Introductions to Cos

Introduction to the trigonometric functions

General

The six trigonometric functions sine $\sin(z)$, cosine $\cos(z)$, tangent $\tan(z)$, cotangent $\cot(z)$, cosecant $\csc(z)$, and secant $\sec(z)$ are well known and among the most frequently used elementary functions. The most popular functions $\sin(z)$, $\cos(z)$, $\tan(z)$, and $\cot(z)$ are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of $i z$:

$$\sin(z) = \frac{e^{i z} - e^{-i z}}{2 i}$$

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}$$

$$\tan(z) = -\frac{i \left(e^{i z} - e^{-i z}\right)}{e^{i z} + e^{-i z}}$$

$$\cot(z) = \frac{i \left(e^{i z} + e^{-i z}\right)}{e^{i z} - e^{-i z}}$$

$$\csc(z) = \frac{2 i}{e^{i z} - e^{-i z}}$$

$$\sec(z) = \frac{2}{e^{i z} + e^{-i z}}.$$

The functions $\tan(z)$, $\cot(z)$, $\csc(z)$, and $\sec(z)$ can also be defined through the functions $\sin(z)$ and $\cos(z)$ using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

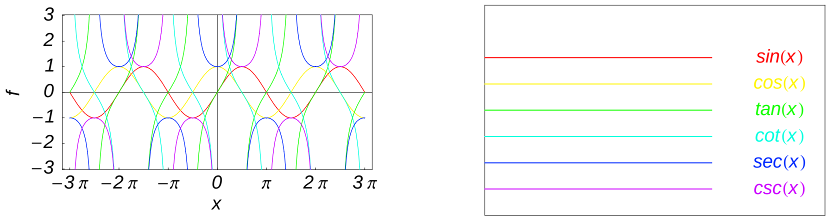
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.



Connections within the group of trigonometric functions and with other function groups

Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sin(z)$ and $\cos(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) = \sqrt{\frac{\pi z}{2}}\, J_{1/2}(z) \quad \sin(z) = -i\sqrt{\frac{\pi i z}{2}}\, I_{1/2}(i z) \quad \sin(z) = \sqrt{\frac{\pi z}{2}}\, Y_{-1/2}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}}\left(\sqrt{i z}\, K_{1/2}(i z) - \sqrt{-i z}\, K_{1/2}(-i z)\right)$$

$$\cos(z) = \sqrt{\frac{\pi z}{2}}\, J_{-1/2}(z) \quad \cos(z) = \sqrt{\frac{\pi i z}{2}}\, I_{-1/2}(i z) \quad \cos(z) = -\sqrt{\frac{\pi z}{2}}\, Y_{1/2}(z) \quad \cos(z) = \sqrt{\frac{i z}{2\pi}}\, K_{1/2}(i z) + \sqrt{-\frac{i z}{2\pi}}\, K_{1/2}(-i z)$$

$$\sin(z) = \mathrm{Se}(1, 0, z) \quad \cos(z) = \mathrm{Ce}(1, 0, z)$$

$$\sin(z) = z\, {}_0F_1\left(; \tfrac{3}{2}; -\tfrac{z^2}{4}\right) \quad \cos(z) = {}_0F_1\left(; \tfrac{1}{2}; -\tfrac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{array}{ll}
\sin(z) = \mathrm{sd}(z \mid 0) = \mathrm{sn}(z \mid 0) & \sin(z) = -i\,\mathrm{sc}(i z \mid 1) = -i\,\mathrm{sd}(i z \mid 1) \\
\cos(z) = \mathrm{cd}(z \mid 0) = \mathrm{cn}(z \mid 0) & \cos(z) = \mathrm{nc}(i z \mid 1) = \mathrm{nd}(i z \mid 1) \\
\tan(z) = \mathrm{sc}(z \mid 0) & \tan(z) = -i\,\mathrm{sn}(i z \mid 1) \\
\cot(z) = \mathrm{cs}(z \mid 0) & \cot(z) = i\,\mathrm{ns}(i z \mid 1) \\
\csc(z) = \mathrm{ds}(z \mid 0) = \mathrm{ns}(z \mid 0) & \csc(z) = i\,\mathrm{cs}(i z \mid 1) = i\,\mathrm{ds}(i z \mid 1) \\
\sec(z) = \mathrm{dc}(z \mid 0) = \mathrm{nc}(z \mid 0) & \sec(z) = \mathrm{cn}(i z \mid 1) = \mathrm{dn}(i z \mid 1).
\end{array}$$

Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\begin{array}{ll}
\sin(z) = -i \sinh(i z) & \sin(i z) = i \sinh(z) \\
\cos(z) = \cosh(i z) & \cos(i z) = \cosh(z) \\
\tan(z) = -i \tanh(i z) & \tan(i z) = i \tanh(z) \\
\cot(z) = i \coth(i z) & \cot(i z) = -i \coth(z) \\
\csc(z) = i\, \mathrm{csch}(i z) & \csc(i z) = -i\, \mathrm{csch}(z) \\
\sec(z) = \mathrm{sech}(i z) & \sec(i z) = \mathrm{sech}(z).
\end{array}$$

Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\sin(\sin^{-1}(z)) = z \quad \sin^{-1}(\sin(z)) = z \;/;\; -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0$$

$$\cos(\cos^{-1}(z)) = z \quad \cos^{-1}(\cos(z)) = z \;/;\; 0 < \mathrm{Re}(z) < \pi \lor \mathrm{Re}(z) = 0 \land \mathrm{Im}(z) \geq 0 \lor \mathrm{Re}(z) = \pi \land \mathrm{Im}(z) \leq 0$$

$$\tan(\tan^{-1}(z)) = z \quad \tan^{-1}(\tan(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) > 0$$

$$\cot(\cot^{-1}(z)) = z \quad \cot^{-1}(\cot(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$

$$\csc(\csc^{-1}(z)) = z \quad \csc^{-1}(\csc(z)) = z \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0$$

$$\sec(\sec^{-1}(z)) = z \quad \sec^{-1}(\sec(z)) = z \;/;\; 0 < \mathrm{Re}(z) < \pi \lor \mathrm{Re}(z) = 0 \land \mathrm{Im}(z) \geq 0 \lor \mathrm{Re}(z) = \pi \land \mathrm{Im}(z) \leq 0.$$

Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) \qquad \cos^2(z) = 1 - \sin^2(z)$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\sin(z)}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \tan^2(z) = \frac{\sin^2(z)}{1 - \sin^2(z)}$$

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} \qquad \cot^2(z) = \frac{1 - \sin^2(z)}{\sin^2(z)}$$

$$\csc(z) = \frac{1}{\sin(z)} \qquad \csc^2(z) = \frac{1}{\sin^2(z)}$$

$$\sec(z) = \frac{1}{\cos(z)} = \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} \qquad \sec^2(z) = \frac{1}{1 - \sin^2(z)}.$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument z is replaced by $p\,\pi/2 + q\,z$ with $q^2 = 1 \bigwedge p \in \mathbb{Z}$:

$$\sin(-z - 2\pi) = -\sin(z) \qquad \sin(z - 2\pi) = \sin(z)$$

$$\sin\left(-z - \frac{3\pi}{2}\right) = \cos(z) \qquad \sin\left(z - \frac{3\pi}{2}\right) = \cos(z)$$

$$\sin(-z - \pi) = \sin(z) \qquad \sin(z - \pi) = -\sin(z)$$

$$\sin\left(-z - \frac{\pi}{2}\right) = -\cos(z) \qquad \sin\left(z - \frac{\pi}{2}\right) = -\cos(z)$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos(z) \qquad \sin\left(\frac{\pi}{2} - z\right) = \cos(z)$$

$$\sin(z + \pi) = -\sin(z) \qquad \sin(\pi - z) = \sin(z)$$

$$\sin\left(z + \frac{3\pi}{2}\right) = -\cos(z) \qquad \sin\left(\frac{3\pi}{2} - z\right) = -\cos(z)$$

$$\sin(z + 2\pi) = \sin(z) \qquad \sin(2\pi - z) = -\sin(z)$$

$$\cos(-z-2\pi) = \cos(z) \quad \cos(z-2\pi) = \cos(z)$$
$$\cos\left(-z-\frac{3\pi}{2}\right) = \sin(z) \quad \cos\left(z-\frac{3\pi}{2}\right) = -\sin(z)$$
$$\cos(-z-\pi) = -\cos(z) \quad \cos(z-\pi) = -\cos(z)$$
$$\cos\left(-z-\frac{\pi}{2}\right) = -\sin(z) \quad \cos\left(z-\frac{\pi}{2}\right) = \sin(z)$$
$$\cos\left(z+\frac{\pi}{2}\right) = -\sin(z) \quad \cos\left(\frac{\pi}{2}-z\right) = \sin(z)$$
$$\cos(z+\pi) = -\cos(z) \quad \cos(\pi-z) = -\cos(z)$$
$$\cos\left(z+\frac{3\pi}{2}\right) = \sin(z) \quad \cos\left(\frac{3\pi}{2}-z\right) = -\sin(z)$$
$$\cos(z+2\pi) = \cos(z) \quad \cos(2\pi-z) = \cos(z)$$

$$\tan(-z-\pi) = -\tan(z) \quad \tan(z-\pi) = \tan(z)$$
$$\tan\left(-z-\frac{\pi}{2}\right) = \cot(z) \quad \tan\left(z-\frac{\pi}{2}\right) = -\cot(z)$$
$$\tan\left(z+\frac{\pi}{2}\right) = -\cot(z) \quad \tan\left(\frac{\pi}{2}-z\right) = \cot(z)$$
$$\tan(z+\pi) = \tan(z) \quad \tan(\pi-z) = -\tan(z)$$

$$\cot(-z-\pi) = -\cot(z) \quad \cot(z-\pi) = \cot(z)$$
$$\cot\left(-z-\frac{\pi}{2}\right) = \tan(z) \quad \cot\left(z-\frac{\pi}{2}\right) = -\tan(z)$$
$$\cot\left(z+\frac{\pi}{2}\right) = -\tan(z) \quad \cot\left(\frac{\pi}{2}-z\right) = \tan(z)$$
$$\cot(z+\pi) = \cot(z) \quad \cot(\pi-z) = -\cot(z)$$

