ZernikeR

Notations

Traditional name

Zernike polynomials

Traditional notation

$R_n^m(z)$

Mathematica StandardForm notation

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ZernikeR[n, m, z]
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Primary definition

05.18.02.0001.01

$$R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right) z^m P_{\frac{n-m}{2}}^{(m,0)}\left(1-2z^2\right) /; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Specific values

Specialized values

For fixed *n*, *m*

05.18.03.0001.01

$$R_n^m(0) = \begin{cases} (-1)^{n/2} & \frac{n}{2} \in \mathbb{Z} \bigwedge m = 0 \end{cases} /; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.03.0002.01

$$R_n^m(1) = \begin{cases} 1 & \frac{n-m}{2} \in \mathbb{Z} \end{cases} /; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

For fixed *n*, *z*

05.18.03.0003.01

$$R_n^0(z) = \cos\left(\frac{\pi n}{2}\right) P_{\frac{n}{2}}^{(0,0)}\left(1-2z^2\right) /; n \in \mathbb{N}$$

05.18.03.0004.01

$$R_n^0(z) = \begin{cases} \sum_{j=0}^{\frac{n}{2}} \frac{(-1)^j (n-j)!}{j! \left(\left(\frac{n}{2}-j\right)!\right)^2} z^{n-2j} & \frac{n}{2} \in \mathbb{Z} \end{cases} /; n \in \mathbb{N}$$

05.18.03.0005.01

$$R_n^n(z) = z^n /; n \in \mathbb{N}$$

05.18.03.0006.01

$R_n^{n-2\left\lfloor\frac{n}{2}\right\rfloor}(z) = 1 /; n \in \mathbb{N}$

05.18.03.0007.01

$R_n^{n-1}(z) = 0 /; n \in \mathbb{N}^+$

05.18.03.0008.01

$R_n^{n-2}(z) = n\, z^n - (n-1)\, z^{n-2} /; n \in \mathbb{Z} \land n \geq 2$

05.18.03.0009.01

$R_n^{n-3}(z) = 0 /; n \in \mathbb{Z} \land n \geq 3$

05.18.03.0010.01

$$R_n^{n-4}(z) = \frac{n\,(n-1)}{2}\, z^n - (n-1)\,(n-2)\, z^{n-2} + \frac{(n-2)\,(n-3)}{2}\, z^{n-4} /; n \in \mathbb{Z} \land n \geq 4$$

05.18.03.0011.01

$R_n^{n-5}(z) = 0 /; n \in \mathbb{Z} \bigwedge n \geq 5$

05.18.03.0012.01

$$R_n^{n-6}(z) = \frac{1}{6}\,(n-2)\,(n-1)\, n\, z^n - \frac{1}{2}\,(n-3)\,(n-2)\,(n-1)\, z^{n-2} + \frac{1}{2}\,(n-4)\,(n-3)\,(n-2)\, z^{n-4} - \frac{1}{6}\,(n-5)\,(n-4)\,(n-3)\, z^{n-6} /; n \in \mathbb{Z} \land n \geq 6$$

For fixed *z*

05.18.03.0013.01

$R_{m+2k+1}^m(z) = 0 /; m \in \mathbb{N} \bigwedge k \in \mathbb{N}$

05.18.03.0014.01

$$R_{m+2k}^m(z) = \sum_{j=0}^{k} \frac{(-1)^j\,(2\,k+m-j)!}{j!\,(k+m-j)!\,(k-j)!}\, z^{m+2k-2j} /; m \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

