

# WeierstrassZetaHalfPeriodValues

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## Notations

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### Traditional name

Weierstrass zeta function values at half periods

### Traditional notation

$$\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$$

### Mathematica StandardForm notation

$$\text{WeierstrassZetaHalfPeriodValues}[\{g_2, g_3\}]$$

## Primary definition

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09.21.02.0001.01

$$\{\eta_1, \eta_2, \eta_3\} = \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} /;$$

$$\{\eta_1, \eta_2, \eta_3\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} \wedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \omega_2 = -\omega_1 - \omega_3$$

Special notations for this file:

09.21.02.0002.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), \omega_2(g_2, g_3), \omega_3(g_2, g_3)\} /; \omega_2 = \omega_2(g_2, g_3) = -\omega_1(g_2, g_3) - \omega_3(g_2, g_3)$$

09.21.02.0003.01

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.21.02.0004.01

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.21.02.0008.01

$$\{e_1, e_2, e_3\} = \{e_1(g_2, g_3), e_2(g_2, g_3), e_3(g_2, g_3)\}$$

09.21.02.0009.01

$$\{e'_1, e'_2, e'_3\} = \{e'_1(g_2, g_3), e'_2(g_2, g_3), e'_3(g_2, g_3)\}$$

09.21.02.0005.01

$$e_n = \wp(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.21.02.0010.01

$$e'_n = \wp'(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.21.02.0006.01

$$\eta_n = \zeta(\omega_n; g_2, g_3) \wedge n \in \{1, 2, 3\}$$

09.21.02.0007.01

$$q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

## Specific values

### Values at fixed points

Equianharmonic case  $\{g_2, g_3\} = \{0, 1\}$

09.21.03.0001.01

$$\{\eta_1(0, 1), \eta_2(0, 1), \eta_3(0, 1)\} = \left\{ \frac{\pi}{2 \omega_1 \sqrt{3}}, \frac{\pi}{2 \omega_1 \sqrt{3}} e^{2\pi i/3}, \frac{\pi}{2 \omega_1 \sqrt{3}} e^{4\pi i/3} \right\}; \{g_2, g_3\} = \{0, 1\}$$

Lemniscatic case  $\{g_2, g_3\} = \{1, 0\}$

09.21.03.0002.01

$$\{\eta_1(1, 0), \eta_2(1, 0), \eta_3(1, 0)\} = \left\{ \frac{\pi}{4 \omega_1}, \frac{\pi}{4 \omega_1} (1 - i), -\frac{\pi i}{4 \omega_1} \right\}; \{g_2, g_3\} = \{1, 0\}$$

### Values at infinities

09.21.03.0003.01

$$\eta_1(g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) = \frac{\pi^2}{12 \omega_1}$$

09.21.03.0004.01

$$\eta_3(g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) = \frac{\pi^2}{12 \omega_3}$$

## General characteristics

### Domain and analyticity

$\{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\}$  is an vector-valued function of  $g_2$  and  $g_3$  that is analytic in each component, and it is defined over  $\mathbb{C}^2$ .

09.21.04.0001.01

$$\{g_2 * g_3\} \rightarrow \{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} :: \{\mathbb{C} \otimes \mathbb{C}\} \rightarrow \{\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}\}$$

### Symmetries and periodicities

#### Mirror symmetry

09.21.04.0002.01

$$\eta_1(\overline{g_2}, \overline{g_3}) = \overline{\eta_1(g_2, g_3)}$$

09.21.04.0003.01

$$\eta_3(\overline{g_2}, \overline{g_3}) = -\overline{\eta_3(g_2, g_3)}$$

#### Transformation of half-periods

09.21.04.0005.01

$$\{\{\eta_1(g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3)), \eta_2(g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3)), \eta_3(g_2(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3), g_3(a\omega_1 + b\omega_3, c\omega_1 + d\omega_3))\}\} =$$

$$\{\{\eta_1(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), \eta_2(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)), \eta_3(g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\}\} /; \{a, b, c, d\} \in \mathbb{Z} \wedge ad - bc = \pm 1$$

## Series representations

### q-series

09.21.06.0001.01

$$\eta_1 = \frac{\pi^2}{12\omega_1} - \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}}$$

09.21.06.0002.01

$$\eta_1 = \frac{\pi^2}{2\omega_1} \left( 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{(q^{2k} + 1)^2} + \frac{1}{2} \right) - e_1 \omega_1$$

09.21.06.0003.01

$$\eta_1 = \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(q^{2k-1} + 1)^2} - e_2 \omega_1$$

09.21.06.0004.01

$$\eta_1 = -e_3 \omega_1 - \frac{2\pi^2}{\omega_1} \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(1 - q^{2k-1})^2}$$

09.21.06.0005.01

$$\eta_1 = \frac{\pi^2}{2\omega_1} \left( \frac{1}{6} - 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1 - q^{2k})^2} \right)$$

### Other series representations

09.21.06.0006.01

$$\eta_j = \frac{\pi^2}{2\omega_j} \left( \frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{\sin^2\left(\frac{\pi n \omega_k}{\omega_j}\right)} \right) /; \{j, k\} \in \{1, 2, 3\} \wedge j \neq k$$

09.21.06.0007.01

$$\eta_i + e_i \omega_i = \frac{\pi^2}{4\omega_i} \left( 1 + 2 \sum_{n=1}^{\infty} \sec^2\left(\frac{\pi n \omega_j}{\omega_i}\right) \right) /; \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

09.21.06.0008.01

$$\eta_i + e_j \omega_i = \frac{\pi^2}{2\omega_i} \sum_{n=1}^{\infty} \sec^2\left(\pi \frac{2n-1}{2} \frac{\omega_k}{\omega_i}\right) /; \{i, j, k\} \in \{1, 2, 3\} \wedge i \neq j \neq k$$