$$\csc(-z-2\pi) = -\csc(z) \quad \csc(z-2\pi) = \csc(z)$$
$$\csc\left(-z-\frac{3\pi}{2}\right) = \sec(z) \quad \csc\left(z-\frac{3\pi}{2}\right) = \sec(z)$$
$$\csc(-z-\pi) = \csc(z) \quad \csc(z-\pi) = -\csc(z)$$
$$\csc\left(-z-\frac{\pi}{2}\right) = -\sec(z) \quad \csc\left(z-\frac{\pi}{2}\right) = -\sec(z)$$
$$\csc\left(z+\frac{\pi}{2}\right) = \sec(z) \quad \csc\left(\frac{\pi}{2}-z\right) = \sec(z)$$
$$\csc(z+\pi) = -\csc(z) \quad \csc(\pi-z) = \csc(z)$$
$$\csc\left(z+\frac{3\pi}{2}\right) = -\sec(z) \quad \csc\left(\frac{3\pi}{2}-z\right) = -\sec(z)$$
$$\csc(z+2\pi) = \csc(z) \quad \csc(2\pi-z) = -\csc(z)$$

$$\sec(-z-2\pi) = \sec(z) \quad \sec(z-2\pi) = \sec(z)$$
$$\sec\left(-z-\frac{3\pi}{2}\right) = \csc(z) \quad \sec\left(z-\frac{3\pi}{2}\right) = -\csc(z)$$
$$\sec(-z-\pi) = -\sec(z) \quad \sec(z-\pi) = -\sec(z)$$
$$\sec\left(-z-\frac{\pi}{2}\right) = -\csc(z) \quad \sec\left(z-\frac{\pi}{2}\right) = \csc(z)$$
$$\sec\left(z+\frac{\pi}{2}\right) = -\csc(z) \quad \sec\left(\frac{\pi}{2}-z\right) = \csc(z)$$
$$\sec(z+\pi) = -\sec(z) \quad \sec(\pi-z) = -\sec(z)$$
$$\sec\left(z+\frac{3\pi}{2}\right) = \csc(z) \quad \sec\left(\frac{3\pi}{2}-z\right) = -\csc(z)$$
$$\sec(z+2\pi) = \sec(z) \quad \sec(2\pi-z) = \sec(z).$$

The best-known properties and formulas for trigonometric functions

Real values for real arguments

For real values of argument z, the values of all the trigonometric functions are real (or infinity).

In the points $z = 2\pi n/m$ /; $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like $\sin(\pi/2) = 1$ or $\sin(\pi/6) = 1/2$). The values of trigonometric functions can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

Simple values at zero

All trigonometric functions have rather simple values for arguments $z == 0$ and $z == \pi/2$:

$\sin(0) == 0 \quad \sin\left(\frac{\pi}{2}\right) == 1$

$\cos(0) == 1 \quad \cos\left(\frac{\pi}{2}\right) == 0$

$\tan(0) == 0 \quad \tan\left(\frac{\pi}{2}\right) == \tilde{\infty}$

$\cot(0) == \tilde{\infty} \quad \cot\left(\frac{\pi}{2}\right) == 0$

$\csc(0) == \tilde{\infty} \quad \csc\left(\frac{\pi}{2}\right) == 1$

$\sec(0) == 1 \quad \sec\left(\frac{\pi}{2}\right) == \tilde{\infty}.$

Analyticity

All trigonometric functions are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. The two functions $\sin(z)$ and $\cos(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other trigonometric functions are meromorphic functions with simple poles at points $z == \pi k$ /; $k \in \mathbb{Z}$ for $\csc(z)$ and $\cot(z)$, and at points $z == \pi/2 + \pi k$ /; $k \in \mathbb{Z}$ for $\sec(z)$ and $\tan(z)$.

Periodicity

All trigonometric functions are periodic functions with a real period (2π or π):

$\sin(z) == \sin(z + 2\pi) \quad \sin(z + 2\pi k) == \sin(z)$ /; $k \in \mathbb{Z}$

$\cos(z) == \cos(z + 2\pi) \quad \cos(z + 2\pi k) == \cos(z)$ /; $k \in \mathbb{Z}$

$\tan(z) == \tan(z + \pi) \quad \tan(z + \pi k) == \tan(z)$ /; $k \in \mathbb{Z}$

$\cot(z) == \cot(z + \pi) \quad \cot(z + \pi k) == \cot(z)$ /; $k \in \mathbb{Z}$

$\csc(z) == \csc(z + 2\pi) \quad \csc(z + 2\pi k) == \csc(z)$ /; $k \in \mathbb{Z}$

$\sec(z) == \sec(z + 2\pi) \quad \sec(z + 2\pi k) == \sec(z)$ /; $k \in \mathbb{Z}$.

Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$\sin(-z) == -\sin(z) \quad \sin(\bar{z}) == \overline{\sin(z)}$

$\cos(-z) == \cos(z) \quad \cos(\bar{z}) == \overline{\cos(z)}$

$\tan(-z) == -\tan(z) \quad \tan(\bar{z}) == \overline{\tan(z)}$

$\cot(-z) == -\cot(z) \quad \cot(\bar{z}) == \overline{\cot(z)}$

$\csc(-z) == -\csc(z) \quad \csc(\bar{z}) == \overline{\csc(z)}$

$\sec(-z) == \sec(z) \quad \sec(\bar{z}) == \overline{\sec(z)}.$

Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\frac{\partial \sin(z)}{\partial z} == \cos(z) \qquad \frac{\partial \cos(z)}{\partial z} == -\sin(z) \qquad \frac{\partial \tan(z)}{\partial z} == \sec^2(z)$$
$$\frac{\partial \cot(z)}{\partial z} == -\csc^2(z) \qquad \frac{\partial \csc(z)}{\partial z} == -\cot(z)\csc(z) \qquad \frac{\partial \sec(z)}{\partial z} == \sec(z)\tan(z).$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$:

$$w''(z) + w(z) == 0 /; w(z) == \cos(z) \land w(0) == 1 \land w'(0) == 0$$
$$w''(z) + w(z) == 0 /; w(z) == \sin(z) \land w(0) == 0 \land w'(0) == 1$$
$$w''(z) + w(z) == 0 /; w(z) == c_1 \cos(z) + c_2 \sin(z).$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 - (w(z))^2} == 0 /; w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{1 - (w(z))^2} == 0 /; w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - w(z)^2 - 1 == 0 /; w(z) == \tan(z) \land w(0) == 0$$
$$w'(z) + w(z)^2 + 1 == 0 /; w(z) == \cot(z) \bigwedge w\left(\frac{\pi}{2}\right) == 0$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 /; w(z) == \csc(z)$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 /; w(z) == \sec(z).$$

Applications of trigonometric functions

Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle, a, b, and c represent the lengths of the sides opposite to the angles, Δ the area , R the circumradius, and r the inradius. Then the following identities hold:

$$\alpha + \beta + \gamma == \pi$$
$$\frac{\sin(\alpha)}{a} == \frac{\sin(\beta)}{b} == \frac{\sin(\gamma)}{c}$$

$$\sin(\alpha)\sin(\beta)\sin(\gamma) == \frac{\Delta}{2R^2} \qquad \sin(\alpha) == \frac{2\Delta}{bc}$$
$$\cos(\alpha) == \frac{b^2 + c^2 - a^2}{2bc} \qquad \cot(\alpha) == \frac{b^2 + c^2 - a^2}{4\Delta}$$
$$\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) == \frac{r}{4R} \qquad \cos(\alpha) + \cos(\beta) + \cos(\gamma) == 1 + \frac{r}{R}$$

$$\cot(\alpha)+\cot(\beta)+\cot(\gamma)=\frac{a^2+b^2+c^2}{4\,\Delta}$$

$$\tan(\alpha)+\tan(\beta)+\tan(\gamma)=\tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta)+\cot(\alpha)\cot(\gamma)+\cot(\beta)\cot(\gamma)=1$$

$$\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)=1-2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)}=\frac{r}{c}.$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha)=\frac{a}{c}\ /;\ \gamma=\frac{\pi}{2}\qquad \cos(\alpha)=\frac{b}{c}\ /;\ \gamma=\frac{\pi}{2}$$

$$\tan(\alpha)=\frac{a}{b}\ /;\ \gamma=\frac{\pi}{2}\qquad \cot(\alpha)=\frac{b}{a}\ /;\ \gamma=\frac{\pi}{2}$$

$$\csc(\alpha)=\frac{c}{a}\ /;\ \gamma=\frac{\pi}{2}\qquad \sec(\alpha)=\frac{c}{b}\ /;\ \gamma=\frac{\pi}{2}.$$

Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Cosine Function

Defining the cosine function

The cosine function is one of the oldest mathematical functions. It was first used in ancient Egypt in the book of Ahmes (c. 2000 B.C.). Much later F. Viète (1590) evaluated some values of $\cos(n\,z)$, E. Gunter (1636) introduced the notation "Cosi" and the word "cosinus" (replacing "complementi sinus"), and I. Newton (1658, 1665) found the series expansion for $\cos(z)$.

The classical definition of the cosine function for real arguments is: "the cosine of an angle α in a right-angle triangle is the ratio of the length of the adjacent leg to the length of the hypotenuse." This description of $\cos(\alpha)$ is valid for $0<\alpha<\pi/2$ when the triangle is nondegenerate. This approach to the cosine can be expanded to arbitrary real values of α if consideration is given to the arbitrary point $\{x,\ y\}$ in the x,y-Cartesian plane and $\cos(\alpha)$ is defined as the ratio $x\Big/\left(x^2+y^2\right)^{1/2}$, assuming that α is the value of the angle between the positive direction of the x-axis and the direction from the origin to the point $\{x,\ y\}$.