05.18.03.0015.01

$R_0^0(z) = 1$

05.18.03.0016.01

$R_1^0(z) = 0$

05.18.03.0017.01

$R_1^1(z) = z$

05.18.03.0018.01

$R_2^0(z) = 2\, z^2 - 1$

05.18.03.0019.01

$R_2^1(z) = 0$

05.18.03.0020.01

$R_2^2(z) = z^2$

05.18.03.0021.01

$R_3^0(z) = 0$

05.18.03.0022.01

$R_3^1(z) = 3\, z^3 - 2\, z$

05.18.03.0023.01

$R_3^2(z) = 0$

05.18.03.0024.01

$R_3^3(z) = z^3$

05.18.03.0025.01

$R_4^0(z) = 6 z^4 - 6 z^2 + 1$

05.18.03.0026.01

$R_4^1(z) = 0$

05.18.03.0027.01

$R_4^2(z) = 4 z^4 - 3 z^2$

05.18.03.0028.01

$R_4^3(z) = 0$

05.18.03.0029.01

$R_4^4(z) = z^4$

05.18.03.0030.01

$R_5^9(z) = 0$

05.18.03.0031.01

$R_5^1(z) = 10 z^5 - 12 z^3 + 3 z$

05.18.03.0032.01

$R_5^2(z) = 0$

05.18.03.0033.01

$R_5^3(z) = 5 z^5 - 4 z^3$

05.18.03.0034.01

$R_5^4(z) = 0$

05.18.03.0035.01

$R_5^5(z) = z^5$

General characteristics

Domain and analyticity

The function $R_n^m(z)$is defined over $\mathbb{N} \otimes \mathbb{N} \otimes \mathbb{C}$. For fixed n, m /; $\frac{n-m}{2} \in \mathbb{N}$, the function $R_n^m(z)$ is a polynomial in z of degree n.

05.18.04.0001.01

$(n * m * z) \longrightarrow R_n^m(z) :: (\mathbb{N} \otimes \mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

05.18.04.0002.01

$R_n^m(\bar{z}) = \overline{R_n^m(z)}$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed n, m /; $\frac{n-m}{2} \in \mathbb{N}$, the function $R_n^m(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.18.04.0003.01

$$\mathcal{S}ing_z\left(R_n^m(z)\right) = \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to *z*

For fixed n, m /; $\frac{n-m}{2} \in \mathbb{N}$, the function $R_n^m(z)$ does not have branch points.

05.18.04.0004.01

$$\mathcal{BP}_z\left(R_n^m(z)\right) = \{\}$$

Branch cuts

With respect to *z*

For fixed n, m /; $\frac{n-m}{2} \in \mathbb{N}$, the function $R_n^m(z)$ does not have branch cuts.

05.18.04.0005.01

$$\mathcal{BC}_z\left(R_n^m(z)\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

05.18.06.0001.01

$$R_n^m(z) \propto R_n^m(z_0) + \frac{1}{z_0\left(z_0^2-1\right)}\left(-\left((n+2)z_0^2+m\right)R_n^m(z_0) + (m+n+2)\,z_0\,R_{n+1}^{m+1}(z_0)\right)(z-z_0) + \frac{1}{2\,z_0^2\left(z_0^2-1\right)^2}\left(\left(m^2+\left((2n+7)z_0^2-1\right)m + z_0^2\left(n^2 z_0^2 + 5\,n\,z_0^2 + 6\,z_0^2 + n + 2\right)\right)R_n^m(z_0) - (m+n+2)\,z_0\left(\left((2n+7)z_0^2 + 2m + 1\right)R_{n+1}^{m+1}(z_0) - (m+n+4)\,z_0\,R_{n+2}^{m+2}(z_0)\right)\right)(z-z_0)^2 + \ldots \;/;\; (z \to z_0) \bigwedge n \in \mathbb{N} \bigwedge m \in \mathbb{N} \bigwedge n \geq m$$

05.18.06.0002.01

$$R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right) \frac{(-1)^{\frac{n-m}{2}} \Gamma\left(\frac{m+n+2}{2}\right)}{\Gamma\left(\frac{2-m+n}{2}\right) m!} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (i-k+m+1)_{k-i}\, 2^{-i}\, z_0^{m-k} \sum_{j=0}^{i} \frac{(1-i+2j)_{2(i-j)} \left(\frac{m+n+2}{2}\right)_j \left(\frac{m-n}{2}\right)_j}{(i-j)!\,(m+1)_j} (2 z_0)^{2j}\, {}_2F_1\left(j+\frac{m+n+2}{2}, j+\frac{m-n}{2}; j+m+1; z_0^2\right) (z-z_0)^k \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0003.01

$$R_n^m(z) \propto R_n^m(z_0)\,(1+O(z-z_0)) \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Expansions at $z == 0$