09.21.06.0009.01

$$\eta_i + e_j \omega_i = \frac{\pi^2}{2\omega_i} \sum_{n=1}^{\infty} \csc^2\left(\pi \frac{2n-1}{2} \frac{\omega_j}{\omega_i}\right) /; \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

## Differential equations

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### Partial differential equations

$$09.21.13.0001.01$$

$$72 g_3 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_2} + 4 g_2^2 \frac{\partial \zeta(\omega_1; g_2, g_3)}{\partial g_3} - \omega_1 g_2 = 0$$

$$09.21.13.0002.01$$

$$72 g_3 \frac{\partial \zeta(\omega_2; g_2, g_3)}{\partial g_2} + 4 g_2^2 \frac{\partial \zeta(\omega_2; g_2, g_3)}{\partial g_3} - \omega_2 g_2 = 0$$

## Identities

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### Functional identities

$$09.21.17.0001.01$$

$$\eta_1 + \eta_2 + \eta_3 = 0$$

Legendre's relation:

$$09.21.17.0002.01$$

$$\eta_1 \omega_3 - \eta_3 \omega_1 = \frac{\pi i}{2}$$

Legendre's relation:

$$09.21.17.0003.01$$

$$\eta_i \omega_j - \eta_j \omega_i = \operatorname{sgn} \left( \operatorname{Re} \left( -i \frac{\omega_j}{\omega_i} \right) \right) \frac{\pi i}{2} /; \{i, j\} \in \{1, 2, 3\} \wedge i \neq j$$

## Differentiation

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### Low-order differentiation

#### With respect to $g_2$

$$09.21.20.0001.01$$

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_2} =$$

$$\frac{1}{8(g_2^3 - 27g_3^2)} (18g_3 \{e'_1, e'_2, e'_3\} - g_2(3g_3 + 2g_2\{e_1, e_2, e_3\})\{\omega_1, \omega_2, \omega_3\} + 2(g_2^2 + 18g_3\{e_1, e_2, e_3\})\{\eta_1, \eta_2, \eta_3\})$$

#### With respect to $g_3$

$$09.21.20.0002.01$$

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_3} = \frac{1}{4(g_2^3 - 27g_3^2)} ((g_2^2 + 18g_3\{e_1, e_2, e_3\})\{\omega_1, \omega_2, \omega_3\} - 6g_2\{e'_1, e'_2, e'_3\} - 6(3g_3 + 2g_2\{e_1, e_2, e_3\})\{\eta_1, \eta_2, \eta_3\})$$

#### With respect to $\omega_1$

09.21.20.0003.01

$$\frac{\partial \eta_1}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left( 2 \eta_1 \eta_3 - \frac{1}{6} g_2 \omega_1 \omega_3 \right)$$

09.21.20.0004.01

$$\frac{\partial \eta_3}{\partial \omega_1} = \frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left( 2 \eta_3^2 - \frac{1}{6} g_2 \omega_3^2 \right)$$

**With respect to  $\omega_3$**

09.21.20.0005.01

$$\frac{\partial \eta_1}{\partial \omega_3} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left( 2 \eta_1^2 - \frac{1}{6} g_2 \omega_1^2 \right)$$

09.21.20.0006.01

$$\frac{\partial \eta_3}{\partial \omega_1} = -\frac{\omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left( 2 \eta_1 \eta_3 - \frac{1}{6} g_2 \omega_1 \omega_3 \right)$$

## Representations through more general functions

### Through other functions

09.21.26.0001.01

$$\{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_2; g_2, g_3), \zeta(\omega_3; g_2, g_3)\} /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \wedge \omega_2 = -\omega_1 - \omega_3$$

## Representations through equivalent functions

### With related functions

#### Involving theta functions

09.21.27.0001.01

$$\eta_1 = -\frac{\pi^2}{12 \omega_1} \frac{\vartheta_1^{(3)}(0, q)}{\vartheta_1'(0, q)}$$

09.21.27.0002.01

$$\eta_1 = -e_i \omega_1 - \frac{\pi^2}{4 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} /; i \in \{1, 2, 3\}$$

09.21.27.0003.01

$$\eta_1^2 = \left( \frac{g_2}{6} - e_i^2 \right) \omega_1^2 - e_i - \frac{\pi^2 \eta_1}{2 \omega_1} \frac{\vartheta_{i+1}''(0, q)}{\vartheta_{i+1}(0, q)} - \frac{\pi^4}{48 \omega_1^2} \frac{\vartheta_{i+1}^{(4)}(0, q)}{\vartheta_{i+1}(0, q)} /; i \in \{1, 2, 3\}$$

#### Involving elliptic integrals and modular functions

09.21.27.0004.01

$$\{\eta_1(g_2, g_3), \eta_2(g_2, g_3), \eta_3(g_2, g_3)\} = \left\{ \sqrt{e_1 - e_3} \left( E(m) - \frac{e_1}{e_1 - e_3} K(m) \right), \right. \\ \left. \frac{K(m) e_1 - (e_1 - e_3) (E(m) - i E(1 - m)) + i K(1 - m) e_3}{\sqrt{e_1 - e_3}}, -i \sqrt{e_1 - e_3} \left( E(1 - m) + \frac{e_3}{e_1 - e_3} K(1 - m) \right) \right\} /; m = \lambda \left( \frac{\omega_3}{\omega_1} \right)$$

09.21.27.0005.01

$$\eta_1 = \frac{K(m) (3 E(m) + (m - 2) K(m))}{3 \omega_1} /; m = \lambda \left( \frac{\omega_3}{\omega_1} \right)$$

## History

– K. Weierstrass (1862)

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