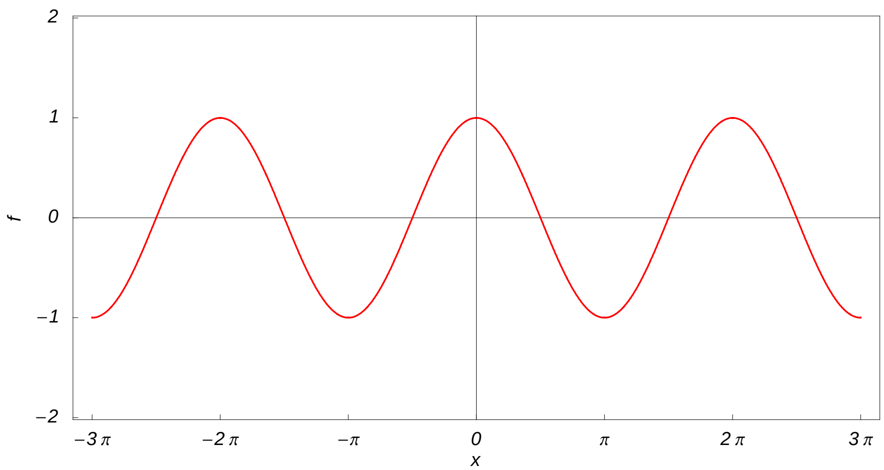
The following formula can also be used as a definition of the cosine function:

$$\cos(z)=1-\frac{z^2}{2}+\frac{z^4}{24}-\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k\,z^{2k}}{(2\,k)!}.$$

This series converges for all finite numbers z.

A quick look at the cosine function

Here is a graphic of the cosine function $f(x) = \cos(x)$ for real values of its argument x.



Representation through more general functions

The function $\cos(z)$ is a particular case of more complicated mathematical functions. For example, it is a special case of the generalized hypergeometric function ${}_0F_1(;a;w)$ with the parameter $a == \frac{1}{2}$ at $w == -\frac{z^2}{4}$:

$$\cos(z) == {}_0F_1\left(;\frac{1}{2};-\frac{z^2}{4}\right).$$

It is also a particular case of the Bessel function $J_\nu(z)$ with the parameter $\nu == -\frac{1}{2}$, multiplied by $\sqrt{\frac{\pi z}{2}}$:

$$\cos(z) == \sqrt{\frac{\pi z}{2}}\, J_{-\frac{1}{2}}(z).$$

Other Bessel functions can also be expressed through cosine functions for similar values of the parameter:

$$\cos(z) == \sqrt{\frac{\pi i z}{2}}\, I_{-\frac{1}{2}}(i z) \quad \cos(z) == -\sqrt{\frac{\pi z}{2}}\, Y_{\frac{1}{2}}(z) \quad \cos(z) == \frac{1}{\sqrt{2\pi}}\left(\sqrt{i z}\, K_{\frac{1}{2}}(i z) + \sqrt{-i z}\, K_{\frac{1}{2}}(-i z)\right).$$

Struve functions can also degenerate into the cosine function for similar values of the parameter:

$$\cos(z) == 1 - \sqrt{\frac{\pi z}{2}}\, \boldsymbol{H}_{\frac{1}{2}}(z) \quad \cos(z) == 1 + \sqrt{\frac{\pi i z}{2}}\, \boldsymbol{L}_{\frac{1}{2}}(i z).$$

But the function $\cos(z)$ is also a degenerate case of the doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\cos(z) == \mathrm{cd}(z \mid 0) == \mathrm{cn}(z \mid 0)$$
$$\cos(z) == \mathrm{nc}(i z \mid 1) == \mathrm{nd}(i z \mid 1).$$

Finally, the function $\cos(z)$ is the particular case of another class of functions—the Mathieu functions:

$$\cos(z) == \mathrm{Ce}(1, 0, z).$$

Definition of the cosine function for a complex argument

In the complex z-plane, the function $\cos(z)$ is defined using the exponential function e^w in the points $w = i\,z$ and $w = -i\,z$ through the formula:

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}.$$

The key role in this definition of $\cos(z)$ belongs to the famous Euler formula that connects the exponential, the sine, and the cosine functions:

$$e^{i z} = \cos(z) + i \sin(z).$$

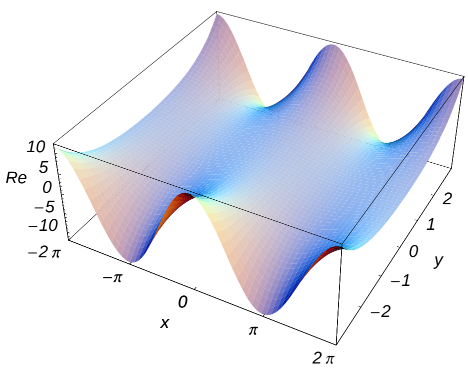
Changing z to $-z$, the Euler formula can be converted into the following modification:

$$e^{-i z} = \cos(z) - i \sin(z).$$

Adding the preceding formulas gives the following result:

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}.$$

Here are two graphics showing the real and imaginary parts of the cosine function over the complex plane.



The best-known properties and formulas for the cosine function

Values in points

Students usually learn the following basic table of cosine function values for special points of the circle:

$\cos(0) = 1$	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\cos\left(\frac{\pi}{2}\right) = 0$	$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$	$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$	$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
$\cos(\pi) = -1$	$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$	$\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$	$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$
$\cos\left(\frac{3\pi}{2}\right) = 0$	$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$	$\cos\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\cos(2\pi) = 1 \quad \cos(\pi m) = (-1)^m \text{ /; } m \in \mathbb{Z} \quad \cos\left(\pi\left(\frac{1}{2} + m\right)\right) = 0 \text{ /; } m \in \mathbb{Z}$.

General characteristics

For real values of argument z, the values of $\cos(z)$ are real.

In the points $z = 2\pi n/m$ /; $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\cos(z)$ are algebraic. In several cases they can even be rational numbers, 0, or 1. Here are some examples:

$\cos(0) = 1 \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \cos\left(\frac{\pi}{2}\right) = 0.$

The values of $\cos\left(\frac{n\pi}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

The function $\cos(z)$ is an entire analytical function of z that is defined over the whole complex z-plane and does not have branch cuts and branch points. It has an essential singular point at $z = \tilde{\infty}$. It is a periodic function with the real period 2π:

$\cos(z + 2\pi) = \cos(z)$

$\cos(z) = \cos(z + 2\pi k) \text{ /; } k \in \mathbb{Z} \quad \cos(z) = (-1)^k \cos(z + \pi k) \text{ /; } k \in \mathbb{Z}$.

The function $\cos(z)$ is an even function with mirror symmetry:

$\cos(-z) = \cos(z) \quad \cos(\bar{z}) = \overline{\cos(z)}$.

Differentiation

The derivatives of $\cos(z)$ have simple representations using either the $\sin(z)$ function or the $\cos(z)$ function:

$\frac{\partial \cos(z)}{\partial z} = -\sin(z) \quad \frac{\partial^n \cos(z)}{\partial z^n} = \cos\left(z + \frac{\pi n}{2}\right) \text{ /; } n \in \mathbb{N}^+$.

Ordinary differential equation

The function $\cos(z)$ satisfies the simplest possible linear differential equation with constant coefficients:

$w''(z) + w(z) = 0 \text{ /; } w(z) = \cos(z) \wedge w(0) = 1 \wedge w'(0) = 0.$

The complete solution of this equation can be represented as a linear combination of $\sin(z)$ and $\cos(z)$ with arbitrary constant coefficients c_1 and c_2:

$w''(z) + w(z) = 0 \text{ /; } w(z) = c_1 \cos(z) + c_2 \sin(z).$

The function $\cos(z)$ also satisfies first-order nonlinear differential equations:

$w'(z) - \sqrt{1 - (w(z))^2} = 0 \text{ /; } w(z) = \cos(z) \bigwedge w(0) = 1 \bigwedge \operatorname{Re}(z) < \frac{\pi}{2}.$

Series representation

The function $\cos(z)$ has a simple series expansion at the origin that converges in the whole complex z-plane:

$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}.$$

For real $z = x$ this series can be interpreted as the real part of the series expansion for the exponential function e^{ix}:

$$\cos(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \ldots = \operatorname{Re}\left(1 + i x + \frac{(i x)^2}{2!} + \frac{(i x)^3}{3!} + \ldots\right) = \operatorname{Re}\left(\sum_{k=0}^{\infty} \frac{(i x)^k}{k!}\right) = \operatorname{Re}\left(e^{i x}\right).$$

Product representation

The following famous infinite product representation for $\cos(z)$ clearly illustrates that $\cos(z) = 0$ at $z = \pi k - \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$:

$$\cos(z) = \prod_{k=1}^{\infty} \left(1 - \frac{4 z^2}{\pi^2 (2k-1)^2}\right).$$

Indefinite integration

Indefinite integrals of expressions involving the cosine function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \cos(z) \, dz = \sin(z)$$

$$\int \sqrt{\cos(z)} \, dz = 2 E\left(\frac{z}{2} \middle| 2\right)$$

$$\int z^{\alpha-1} \cos^{\nu}(a z) \, dz = \frac{2^{-\nu} (1 - \nu \bmod 2)}{\alpha} z^{\alpha} \binom{\nu}{\frac{\nu}{2}} -$$
$$2^{-\nu} z^{\alpha} \sum_{j=0}^{\left\lfloor \frac{\nu-1}{2} \right\rfloor} \binom{\nu}{j} \left(\Gamma(\alpha, i a (2j - \nu) z) (i a (2j - \nu) z)^{-\alpha} + \Gamma(\alpha, i a (\nu - 2j) z) (i a (\nu - 2j) z)^{-\alpha}\right) /; \alpha \neq 0 \bigwedge \nu \in \mathbb{N}^{+}.$$

The last integral cannot be evaluated in closed form using the known classical special functions for arbitrary values of parameters α and ν.

Definite integration

Definite integrals that contain the cosine function are sometimes simple. For example, the famous Dirichlet type and Fresnel integrals have the following values:

$$\int_0^{\infty} \frac{\cos(t) - 1}{t} \, dt = -\gamma$$

$$\int_0^{\infty} \cos\left(t^2\right) dt = \frac{1}{2} \sqrt{\frac{\pi}{2}},$$

where γ is the Euler-Mascheroni constant $\gamma = 0.577216 \ldots$.