05.18.06.0004.01

$$R_n^m(z) \propto \frac{\cos\left(\frac{n-m}{2}\pi\right) \frac{m+n}{2}!}{m!\, \frac{n-m}{2}!} z^m \left(1 + \frac{(m-n)(m+n+2)\, z^2}{4(m+1)} + \frac{(m-n)(m-n+2)(m+n+2)(m+n+4)\, z^4}{32(m+1)(m+2)} + \ldots\right) /.\, (z \to 0) \land n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0005.01

$$R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right) z^m \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j \left(j+\frac{m+n}{2}\right)!}{j!\,(j+m)!\left(\frac{n-m}{2}-j\right)!} z^{2j} \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0006.01

$$R_n^m(z) = \begin{cases} (-1)^{\frac{n-m}{2}} \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j \left(j+\frac{m+n}{2}\right)!}{j!\,(j+m)!\left(\frac{n-m}{2}-j\right)!} z^{m+2j} & \frac{n-m}{2} \in \mathbb{Z} \end{cases} /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0007.01

$$R_n^m(z) \propto \frac{\cos\left(\frac{n-m}{2}\pi\right) \frac{m+n}{2}!}{m!\, \frac{n-m}{2}!} z^m \left(1+O\left(z^2\right)\right) /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Expansions at $z == \infty$

05.18.06.0008.01

$$R_n^m(z) \propto \frac{(-1)^{\frac{n-m}{2}} \cos\left(\frac{n-m}{2}\pi\right) n!}{\frac{n-m}{2}!\, \frac{m+n}{2}!} z^n \left(1 + \frac{(m-n)(m+n)}{4 n z^2} + \frac{(m-n)(m-n+2)(m+n-2)(m+n)}{32(n-1)\, n\, z^4} + \ldots\right) /;\, (|z| \to \infty) \land n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0009.01

$$R_n^m(z) = (-1)^{\frac{n-m}{2}} \cos\left(\frac{n-m}{2}\pi\right) \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j (n-j)!}{j!\left(\frac{m+n}{2}-j\right)!\left(\frac{n-m}{2}-j\right)!} z^{n-2j} \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0010.01

$$R_n^m(z) = \begin{cases} \sum_{j=0}^{\frac{n-m}{2}} \frac{(-1)^j (n-j)!}{j!\left(\frac{m+n}{2}-j\right)!\left(\frac{n-m}{2}-j\right)!} z^{n-2j} & \frac{n-m}{2} \in \mathbb{Z} \end{cases} /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.06.0011.01

$$R_n^m(z) \propto \frac{(-1)^{\frac{n-m}{2}} \cos\left(\frac{n-m}{2}\pi\right) n!}{\frac{n-m}{2}!\, \frac{m+n}{2}!} z^n \left(1+O\left(\frac{1}{z^2}\right)\right) /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Integral representations

On the real axis

05.18.07.0001.01

$$R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right)\int_0^\infty J_{n+1}(t)\, J_m(z\, t)\, dt \;/;\; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.07.0002.01

$$R_n^m(z) = \begin{cases} (-1)^{\frac{n-m}{2}} \int_0^\infty J_{n+1}(t)\, J_m(z\, t)\, dt & \frac{n-m}{2} \in \mathbb{Z} \end{cases} \;/;\; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Integral representations of negative integer order

Rodrigues-type formula.

05.18.07.0003.01

$$R_n^m(z) = \frac{(-1)^{\frac{n-m}{2}} \cos\left(\frac{n-m}{2}\pi\right) z^{-m}}{\Gamma\left(\frac{n-m}{2}+1\right)} \text{Function}\left[z, z^{\frac{m+n}{2}} (z-1)^{\frac{n-m}{2}}\right]^{\left(\frac{n-m}{2}\right)}\left(z^2\right) \;/;\; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

05.18.07.0004.01

$$R_n^m(z) = \begin{cases} \frac{z^{-m}}{\Gamma\left(\frac{n-m}{2}+1\right)} \text{Function}\left[z, z^{\frac{m+n}{2}} (z-1)^{\frac{n-m}{2}}\right]^{\left(\frac{n-m}{2}\right)}\left(z^2\right) & \frac{n-m}{2} \in \mathbb{Z} \end{cases} \;/;\; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

Generating functions

05.18.11.0001.01

$$R_{m+2k}^m(z) = \left(\left[w^k\right] \frac{\left(w - \sqrt{w^2 + 2\left(1 - 2z^2\right) w + 1} + 1\right)^m}{(2\, z\, w)^m \sqrt{w^2 + 2\left(1 - 2z^2\right) w + 1}}\right) \;/;\; m \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.18.16.0001.01

$$R_n^m(-z) = -R_n^m(z) \;/;\; \frac{m-1}{2} \in \mathbb{Z}$$

05.18.16.0002.01

$$R_n^m(-z) = R_n^m(z) \;/;\; \frac{m}{2} \in \mathbb{Z}$$

Identities

Recurrence identities

Consecutive neighbors

05.18.17.0001.01

$$R_n^m(z) = -\frac{2(n+3)\left(m^2-\left(n^2+6n+8\right)\left(2z^2-1\right)\right)}{(n+4)\left((n+2)^2-m^2\right)} R_{n+2}^m(z) - \frac{(n+2)\left((n+4)^2-m^2\right)}{(n+4)\left((n+2)^2-m^2\right)} R_{n+4}^m(z) \;/;\; n\in\mathbb{N}\land m\in\mathbb{N}\land m\leq n$$

