 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} - \frac{\pi^4}{48 \omega_1^2} \frac{\vartheta_{i+1}^{(4)}(0, q)}{\vartheta_{i+1}(0, q)} /; i \in \{1, 2, 3\}$$

Involving elliptic integrals and modular functions

09.21.27.0004.01

$$\{\eta_1(g_2,\, g_3),\, \eta_2(g_2,\, g_3),\, \eta_3(g_2,\, g_3)\} == \left\{\sqrt{e_1 - e_3}\left(E(m) - \frac{e_1}{e_1 - e_3} K(m)\right),\right.$$
$$\left.\frac{K(m)\, e_1 - (e_1 - e_3)\,(E(m) - i\, E(1-m)) + i\, K(1-m)\, e_3}{\sqrt{e_1 - e_3}},\, -i\sqrt{e_1 - e_3}\left(E(1-m) + \frac{e_3}{e_1 - e_3} K(1-m)\right)\right\} /;\, m == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

09.21.27.0005.01

$$\eta_1 == \frac{K(m)\,(3\, E(m) + (m-2)\, K(m))}{3\, \omega_1} /;\, m == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

History

– K. Weierstrass (1862)

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