Some special functions can be used to evaluate more complicated definite integrals. For example, elliptic integrals and gamma functions are needed to express the following integrals:

$$\int_0^\pi \cos^{\frac{1}{2}}(z)\, dz = 2\, E(2)$$

$$\int_0^\pi z^{\alpha-1} \cos(z)\, dz = \cos\left(\frac{\pi\,\alpha}{2}\right) \Gamma(\alpha) \;/;\; 0 < \operatorname{Re}(\alpha) < 1.$$

Integral transforms

Integral transforms of expressions involving the cosine function may not be classically convergent but can be interpreted in a generalized functions (distributions) sense. For example, the exponential Fourier transform of the cosine function $\cos(z)$ does not exist in the classical sense but can be expressed using the Dirac delta function.

$$\mathcal{F}_t[\cos(t)](z) = \sqrt{\frac{\pi}{2}}\; \delta(z-1) + \sqrt{\frac{\pi}{2}}\; \delta(z+1).$$

Among other integral transforms of the cosine function, the best known are the Fourier cosine and sine transforms, and the Laplace, Mellin, Hilbert, and Hankel transforms:

$$\mathcal{F}c_t[\cos(a\,t)](z) = \sqrt{\frac{\pi}{2}}\; (\delta(a-z) + \delta(a+z)) \;/;\; a \in \mathbb{R}$$

$$\mathcal{F}s_t[\cos(a\,t)](z) = -\sqrt{\frac{2}{\pi}}\; \frac{z}{a^2 - z^2} \;/;\; a \in \mathbb{R}$$

$$\mathcal{L}_t[\cos(t)](z) = \frac{z}{1+z^2}$$

$$\mathcal{M}_t[\cos(t)](z) = \Gamma(z) \cos\left(\frac{\pi\, z}{2}\right) \;/;\; 0 < \operatorname{Re}(z) < 1$$

$$\mathcal{H}_t[\cos(t)](x) = -\sin(x)$$

$$\mathcal{H}_{t;\nu}\left[t^{\alpha-1} \cos(t)\right](z) = \frac{1}{\Gamma(\nu+1)} \left(2^{-\nu} z^{\nu+\frac{1}{2}} \cos\left(\frac{1}{4}\,\pi\,(2\,\alpha + 2\,\nu + 1)\right) \Gamma\left(\alpha + \nu + \frac{1}{2}\right) {}_2F_1\left(\frac{1}{4}\,(2\,\alpha + 2\,\nu + 1),\; \frac{1}{4}\,(2\,\alpha + 2\,\nu + 3);\; \nu + 1;\; z^2\right)\right) /;$$
$$\operatorname{Re}(\alpha + \nu) > -\frac{1}{2} \bigwedge \operatorname{Re}(\alpha) < 1.$$

Finite summation

The following finite sums from the cosine can be expressed using the trigonometric functions:

$$\sum_{k=0}^{n} \cos(a\,k) = \csc\left(\frac{a}{2}\right) \sin\left(\frac{a\,(n+1)}{2}\right) \cos\left(\frac{a\,n}{2}\right)$$

$$\sum_{k=0}^{n} (-1)^k \cos(a\,k) = \cos\left(\frac{1}{2}\,(a + n\,(a+\pi))\right) \sec\left(\frac{a}{2}\right) \cos\left(\frac{1}{2}\,n\,(a+\pi)\right)$$

$$\sum_{k=0}^{n} \cos(a\,k+z) == \csc\left(\frac{a}{2}\right) \sin\left(\frac{1}{2}\,a\,(n+1)\right) \cos\left(\frac{a\,n}{2}+z\right)$$

$$\sum_{k=0}^{n} (-1)^k \cos(a\,k+z) == \cos\left(\frac{1}{2}\,(a+n\,(a+\pi))\right) \sec\left(\frac{a}{2}\right) \cos\left(\frac{1}{2}\,n\,(a+\pi)+z\right)$$

$$\sum_{k=1}^{n} z^k \cos(k\,a) == \frac{z\left(\cos(a\,n)\,z^{n+1}-\cos(n\,a+a)\,z^n-z+\cos(a)\right)}{z^2-2\cos(a)\,z+1}.$$

Infinite summation

The following infinite sums can be expressed using elementary functions:

$$\sum_{k=1}^{\infty} \frac{\cos(a\,k)}{k} == \frac{1}{2} \log\left(\frac{1}{2\,(1-\cos(a))}\right) /;\ 0 < \mathrm{Re}(a) < 2\,\pi$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos(a\,k)}{k} == \log\left(2\cos\left(\frac{a}{2}\right)\right) /;\ |\mathrm{Re}(a)| < \pi$$

$$\sum_{k=0}^{\infty} \frac{\cos(a\,k)}{k!} == e^{\cos(a)} \cos(\sin(a))$$

$$\sum_{k=0}^{\infty} \frac{z^k \cos(a\,k)}{k!} == e^{z\cos(a)} \cos(z\sin(a))$$

$$\sum_{k=0}^{\infty} z^k \cos(a\,k) == \frac{1-z\cos(a)}{z^2-2\cos(a)\,z+1} /;\ |z| < 1.$$

Finite products

The following finite products from the cosine can be expressed using trigonometric functions:

$$\prod_{k=1}^{2n-1} \cos\left(\frac{k\,\pi}{n}\right) == 2^{1-2n}\,((-1)^n-1) /;\ n \in \mathbb{N}^+$$

$$\prod_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \cos\left(\frac{\pi\,k}{n}\right) == 2^{\frac{1-n}{2}} \left(\frac{1}{2}\,(1-(-1)^n) + \frac{1}{2}\,(1+(-1)^n)\sqrt{n}\right) /;\ n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \cos\left(\frac{\pi\,k}{n}+z\right) == \frac{2^{1-n}}{\cos(z)} \sin\left(n\left(z+\frac{\pi}{2}\right)\right) /;\ n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \cos\left(\frac{2\,\pi\,k}{n}+z\right) == (-2)^{1-n} \sec(z) \left(\cos(n\,z) - \cos\left(\frac{n\,\pi}{2}\right)\right) /;\ n \in \mathbb{N}^+.$$

Infinite products

The following infinite product that contains the cosine function can be expressed using the sine function:

$$\prod_{k=1}^{\infty} \cos\left(\frac{z}{2^k}\right) == \frac{\sin(z)}{z}.$$

Addition formulas

The cosine of a sum can be represented by the rule: "the cosine of a sum is equal to the product of the cosines minus the product of the sines." A similar rule is valid for the cosine of the difference:

$$\cos(a+b) == \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) == \cos(a)\cos(b) + \sin(a)\sin(b).$$

Multiple arguments

In the case of multiple arguments z, $2z$, $3z$, …, the function $\cos(z)$ can be represented as the finite sum of terms that include powers of the sine and the cosine:

$$\cos(2z) == \cos^2(z) - \sin^2(z)$$

$$\cos(3z) == 4\cos^3(z) - 3\cos(z)$$

$$\cos(nz) == \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{2k} \sin^{2k}(z) \cos^{n-2k}(z) \text{ /; } n \in \mathbb{N}^+.$$

The function $\cos(nz)$ can also be represented as the finite sum including only the cosine of z:

$$\cos(nz) == n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k-1)!\, 2^{n-2k-1}}{k!\,(n-2k)!} \cos^{n-2k}(z) \text{ /; } n \in \mathbb{N}^+.$$

Half-angle formulas

The cosine of the half-angle can be represented by the following simple formula that is valid in some vertical strips:

$$\cos\left(\frac{z}{2}\right) == \sqrt{\frac{1+\cos(z)}{2}} \text{ /; } |\mathrm{Re}(z)| < \pi \lor \mathrm{Re}(z) == -\pi \land \mathrm{Im}(z) \geq 0 \lor \mathrm{Re}(z) == \pi \land \mathrm{Im}(z) \leq 0.$$

To make this formula correct for all complex z, a complicated prefactor is needed:

$$\cos\left(\frac{z}{2}\right) == c(z)\sqrt{\frac{1+\cos(z)}{2}} \text{ /; } c(z) == (-1)^{\left\lfloor \frac{\mathrm{Re}(z)+\pi}{2\pi} \right\rfloor} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)+\pi}{2\pi} \right\rfloor + \left\lfloor -\frac{\mathrm{Re}(z)+\pi}{2\pi} \right\rfloor}\right) \theta(-\mathrm{Im}(z))\right),$$

where $c(z)$ contains the unit step, real part, imaginary part, and the floor functions.

Sums of two direct functions

The sum of two cosine functions can be described by the rule: "the sum of the cosines is equal to two times the cosine of the half-difference multiplied by the cosine of the half-sum." A similar rule is valid for the difference of two cosines:

$$\cos(a)+\cos(b) == 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos(a)-\cos(b) == -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).$$

Products involving the direct function

The product of two cosine functions and the product of the cosine and sine have the following representations:

$$\cos(a)\cos(b) == \frac{1}{2}(\cos(a-b)+\cos(a+b))$$

$$\cos(a)\sin(b) == \frac{1}{2}(\sin(b-a)+\sin(a+b)).$$

Powers of the direct function

The integer powers of the cosine functions can be expanded as finite sums of cosine functions with multiple arguments. These sums include binomial coefficients:

$$\cos^n(z) == 2^{-n}\binom{n}{\frac{n}{2}}(1-n \bmod 2)+2^{1-n}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{k}\cos((n-2k)z)\ /;\ n\in\mathbb{N}^+.$$

Inequalities

The best-known inequalities for cosine functions are the following:

$$|\cos(x)| \leq 1\ /;\ x\in\mathbb{R}$$

$$\cos(x) < \frac{\sin(x)}{x}\ /;\ 0<x<4.493409.$$

Relations with its inverse function

There are simple relations between the function $\cos(z)$ and its inverse function $\cos^{-1}(z)$:

$$\cos^{-1}(\cos(z)) == z \quad \cos^{-1}(\cos(z)) == z\ /;\ 0<\operatorname{Re}(z)<\pi \lor \operatorname{Re}(z)==0 \land \operatorname{Im}(z)\geq 0 \lor \operatorname{Re}(z)==\pi \land \operatorname{Im}(z)\leq 0.$$

The second formula is valid at least in the vertical strip $0<\operatorname{Re}(z)<\pi$. Outside of this strip, a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\cos^{-1}(\cos(z)) == \frac{\pi}{2}\left(1-(-1)^{\left\lfloor-\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\right)+(-1)^{\left\lfloor-\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\left(\left(1+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rfloor+\left\lfloor-\frac{\operatorname{Re}(z)}{\pi}\right\rfloor}\right)\theta(\operatorname{Im}(z))-1\right)\left(z+\pi\left\lfloor-\frac{\operatorname{Re}(z)}{\pi}\right\rfloor\right).$$