05.18.17.0002.01

$$R_n^m(z) = \frac{n}{n^2-m^2}\left(\left(4(n-1)z^2-\frac{(n-m)^2}{n}-\frac{(m+n-2)^2}{n-2}\right)R_{n-2}^m(z)-\frac{(n-2)^2-m^2}{n-2}R_{n-4}^m(z)\right) \;/;\; n\in\mathbb{Z}\land n\geq 4\land m\in\mathbb{N}\land m\leq n-1$$

Functional identities

Relations between contiguous functions

05.18.17.0003.01

$$R_{n+2}^m(z) = \frac{n+2}{(n+2)^2-m^2}\left(\left(4(n+1)z^2-\frac{(2-m+n)^2}{n+2}-\frac{(m+n)^2}{n}\right)R_n^m(z)-\frac{n^2-m^2}{n}R_{n-2}^m(z)\right) \;/;\; n\in\mathbb{Z}\land n\geq 2\land m\in\mathbb{N}\land m\leq n$$

05.18.17.0004.01

$$R_{n+2}^m(z) = \frac{2(n+1)\left(2n(n+2)z^2-\left(m^2+(n+2)n\right)\right)}{n(-m+n+2)(m+n+2)} R_n^m(z) - \frac{(n-m)(m+n)(n+2)}{n(-m+n+2)(m+n+2)} R_{n-2}^m(z) \;/;\; n\in\mathbb{Z}\land n\geq 2\land m\in\mathbb{N}\land m\leq n$$

05.18.17.0005.01

$$R_n^m(z) = \frac{m+n}{2(n+1)z} R_{n-1}^{m-1}(z) + \frac{2-m+n}{2(n+1)z} R_{n+1}^{m-1}(z) \;/;\; n\in\mathbb{N}^+ \bigwedge m\in\mathbb{N}^+ \bigwedge m\leq n$$

05.18.17.0006.01

$$R_n^m(z) = \frac{n-m}{2(n+1)z} R_{n-1}^{m+1}(z) + \frac{m+n+2}{2(n+1)z} R_{n+1}^{m+1}(z) \;/;\; n\in\mathbb{N}^+ \bigwedge m\in\mathbb{N} \bigwedge m\leq n$$

05.18.17.0007.01

$$R_n^m(z) = \frac{(m+n+2)z}{2m} R_{n+1}^{m+1}(z) - \frac{(2-m+n)z}{2m} R_{n+1}^{m-1}(z) \;/;\; n\in\mathbb{N} \bigwedge m\in\mathbb{N}^+ \bigwedge m\leq n$$

05.18.17.0008.01

$$R_n^m(z) = \frac{(m+n)z}{2m} R_{n-1}^{m-1}(z) - \frac{(n-m)z}{2m} R_{n-1}^{m+1}(z) \;/;\; n\in\mathbb{N} \bigwedge m\in\mathbb{N}^+ \bigwedge m\leq n$$

05.18.17.0009.01

$$R_n^m(z) = \frac{(n-m)R_{n-1}^{m+1}(z)+(m+n+2)R_{n+1}^{m+1}(z)}{2(n+1)z} \;/;\; n\in\mathbb{N}\land m\in\mathbb{N}\land m\leq n$$

05.18.17.0010.01

$$R_n^{m+2}(z) = \frac{1}{n+1}\frac{\partial\left(R_{n+1}^{m+1}(z)-R_{n-1}^{m+1}(z)\right)}{\partial z} - R_n^m(z) \;/;\; n\in\mathbb{N}^+ \bigwedge m\in\mathbb{N} \bigwedge m<n$$

Differentiation

Low-order differentiation

05.18.20.0001.01

$$\frac{\partial R_n^m(z)}{\partial z} = \frac{(n+2)z^2+m}{z\left(1-z^2\right)} R_n^m(z) - \frac{m+n+2}{1-z^2} R_{n+1}^{m+1}(z) \;/;\; n\in\mathbb{N}\land m\in\mathbb{N}\land m\leq n$$

05.18.20.0002.01

$$\frac{\partial R_n^m(z)}{\partial z} = \frac{m^2+n^2-2\,n^2\,z^2}{2\,n\,z\left(1-z^2\right)}\,R_n^m(z) - \frac{(m+n)\,(n-m)}{2\,n\,z\left(z^2-1\right)}\,R_{n-2}^m(z)\;/;\;n\in\mathbb{Z}\land n\geq 2\land m\in\mathbb{N}\land m\leq n$$

05.18.20.0003.01

$$\frac{\partial^2 R_n^m(z)}{\partial z^2} = \frac{m^2+\left((2\,n+7)\,z^2-1\right)m+z^2\left(n^2\,z^2+5\,n\,z^2+6\,z^2+n+2\right)}{z^2\left(z^2-1\right)^2}\,R_n^m(z) - \frac{(m+n+2)\left((2\,n+7)\,z^2+2\,m+1\right)}{z\left(z^2-1\right)^2}\,R_{n+1}^{m+1}(z) + \frac{(m+n+2)\,(m+n+4)}{\left(z^2-1\right)^2}\,R_{n+2}^{m+2}(z)\;/;\;n\in\mathbb{N}\land m\in\mathbb{N}\land m\leq n$$

Symbolic differentiation

05.18.20.0004.01

$$\frac{\partial^h R_n^m(z)}{\partial z^h} = \cos\left(\frac{n-m}{2}\,\pi\right)\frac{\Gamma\left(\frac{m+n+2}{2}\right)}{\Gamma\left(\frac{2-m+n}{2}\right)m\,!}\sum_{k=0}^{h}\binom{h}{k}(k+m-h+1)_{h-k}\,2^{-k}\,z^{m-h}\sum_{j=0}^{k}\frac{(2\,j-k+1)_{2\,(k-j)}\left(\frac{m+n+2}{2}\right)_j\left(\frac{m-n}{2}\right)_j}{(k-j)\,!\,(m+1)_j}\,(2\,z)^{2\,j}\,{}_2F_1\left(j+\frac{m+n+2}{2},\,j+\frac{m-n}{2};\,j+m+1;\,z^2\right)\;/;\;h\in\mathbb{N}\bigwedge n\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge m\leq n$$

05.18.20.0005.01

$$\frac{\partial^h R_n^m(z)}{\partial z^h} = \left\{\;\frac{(-1)^{\frac{n-m}{2}}\,\Gamma\left(\frac{m+n+2}{2}\right)}{\Gamma\left(\frac{2-m+n}{2}\right)m!}\sum_{k=0}^{h}\binom{h}{k}(k+m-h+1)_{h-k}\,2^{-k}\,z^{m-h}\sum_{j=0}^{k}\frac{(2\,j-k+1)_{2\,(k-j)}\left(\frac{m+n+2}{2}\right)_j\left(\frac{m-n}{2}\right)_j}{(k-j)!\,(m+1)_j}\,(2\,z)^{2\,j}\,{}_2F_1\left(j+\frac{m+n+2}{2},\,j+\frac{m-n}{2};\,j+m+1;\,z^2\right.\right.$$

$$/;\;h\in\mathbb{N}\bigwedge n\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge m\leq n$$

Fractional integro-differentiation

05.18.20.0006.01

$$\frac{\partial^\alpha R_n^m(z)}{\partial z^\alpha} = \cos\left(\frac{n-m}{2}\,\pi\right)\frac{\Gamma\left(\frac{m+n}{2}+1\right)}{\Gamma\left(\frac{n-m}{2}+1\right)\Gamma(m-\alpha+1)}\,z^{m-\alpha}\,{}_4F_3\left(-\frac{n-m}{2},\,\frac{m+1}{2},\,\frac{m}{2}+1,\,\frac{m+n}{2}+1;\,m+1,\,\frac{m-\alpha+1}{2},\,\frac{m-\alpha}{2}+1;\,z^2\right)$$

05.18.20.0007.01

$$\frac{\partial^\alpha R_n^m(z)}{\partial z^\alpha} = \left\{\;\frac{(-1)^{\frac{n-m}{2}}\,\Gamma\left(\frac{m+n}{2}+1\right)}{\Gamma\left(\frac{n-m}{2}+1\right)\Gamma(m-\alpha+1)}\,z^{m-\alpha}\,{}_4F_3\left(-\frac{n-m}{2},\,\frac{m+1}{2},\,\frac{m}{2}+1,\,\frac{m+n}{2}+1;\,m+1,\,\frac{m-\alpha+1}{2},\,\frac{m-\alpha}{2}+1;\,z^2\right)\quad\frac{n-m}{2}\in\mathbb{Z}\right.$$