Representations through other trigonometric functions

Cosine and sine functions are connected by a very simple formula including the linear function in the argument:

$$\cos(z) == \sin\left(\frac{\pi}{2}-z\right).$$

Another famous formula, connecting $\cos(z)$ and $\sin(z)$, is shown in the well-known Pythagorean theorem:

$$\cos^2(z) = 1 - \sin^2(z)$$

$$\cos(z) = \sqrt{1 - \sin^2(z)} \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) \geq 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) \leq 0.$$

The last restriction on z can be removed, but the formula will get a complicated coefficient $c(z)$ with $|c(z)| = 1$, that contains the unit step, real part, imaginary part, and the floor function:

$$\cos(z) = c(z) \sqrt{1 - \sin^2(z)} \;/;\; c(z) = (-1)^{\left\lfloor \frac{1}{2} - \frac{\mathrm{Re}(z)}{\pi} \right\rfloor} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} - \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{\mathrm{Re}(z)}{\pi} \right\rfloor}\right) \theta(\mathrm{Im}(z))\right).$$

The cosine function can also be represented using other trigonometric functions by the following formulas:

$$\cos(z) = \frac{1 - \tan^2\left(\frac{z}{2}\right)}{1 + \tan^2\left(\frac{z}{2}\right)} \quad \cos(z) = \frac{\cot^2\left(\frac{z}{2}\right) - 1}{\cot^2\left(\frac{z}{2}\right) + 1}$$

$$\cos(z) = \frac{1}{\csc\left(\frac{\pi}{2} - z\right)} \quad \cos(z) = \frac{1}{\sec(z)}.$$

Representations through hyperbolic functions

The cosine function has representations using the hyperbolic functions:

$$\cos(z) = \cosh(i\, z) \quad \cos(i\, z) = \cosh(z) \quad \cos(z) = -i \sinh\left(\frac{\pi\, i}{2} - i\, z\right)$$

$$\cos(z) = \frac{1 + \tanh^2\left(\frac{z\, i}{2}\right)}{1 - \tanh^2\left(\frac{z\, i}{2}\right)} \quad \cos(z) = \frac{\coth^2\left(\frac{i\, z}{2}\right) + 1}{\coth^2\left(\frac{i\, z}{2}\right) - 1} \quad \cos(z) = -\frac{i}{\mathrm{csch}\left(\frac{\pi\, i}{2} - i\, z\right)} \quad \cos(z) = \frac{1}{\mathrm{sech}(i\, z)}.$$

Applications

The cosine function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Trigonometric Functions in *Mathematica*

Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in `StandardForm`.

```
trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

Here is a list `trigFunctions` of the six trigonometric functions in `TraditionalForm`.

```
trigFunctions // TraditionalForm
```

{sin(*z*), cos(*z*), tan(*z*), cot(*z*), sec(*z*), cos(*z*)}

Additional forms of notations

Mathematica also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
trigFunctions /. {z → 2 π z} // (CForm /@ #) &
```

```
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}
```

```
trigFunctions /. {z → 2 π z} // (TeXForm /@ #) &
```

```
{\sin (2\, \pi \, z), \cos (2\, \pi \, z), \tan (2\, \pi \, z), \cot
  (2\, \pi \, z), \sec (2\, \pi \, z), \cos (2\, \pi \, z)}
```

```
trigFunctions /. {z → 2 π z} // (FortranForm /@ #) &
```

```
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Cos (2 * Pi * z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument $\pi/6$, the `Sin` function evaluates to $1/2$.

$$\text{Sin}\left[\frac{\pi}{6}\right]$$

$$\frac{1}{2}$$

$$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\} \;/.\; z \to \frac{\pi}{6}$$

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cos[3.]
```

```
-0.989992
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 2.
```

```
{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}
```

The next inputs calculate 100-digit approximations of the six trigonometric functions at $z = 1$.

```
N[Tan[1], 40]
```

```
1.557407724654902230506974807458360173087
```

```
Cot[1] // N[#, 50] &
```

```
0.64209261593433070300641998659426562023027811391817
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 100]
```

```
{0.841470984807896506652502321630298999622563060798371065672751709991910404391239668⋱
  9486397435430526959,
 0.540302305868139717400936607442976603732310420617922227670097255381100394774471764⋱
  5179518560871830893,
 1.557407724654902230506974807458360173087250772381520038383946605698861397151727289⋱
  555099965202242984,
 0.642092615934330703006419986594265620230278113918171379101162280426276856839164672⋱
  1984829197601968047,
 1.188395105778121216261599452374551003527829834097962625265253666359184367357190487⋱
  913663568030853023,
 1.850815717680925617911753241398650193470396655094009298835158277858815411261596705⋱
  921841413287306671}
```

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for sin(1), cos(1), tan(1), cot(1), sec(1), and csc(1) and analyzes the frequency of the occurrence of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 10 000]]
```

```
{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},
  {6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},
  {3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},
 {{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},
  {7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},
  {3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},
 {{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},
  {7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},
  {3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}
```

Here are 50-digit approximations to the six trigonometric functions at the complex argument $z = 3 + 5\,i$.

```
N[Csc[3 + 5 ⅈ], 100]
```

```
0.001901970423701089996670017296320805840459252512171274310801719695392870034046820⋱
  296847410109982878354 +
 0.013341591397996678721837322466473194390132347157253190972075437462485814431570118⋱
   6726266448851984033 9 ⅈ
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{10.47250853394039227667332253685350327112641995038 8-
  73.46062169567367636679119250508175040721392281447 5ⅈ,
 -73.46729221264526246774645459483395083081485916529 9-
  10.47155767480557437739446422432953780854833065173 4ⅈ,
 -0.00002536867620767603241780613670742628819556070260 2478+
  0.99991282015135380828209263013972954140566020462086ⅈ,
 -0.00002537310004454597738376334678946965675405003735 5986-
  1.00008718680589677432853168810452185771316128318 91ⅈ,
 0.00190197042370108999667001729632080584045925251217 13+
  0.01334159139799667872183732246647319439013234715725 3ⅈ,
 -0.01334047653054973748736110081110083946847048172503 8+
  0.00190146615169515130895192700132542778675889781334 99ⅈ}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
```

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

```
% // InputForm
```

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
 0.909297426825681695396019865911745`20}
```

```
Precision[%%]
```

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

```
Sin[-z]
```

```
-Sin[z]
```

```
Sin[z + π]
```

```
-Sin[z]
```

```
Sin[z + 2 π]
```

```
Sin[z]
```

```
Sin[z + 34 π]
```

```
Sin[z]
```

```
{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}
```

```
{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}
```

```
{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}
```

$\{-\text{Sin}[z], -\text{Cos}[z], \text{Tan}[z], \text{Cot}[z], -\text{Csc}[z], -\text{Sec}[z]\}$

```
{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}
```

$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\}$

```
{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}
```

$\{\text{Sin}[z], \text{Cos}[z], \text{Tan}[z], \text{Cot}[z], \text{Csc}[z], \text{Sec}[z]\}$

Mathematica automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

```
{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],
 Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}
```

$\{z, z, z, z, z, z\}$

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
 Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}
```

$$\left\{z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}\, z}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}}\right\}$$

```
{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
 Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}
```

$$\left\{\sqrt{1-z^2}, z, \frac{1}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{z}\right\}$$

```
{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
 Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}, \sqrt{1-\frac{1}{z^2}}\, z\right\}$$

```
{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],
 Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}
```

$$\left\{\frac{\sqrt{1-z^2}}{z}, \frac{z}{\sqrt{1-z^2}}, \frac{1}{z}, z, \sqrt{1-\frac{1}{z^2}}\, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}\right\}$$

```
{Csc[ArcSin[z]], Csc[ArcCos[z]], Csc[ArcTan[z]],
 Csc[ArcCot[z]], Csc[ArcCsc[z]], Csc[ArcSec[z]]}
```

$$\left\{\frac{1}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{\sqrt{1+z^2}}{z}, \sqrt{1+\frac{1}{z^2}}\, z, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

```
{Sec[ArcSin[z]], Sec[ArcCos[z]], Sec[ArcTan[z]],
 Sec[ArcCot[z]], Sec[ArcCsc[z]], Sec[ArcSec[z]]}
```

$$\left\{\frac{1}{\sqrt{1-z^2}}, \frac{1}{z}, \sqrt{1+z^2}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tan}\left[\frac{\pi}{2}-z\right]$$

```
Cot[z]
```

```
Csc[ⅈ z]
```

```
-ⅈ Csch[z]
```

$$\left\{\text{Sin}\left[\frac{\pi}{2}-z\right], \text{Cos}\left[\frac{\pi}{2}-z\right], \text{Tan}\left[\frac{\pi}{2}-z\right], \text{Cot}\left[\frac{\pi}{2}-z\right], \text{Csc}\left[\frac{\pi}{2}-z\right], \text{Sec}\left[\frac{\pi}{2}-z\right]\right\}$$

```
{Cos[z], Sin[z], Cot[z], Tan[z], Sec[z], Csc[z]}
```

```
{Sin[ⅈ z], Cos[ⅈ z], Tan[ⅈ z], Cot[ⅈ z], Csc[ⅈ z], Sec[ⅈ z]}
```

```
{ⅈ Sinh[z], Cosh[z], ⅈ Tanh[z], -ⅈ Coth[z], -ⅈ Csch[z], Sech[z]}
```

Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

```
1 / Sec[z]
```

```
Cos[z]
```

```
{1/Sin[z], 1/Cos[z], 1/Tan[z], 1/Cot[z], 1/Csc[z], 1/Sec[z],
 Sin[z]/Cos[z], Cos[z]/Sin[z], Sin[z]/Sin[π/2 - z], Cos[z]/Sin[z]^2}
```

```
{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}
```

Trigonometric functions arising as special cases from more general functions

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin(z) and cos(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

$$\text{BesselJ}\left[\frac{1}{2}, z\right]$$

$$\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[z]}{\sqrt{z}}$$

MathieuC[1, 0, z]

Cos[z]

JacobiSC[z, 0]

Tan[z]

In[14]:= $$\left\{\text{BesselJ}\left[\frac{1}{2}, z\right], \text{MathieuS}[1, 0, z], \text{JacobiSN}[z, 0], \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right]\right\}$$

Out[14]= $$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[z]}{\sqrt{z}}, \text{Sin}[z], \text{Sin}[z], \frac{\text{Sin}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{\sqrt{z^2}\ \text{Sin}[z]}{\sqrt{\pi}\ z}\right\}$$

In[15]:= $$\left\{\text{BesselJ}\left[-\frac{1}{2}, z\right], \text{MathieuC}[1, 0, z], \text{JacobiCD}[z, 0], \text{Hypergeometric0F1}\left[\frac{1}{2}, -\frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, \frac{z^2}{4}\right]\right\}$$

Out[15]= $$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \text{Cos}[z]}{\sqrt{z}}, \text{Cos}[z], \text{Cos}[z], \text{Cos}\left[\sqrt{z^2}\right], \frac{\text{Cos}[z]}{\sqrt{\pi}}\right\}$$

In[16]:= {JacobiSC[z, 0], JacobiCS[z, 0], JacobiDS[z, 0], JacobiDC[z, 0]}

Out[16]= {Tan[z], Cot[z], Csc[z], Sec[z]}

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\sin(z)/2$ instead of $\cos(\pi/2 - z)\sin(\pi/6)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sin(2z)\sin(\pi/16)$ should not be automatically expanded into the more complicated expression $\sin(z)\cos(z)\left(2-\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sin[x - y]]
```

```
Cos[y] Sin[x] - Cos[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

```
Cos[z]^4 - 6 Cos[z]^2 Sin[z]^2 + Sin[z]^4
```

```
TrigExpand[{{Sin[x + y], Sin[3 z]},
            {Cos[x + y], Cos[3 z]},
            {Tan[x + y], Tan[3 z]},
            {Cot[x + y], Cot[3 z]},
            {Csc[x + y], Csc[3 z]},
            {Sec[x + y], Sec[3 z]}}]
```

$$\left\{\left\{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y],\ 3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3\right\},\right.$$
$$\left\{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y],\ \mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2\right\},$$
$$\left\{\frac{\mathrm{Cos}[y]\,\mathrm{Sin}[x]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]}+\frac{\mathrm{Cos}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]},\right.$$
$$\left.\frac{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}-\frac{\mathrm{Sin}[z]^3}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}\right\},$$
$$\left\{\frac{\mathrm{Cos}[x]\,\mathrm{Cos}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]}-\frac{\mathrm{Sin}[x]\,\mathrm{Sin}[y]}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]},\right.$$
$$\left.\frac{\mathrm{Cos}[z]^3}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3}-\frac{3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3}\right\},$$
$$\left\{\frac{1}{\mathrm{Cos}[y]\,\mathrm{Sin}[x]+\mathrm{Cos}[x]\,\mathrm{Sin}[y]},\ \frac{1}{3\,\mathrm{Cos}[z]^2\,\mathrm{Sin}[z]-\mathrm{Sin}[z]^3}\right\},$$
$$\left.\left\{\frac{1}{\mathrm{Cos}[x]\,\mathrm{Cos}[y]-\mathrm{Sin}[x]\,\mathrm{Sin}[y]},\ \frac{1}{\mathrm{Cos}[z]^3-3\,\mathrm{Cos}[z]\,\mathrm{Sin}[z]^2}\right\}\right\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sin[x + y], Sin[3 z]}, {Cos[x + y], Cos[3 z]}, {Tan[x + y], Tan[3 z]},
    {Cot[x + y], Cot[3 z]}, {Csc[x + y], Csc[3 z]}, {Sec[x + y], Sec[3 z]}}]]
```

```
Sin[x + y] == Cos[y] Sin[x] + Cos[x] Sin[y]
Sin[3 z] == 3 Cos[z]^2 Sin[z] - Sin[z]^3
Cos[x + y] == Cos[x] Cos[y] - Sin[x] Sin[y]
Cos[3 z] == Cos[z]^3 - 3 Cos[z] Sin[z]^2
Tan[x + y] == Cos[y] Sin[x]/(Cos[x] Cos[y] - Sin[x] Sin[y]) + Cos[x] Sin[y]/(Cos[x] Cos[y] - Sin[x] Sin[y])
Tan[3 z] == 3 Cos[z]^2 Sin[z]/(Cos[z]^3 - 3 Cos[z] Sin[z]^2) - Sin[z]^3/(Cos[z]^3 - 3 Cos[z] Sin[z]^2)
Cot[x + y] == Cos[x] Cos[y]/(Cos[y] Sin[x] + Cos[x] Sin[y]) - Sin[x] Sin[y]/(Cos[y] Sin[x] + Cos[x] Sin[y])
Cot[3 z] == Cos[z]^3/(3 Cos[z]^2 Sin[z] - Sin[z]^3) - 3 Cos[z] Sin[z]^2/(3 Cos[z]^2 Sin[z] - Sin[z]^3)
Csc[x + y] == 1/(Cos[y] Sin[x] + Cos[x] Sin[y])
Csc[3 z] == 1/(3 Cos[z]^2 Sin[z] - Sin[z]^3)
Sec[x + y] == 1/(Cos[x] Cos[y] - Sin[x] Sin[y])
Sec[3 z] == 1/(Cos[z]^3 - 3 Cos[z] Sin[z]^2)
```

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sin[x] + Cos[y]]
```

```
(Cos[x/2 - y/2] + Sin[x/2 - y/2]) (Cos[x/2 + y/2] + Sin[x/2 + y/2])
```

```
Tan[x] - Cot[y] // TrigFactor
```

```
-Cos[x + y] Csc[y] Sec[x]
```

```
TrigFactor[{Sin[x] + Sin[y],
            Cos[x] + Cos[y],
            Tan[x] + Tan[y],
            Cot[x] + Cot[y],
            Csc[x] + Csc[y],
            Sec[x] + Sec[y]}]
```

```
{2 Cos[x/2 - y/2] Sin[x/2 + y/2], 2 Cos[x/2 - y/2] Cos[x/2 + y/2], Sec[x] Sec[y] Sin[x + y],
 Csc[x] Csc[y] Sin[x + y], 1/2 Cos[x/2 - y/2] Csc[x/2] Csc[y/2] Sec[x/2] Sec[y/2] Sin[x/2 + y/2],
 2 Cos[x/2 - y/2] Cos[x/2 + y/2]/((Cos[x/2] - Sin[x/2]) (Cos[x/2] + Sin[x/2]) (Cos[y/2] - Sin[y/2]) (Cos[y/2] + Sin[y/2]))}
```

TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sin[z]^3]
```

$$\frac{1}{4}\,(3\,\mathrm{Sin}[z]-\mathrm{Sin}[3\,z])$$

```
Sin[x] Cos[y] // TrigReduce
```

$$\frac{1}{2}\,(\mathrm{Sin}[x-y]+\mathrm{Sin}[x+y])$$

```
TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]
```

$$\left\{\frac{1}{2}\,(1-\mathrm{Cos}[2\,z]),\ \frac{1}{2}\,(1+\mathrm{Cos}[2\,z]),\ \frac{1-\mathrm{Cos}[2\,z]}{1+\mathrm{Cos}[2\,z]},\ \frac{-1-\mathrm{Cos}[2\,z]}{-1+\mathrm{Cos}[2\,z]},\ -\frac{2}{-1+\mathrm{Cos}[2\,z]},\ \frac{2}{1+\mathrm{Cos}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sin[x + y], Sin[3 z], Sin[x] Sin[y]},
                       {Cos[x + y], Cos[3 z], Cos[x] Cos[y]},
                       {Tan[x + y], Tan[3 z], Tan[x] Tan[y]},
                       {Cot[x + y], Cot[3 z], Cot[x] Cot[y]},
                       {Csc[x + y], Csc[3 z], Csc[x] Csc[y]},
                       {Sec[x + y], Sec[3 z], Sec[x] Sec[y]}}]]
```

$$\left\{\left\{\mathrm{Sin}[x+y],\ \mathrm{Sin}[3\,z],\ \frac{1}{2}\,(\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y])\right\},\right.$$
$$\left\{\mathrm{Cos}[x+y],\ \mathrm{Cos}[3\,z],\ \frac{1}{2}\,(\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y])\right\},$$
$$\left\{\mathrm{Tan}[x+y],\ \mathrm{Tan}[3\,z],\ \frac{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Cot}[x+y],\ \mathrm{Cot}[3\,z],\ \frac{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left\{\mathrm{Csc}[x+y],\ \mathrm{Csc}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]}\right\},$$
$$\left.\left\{\mathrm{Sec}[x+y],\ \mathrm{Sec}[3\,z],\ \frac{2}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sin[x] + Sin[y], Cos[x] + Cos[y],
   Tan[x] + Tan[y], Cot[x] + Cot[y], Csc[x] + Csc[y], Sec[x] + Sec[y]}]]
```

$$\left\{\mathrm{Sin}[x]+\mathrm{Sin}[y],\ \mathrm{Cos}[x]+\mathrm{Cos}[y],\ \frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]},\right.$$
$$\left.\frac{2\,\mathrm{Sin}[x+y]}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\,(\mathrm{Sin}[x]+\mathrm{Sin}[y])}{\mathrm{Cos}[x-y]-\mathrm{Cos}[x+y]},\ \frac{2\,(\mathrm{Cos}[x]+\mathrm{Cos}[y])}{\mathrm{Cos}[x-y]+\mathrm{Cos}[x+y]}\right\}$$

TrigToExp

The function TrigToExp converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sin[2 z]]
```

$$\frac{1}{2} i e^{-2 i z} - \frac{1}{2} i e^{2 i z}$$

```
Sin[z] Tan[2 z] // TrigToExp
```

$$-\frac{\left(e^{-i z} - e^{i z}\right) \left(e^{-2 i z} - e^{2 i z}\right)}{2 \left(e^{-2 i z} + e^{2 i z}\right)}$$

```
TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]
```

$$\left\{\frac{1}{2} i e^{-i z} - \frac{1}{2} i e^{i z}, \frac{e^{-i z}}{2} + \frac{e^{i z}}{2}, \frac{i \left(e^{-i z} - e^{i z}\right)}{e^{-i z} + e^{i z}}, -\frac{i \left(e^{-i z} + e^{i z}\right)}{e^{-i z} - e^{i z}}, -\frac{2 i}{e^{-i z} - e^{i z}}, \frac{2}{e^{-i z} + e^{i z}}\right\}$$