Integration

Indefinite integration

05.18.21.0001.01

$$\int R_n^m(z)\,dz = \frac{\Gamma(n+1)\,z^{n+1}}{(n+1)\,\Gamma\left(\frac{1}{2}\,(2-m+n)\right)\Gamma\left(\frac{1}{2}\,(2+m+n)\right)}\,{}_3F_2\left(-\frac{n+1}{2},\,-\frac{m+n}{2},\,\frac{m-n}{2};\,\frac{1-n}{2},\,-n;\,\frac{1}{z^2}\right)$$

05.18.21.0002.01

$$\int R_n^m(z)\,dz = \cos\left(\frac{n-m}{2}\pi\right)\frac{\Gamma\left(\frac{1}{2}(2+m+n)\right)}{(m+1)!\,\Gamma\left(\frac{1}{2}(2-m+n)\right)} z^{m+1}\,{}_3F_2\left(\frac{m-n}{2}, \frac{m+1}{2}, \frac{1}{2}(m+n+2); m+1, \frac{m+3}{2}; z^2\right)$$

Definite integration

05.18.21.0003.01

$$\int_0^1 t\,R_n^{n-2k}(t)\,J_{n-2k}(z\,t)\,dt = \frac{(-1)^k}{z} J_{n+1}(z) /; n\in\mathbb{N} \bigwedge k\in\mathbb{N} \bigwedge k \le \left\lfloor \frac{n}{2} \right\rfloor$$

Summation

Infinite summation

05.18.23.0001.01

$$\sum_{k=0}^{\infty} R_{m+2k}^m(z)\,w^k = \frac{\left(w-\sqrt{w^2+2\left(1-2z^2\right)w+1}+1\right)^m}{(2\,z\,w)^m\sqrt{w^2+2\left(1-2z^2\right)w+1}} /; m\in\mathbb{N}$$

Operations

Limit operation

05.18.25.0001.01

$$\lim_{z\to 0} z^{-m} R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right)\frac{\frac{m+n}{2}!}{m!\,\frac{n-m}{2}!} /; n\in\mathbb{N}\land m\in\mathbb{N}\land n\ge m$$

Orthogonality, completeness, and Fourier expansions

05.18.25.0002.01

$$\int_0^1 t\,R_n^{\min(n,p)}(t)\,R_p^{\min(n,p)}(t)\,dt = \frac{\delta_{n,p}}{2(n+1)} /; n\in\mathbb{N} \bigwedge p\in\mathbb{N}$$

Representations through more general functions

Through hypergeometric functions

05.18.26.0001.01

$$R_n^m(z) = \cos\left(\frac{n-m}{2}\pi\right)\frac{\Gamma\left(\frac{m+n+2}{2}\right)z^m}{\Gamma\left(\frac{2-m+n}{2}\right)\Gamma(m+1)}\,{}_2F_1\left(\frac{m+n+2}{2}, \frac{m-n}{2}; m+1; z^2\right) /; n\in\mathbb{N}\land m\in\mathbb{N}\land n\ge m$$

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$$R_n^m(z) = \begin{cases} \frac{z^n\,\Gamma(n+1)}{\Gamma\left(1-\frac{m}{2}+\frac{n}{2}\right)\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}\,{}_2F_1\left(\frac{m-n}{2}, -\frac{m+n}{2}; -n; \frac{1}{z^2}\right) & \frac{n-m}{2}\in\mathbb{Z} \end{cases} /; n\in\mathbb{N}\land m\in\mathbb{N}\land n\ge m$$

Through other functions

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$$R_n^m(z) = \begin{cases} (-1)^{\frac{n-m}{2}} z^m P_{\frac{n-m}{2}}^{(m,0)}(1 - 2 z^2) & \frac{n-m}{2} \in \mathbb{Z} \end{cases} /; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

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$$R_n^0(z) = \begin{cases} C_m^{\frac{1}{2}}(2 z^2 - 1) & \frac{n}{2} \in \mathbb{Z} \end{cases} /; n \in \mathbb{N} \land m \in \mathbb{N} \land n \geq m$$

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$$R_n^0(z) == \begin{cases} P_{\frac{n}{2}}(2 z^2 - 1) & \frac{n}{2} \in \mathbb{Z} \end{cases}$$

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