ExpToTrig

The function ExpToTrig converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to TrigToExp. Here are some examples.

$$\text{ExpToTrig}\left[e^{i x \beta}\right]$$

$$\text{Cos}[x \beta] + i \text{Sin}[x \beta]$$

$$\frac{e^{i x \alpha} - e^{i x \beta}}{e^{i x \gamma} + e^{i x \delta}} \text{ // ExpToTrig}$$

$$\frac{\text{Cos}[x \alpha] - \text{Cos}[x \beta] + i \text{Sin}[x \alpha] - i \text{Sin}[x \beta]}{\text{Cos}[x \gamma] + \text{Cos}[x \delta] + i \text{Sin}[x \gamma] + i \text{Sin}[x \delta]}$$

```
ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

$$\text{ExpToTrig}\left[\left\{\alpha e^{-i x \beta} + \alpha e^{i x \beta}, \alpha e^{-i x \beta} + \gamma e^{i x \beta}\right\}\right]$$

$$\{2 \alpha \text{Cos}[x \beta], \alpha \text{Cos}[x \beta] + \gamma \text{Cos}[x \beta] - i \alpha \text{Sin}[x \beta] + i \gamma \text{Sin}[x \beta]\}$$

ComplexExpand

The function ComplexExpand expands expressions assuming that all the occurring variables are real. The value option TargetFunctions is a list of functions from the set {Re, Im, Abs, Arg, Conjugate, Sign}. ComplexExpand tries to give results in terms of the specified functions. Here are some examples

$$\text{ComplexExpand}[\text{Sin}[x + i y] \text{Cos}[x - i y]]$$

$$\text{Cos}[x]\,\text{Cosh}[y]^2\,\text{Sin}[x]-\text{Cos}[x]\,\text{Sin}[x]\,\text{Sinh}[y]^2+i\left(\text{Cos}[x]^2\,\text{Cosh}[y]\,\text{Sinh}[y]+\text{Cosh}[y]\,\text{Sin}[x]^2\,\text{Sinh}[y]\right)$$

```
Csc[x + ⅈ y] Sec[x - ⅈ y] // ComplexExpand
```

$$-\frac{4\,\text{Cos}[x]\,\text{Cosh}[y]^2\,\text{Sin}[x]}{(\text{Cos}[2x]-\text{Cosh}[2y])\,(\text{Cos}[2x]+\text{Cosh}[2y])}+\frac{4\,\text{Cos}[x]\,\text{Sin}[x]\,\text{Sinh}[y]^2}{(\text{Cos}[2x]-\text{Cosh}[2y])\,(\text{Cos}[2x]+\text{Cosh}[2y])}+i\left(\frac{4\,\text{Cos}[x]^2\,\text{Cosh}[y]\,\text{Sinh}[y]}{(\text{Cos}[2x]-\text{Cosh}[2y])\,(\text{Cos}[2x]+\text{Cosh}[2y])}+\frac{4\,\text{Cosh}[y]\,\text{Sin}[x]^2\,\text{Sinh}[y]}{(\text{Cos}[2x]-\text{Cosh}[2y])\,(\text{Cos}[2x]+\text{Cosh}[2y])}\right)$$

In[17]:=
```
li1 = {Sin[x + ⅈ y], Cos[x + ⅈ y], Tan[x + ⅈ y], Cot[x + ⅈ y], Csc[x + ⅈ y], Sec[x + ⅈ y]}
```

Out[17]= $\{\text{Sin}[x+i\,y],\ \text{Cos}[x+i\,y],\ \text{Tan}[x+i\,y],\ \text{Cot}[x+i\,y],\ \text{Csc}[x+i\,y],\ \text{Sec}[x+i\,y]\}$

In[18]:=
```
ComplexExpand[li1]
```

Out[18]=
$$\left\{\text{Cosh}[y]\,\text{Sin}[x]+i\,\text{Cos}[x]\,\text{Sinh}[y],\ \text{Cos}[x]\,\text{Cosh}[y]-i\,\text{Sin}[x]\,\text{Sinh}[y],\ \frac{\text{Sin}[2x]}{\text{Cos}[2x]+\text{Cosh}[2y]}+\frac{i\,\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ -\frac{\text{Sin}[2x]}{\text{Cos}[2x]-\text{Cosh}[2y]}+\frac{i\,\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ -\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2x]-\text{Cosh}[2y]}+\frac{2\,i\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}+\frac{2\,i\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right\}$$

In[19]:=
```
ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[19]=
$$\left\{\text{Cosh}[y]\,\text{Sin}[x],\ \text{Cos}[x]\,\text{Cosh}[y],\ \frac{\text{Sin}[2x]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ -\frac{\text{Sin}[2x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ -\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right\}$$

In[20]:=
```
ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[20]=
$$\left\{\text{Cos}[x]\,\text{Sinh}[y],\ -\text{Sin}[x]\,\text{Sinh}[y],\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right\}$$

In[21]:=
```
ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[21]= $\left\{\sqrt{\text{Cosh}[y]^2\,\text{Sin}[x]^2+\text{Cos}[x]^2\,\text{Sinh}[y]^2},\ \sqrt{\text{Cos}[x]^2\,\text{Cosh}[y]^2+\text{Sin}[x]^2\,\text{Sinh}[y]^2},\right.$

$\sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x]+\text{Cosh}[2y])^2}+\frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x]+\text{Cosh}[2y])^2}},$

$\sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x]-\text{Cosh}[2y])^2}+\frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x]-\text{Cosh}[2y])^2}},$

$\sqrt{\frac{4\,\text{Cosh}[y]^2\,\text{Sin}[x]^2}{(\text{Cos}[2x]-\text{Cosh}[2y])^2}+\frac{4\,\text{Cos}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2x]-\text{Cosh}[2y])^2}},$

$\left.\sqrt{\frac{4\,\text{Cos}[x]^2\,\text{Cosh}[y]^2}{(\text{Cos}[2x]+\text{Cosh}[2y])^2}+\frac{4\,\text{Sin}[x]^2\,\text{Sinh}[y]^2}{(\text{Cos}[2x]+\text{Cosh}[2y])^2}}\right\}$

In[22]:= **% // Simplify[#, {x, y} ∈ Reals] &**

Out[22]= $\left\{\frac{\sqrt{-\text{Cos}[2x]+\text{Cosh}[2y]}}{\sqrt{2}},\ \frac{\sqrt{\text{Cos}[2x]+\text{Cosh}[2y]}}{\sqrt{2}},\ \frac{\sqrt{\text{Sin}[2x]^2+\text{Sinh}[2y]^2}}{\text{Cos}[2x]+\text{Cosh}[2y]},\right.$

$\left.\sqrt{-\frac{\text{Cos}[2x]+\text{Cosh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]}},\ \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2x]+\text{Cosh}[2y]}},\ \frac{\sqrt{2}}{\sqrt{\text{Cos}[2x]+\text{Cosh}[2y]}}\right\}$

In[23]:= **ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]**

Out[23]= $\{\text{ArcTan}[\text{Cosh}[y]\,\text{Sin}[x],\ \text{Cos}[x]\,\text{Sinh}[y]],\ \text{ArcTan}[\text{Cos}[x]\,\text{Cosh}[y],\ -\text{Sin}[x]\,\text{Sinh}[y]],$

$\text{ArcTan}\left[\frac{\text{Sin}[2x]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right],$

$\text{ArcTan}\left[-\frac{\text{Sin}[2x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]}\right],$

$\text{ArcTan}\left[-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]-\text{Cosh}[2y]}\right],$

$\text{ArcTan}\left[\frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ \frac{2\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right]\}$

In[24]:= **ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify**

Out[24]= $\{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y],\ \text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y],$

$\frac{\text{Sin}[2x]-i\,\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ -\frac{\text{Sin}[2x]+i\,\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]},$

$\frac{1}{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y]},\ \frac{1}{\text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y]}\}$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sin[2 z] / Sin[z]]
```

```
2 Cos[z]
```

```
Sin[2 z] / Cos[z] // Simplify
```

```
2 Sin[z]
```

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#]\ \&\ \text{/@}\ \Big(\cos(z)^2+\sin(z)^2==1 \land$$
$$\sin(z)^2==\frac{1-\cos(2z)}{2} \land \cos(z)^2==\frac{1+\cos(2z)}{2} \land$$
$$\tan(z)^2==\frac{1-\cos(2z)}{1+\cos(2z)} \land \cot(z)^2==\frac{1+\cos(2z)}{1-\cos(2z)} \land$$
$$\sin(2z)==2\sin(z)\cos(z) \land \cos(2z)==\cos(z)^2-\sin(z)^2==2\cos(z)^2-1 \land$$
$$\sin(a+b)==\sin(a)\cos(b)+\cos(a)\sin(b) \land \sin(a-b)==\sin(a)\cos(b)-\cos(a)\sin(b) \land$$
$$\cos(a+b)==\cos(a)\cos(b)-\sin(a)\sin(b) \land \cos(a-b)==\cos(a)\cos(b)+\sin(a)\sin(b) \land$$
$$\sin(a)+\sin(b)==2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \land \sin(a)-\sin(b)==2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \land$$
$$\cos(a)+\cos(b)==2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \land \cos(a)-\cos(b)==2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \land$$
$$\tan(a)+\tan(b)==\frac{\sin(a+b)}{\cos(a)\cos(b)} \land \tan(a)-\tan(b)==\frac{\sin(a-b)}{\cos(a)\cos(b)} \land$$
$$A\sin(z)+B\cos(z)==A\sqrt{1+\frac{B^2}{A^2}}\ \sin\left(z+\tan^{-1}\left(\frac{B}{A}\right)\right) \land$$
$$\sin(a)\sin(b)==\frac{\cos(a-b)-\cos(a+b)}{2} \land$$
$$\cos(a)\cos(b)==\frac{\cos(a-b)+\cos(a+b)}{2} \land \sin(a)\cos(b)==\frac{\sin(a+b)+\sin(a-b)}{2} \land$$
$$\sin\left(\frac{z}{2}\right)^2==\frac{1-\cos(z)}{2} \land \cos\left(\frac{z}{2}\right)^2==\frac{1+\cos(z)}{2} \land$$
$$\tan\left(\frac{z}{2}\right)==\frac{1-\cos(z)}{\sin(z)}==\frac{\sin(z)}{1+\cos(z)} \land \cot\left(\frac{z}{2}\right)==\frac{\sin(z)}{1-\cos(z)}==\frac{1+\cos(z)}{\sin(z)}\Big)$$

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $-1 \leq \sin(x) \leq 1$ for all real x, and uses the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\,\pi$.

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],
  Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],
  Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

$$\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\}$$

Mathematica also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},
 -π / 2 < Re[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

```
{z, z}
```

FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of π, for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cos}\left[\frac{\pi}{32}\right]\right]$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\text{Cot}\left[\frac{\pi}{24}\right] \text{ // FunctionExpand}$$

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{\mathrm{Sin}\left[\frac{\pi}{16}\right],\ \mathrm{Cos}\left[\frac{\pi}{16}\right],\ \mathrm{Tan}\left[\frac{\pi}{16}\right],\ \mathrm{Cot}\left[\frac{\pi}{16}\right],\ \mathrm{Csc}\left[\frac{\pi}{16}\right],\ \mathrm{Sec}\left[\frac{\pi}{16}\right]\right\}$$

$$\left\{\mathrm{Sin}\left[\frac{\pi}{16}\right],\ \mathrm{Cos}\left[\frac{\pi}{16}\right],\ \mathrm{Tan}\left[\frac{\pi}{16}\right],\ \mathrm{Cot}\left[\frac{\pi}{16}\right],\ \mathrm{Csc}\left[\frac{\pi}{16}\right],\ \mathrm{Sec}\left[\frac{\pi}{16}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}},\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

$$\left\{\mathrm{Sin}\left[\frac{\pi}{60}\right],\ \mathrm{Cos}\left[\frac{\pi}{60}\right],\ \mathrm{Tan}\left[\frac{\pi}{60}\right],\ \mathrm{Cot}\left[\frac{\pi}{60}\right],\ \mathrm{Csc}\left[\frac{\pi}{60}\right],\ \mathrm{Sec}\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\mathrm{Sin}\left[\frac{\pi}{60}\right],\ \mathrm{Cos}\left[\frac{\pi}{60}\right],\ \mathrm{Tan}\left[\frac{\pi}{60}\right],\ \mathrm{Cot}\left[\frac{\pi}{60}\right],\ \mathrm{Csc}\left[\frac{\pi}{60}\right],\ \mathrm{Sec}\left[\frac{\pi}{60}\right]\right\}$$

Together[FunctionExpand[%]]

$$\left\{\frac{1}{16}\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),\right.$$

$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$

$$\frac{-1-\sqrt{3}+\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{-1+\sqrt{3}+\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$

$$\frac{16}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$

$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

{Sin[π/9], Cos[π/9], Tan[π/9], Cot[π/9], Csc[π/9], Sec[π/9]}

$$\left\{\text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], \text{Tan}\left[\frac{\pi}{9}\right], \text{Cot}\left[\frac{\pi}{9}\right], \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right]\right\}$$

FunctionExpand[%] // Together

$$\Big\{\frac{1}{8}\Big(-i\,2^{2/3}\big(-1-i\sqrt{3}\big)^{1/3}+2^{2/3}\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+i\,2^{2/3}\big(-1+i\sqrt{3}\big)^{1/3}+2^{2/3}\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}\Big),$$

$$\frac{1}{8}\Big(2^{2/3}\big(-1-i\sqrt{3}\big)^{1/3}+i\,2^{2/3}\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+2^{2/3}\big(-1+i\sqrt{3}\big)^{1/3}-i\,2^{2/3}\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}\Big),$$

$$\frac{-\big(-1-i\sqrt{3}\big)^{1/3}-i\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+\big(-1+i\sqrt{3}\big)^{1/3}-i\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}}{-i\big(-1-i\sqrt{3}\big)^{1/3}+\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}-i\big(-1+i\sqrt{3}\big)^{1/3}-\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}},$$

$$\frac{\big(-1-i\sqrt{3}\big)^{1/3}+i\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+\big(-1+i\sqrt{3}\big)^{1/3}-i\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}}{-i\big(-1-i\sqrt{3}\big)^{1/3}+\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+i\big(-1+i\sqrt{3}\big)^{1/3}+\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}},$$

$$8\Big/\Big(-i\,2^{2/3}\big(-1-i\sqrt{3}\big)^{1/3}+2^{2/3}\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}+i\,2^{2/3}\big(-1+i\sqrt{3}\big)^{1/3}+2^{2/3}\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}\Big),$$

$$-(8\,i)\Big/\Big(-i\,2^{2/3}\big(-1-i\sqrt{3}\big)^{1/3}+2^{2/3}\sqrt{3}\big(-1-i\sqrt{3}\big)^{1/3}-i\,2^{2/3}\big(-1+i\sqrt{3}\big)^{1/3}-2^{2/3}\sqrt{3}\big(-1+i\sqrt{3}\big)^{1/3}\Big)\Big\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$$\{\text{Root}[-3+36\,\#1^2-96\,\#1^4+64\,\#1^6\,\&,\,4],\ \text{Root}[-1-6\,\#1+8\,\#1^3\,\&,\,3],$$
$$\text{Root}[-3+27\,\#1^2-33\,\#1^4+\#1^6\,\&,\,4],\ \text{Root}[-1+33\,\#1^2-27\,\#1^4+3\,\#1^6\,\&,\,6],$$
$$\text{Root}[-64+96\,\#1^2-36\,\#1^4+3\,\#1^6\,\&,\,6],\ \text{Root}[-8+6\,\#1^2+\#1^3\,\&,\,3]\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

```
FunctionExpand[Cot[Sqrt[-z^2]]]
```

$$-\frac{\sqrt{-z}\,\text{Coth}[z]}{\sqrt{z}}$$

```
Tan[Sqrt[I z^2]] // FunctionExpand
```

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4}z}\,\sqrt{(-1)^{3/4}z}\,\text{Tan}\big[(-1)^{1/4}z\big]}{z}$$

$$\left\{\text{Sin}\left[\sqrt{z^2}\right], \text{Cos}\left[\sqrt{z^2}\right], \text{Tan}\left[\sqrt{z^2}\right], \text{Cot}\left[\sqrt{z^2}\right], \text{Csc}\left[\sqrt{z^2}\right], \text{Sec}\left[\sqrt{z^2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z} \sqrt{i z} \,\text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{-i z} \sqrt{i z} \,\text{Tan}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \,\text{Cot}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \,\text{Csc}[z]}{z}, \text{Sec}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2} \,\text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{z^2} \,\text{Tan}[z]}{z}, \frac{\sqrt{z^2} \,\text{Cot}[z]}{z}, \frac{\sqrt{z^2} \,\text{Csc}[z]}{z}, \text{Sec}[z]\right\}$$

Here are some similar examples.

```
Sin[2 ArcTan[z]] // FunctionExpand
```

$$\frac{2 z}{1 + z^2}$$

$$\text{Cos}\left[\frac{\text{ArcCot}[z]}{2}\right] \text{ // FunctionExpand}$$

$$\frac{\sqrt{1 + \frac{\sqrt{-z} \sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

```
{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]],
  Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand
```

$$\left\{2 \sqrt{1 - z}\, z \sqrt{1 + z}, -1 + 2 z^2, -\frac{2 z}{(-1 + z)(1 + z)}, \frac{1}{2}\left(1 + \frac{1}{z^2}\right) z \left(\frac{1}{-1 - z^2} - \frac{z^2}{-1 - z^2}\right), \frac{\sqrt{-i z} \sqrt{i z}\, z}{2 \sqrt{(-1 + z)(1 + z)}}, \frac{z^2}{2 - z^2}\right\}$$

$$\left\{\text{Sin}\left[\frac{\text{ArcSin}[z]}{2}\right], \text{Cos}\left[\frac{\text{ArcCos}[z]}{2}\right], \text{Tan}\left[\frac{\text{ArcTan}[z]}{2}\right], \text{Cot}\left[\frac{\text{ArcCot}[z]}{2}\right], \text{Csc}\left[\frac{\text{ArcCsc}[z]}{2}\right], \text{Sec}\left[\frac{\text{ArcSec}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{1-\sqrt{1-z}\,\sqrt{1+z}}}{\sqrt{2}\,\sqrt{-i z}\,\sqrt{i z}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{i(-i+z)}\,\sqrt{-i(i+z)}},\right.$$
$$\left. z\left(1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\,\sqrt{z}}\right),\ \frac{\sqrt{2}\,\sqrt{-\frac{i}{z}}\,\sqrt{\frac{i}{z}}\,z}{\sqrt{1-\frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-i z}\,\sqrt{i z}}}},\ \frac{\sqrt{2}\,\sqrt{-z}}{\sqrt{-1-z}}\right\}$$

Simplify[%]

$$\left\{\frac{z\sqrt{1-\sqrt{1-z^2}}}{\sqrt{2}\,\sqrt{z^2}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{1+z^2}},\ z+\frac{\sqrt{z}\,\sqrt{-1-z^2}}{\sqrt{-z}},\ \frac{\sqrt{2}\,\sqrt{\frac{1}{z^2}}\,z}{\sqrt{1-\frac{\sqrt{z^2}\,\sqrt{-1+z^2}}{z^2}}},\ \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cos}\left[\frac{1}{2}\,i\,\text{Log}[1-i z]-\frac{1}{2}\,i\,\text{Log}[1+i z]\right]\ //\ \text{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2}\left(\text{Log}[1-i z]-\text{Log}[1+i z]\right)\right]$$

$$\text{Cos}\left[\frac{1}{2}\,i\,\text{Log}[1-i z]-\frac{1}{2}\,i\,\text{Log}[1+i z]\right]\ //\ \text{FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\text{Sin}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Cos}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\right.$$
$$\text{Tan}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Cot}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],$$
$$\left.\text{Csc}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Sec}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \text{Simplify}$$

$$\left\{z,\ \frac{1-z^2+i z\sqrt{1-z^2}}{i z+\sqrt{1-z^2}},\ \frac{z\left(z-i\sqrt{1-z^2}\right)}{-i+i z^2+z\sqrt{1-z^2}},\ \frac{1-z^2+i z\sqrt{1-z^2}}{i z^2+z\sqrt{1-z^2}},\ \frac{1}{z},\ \frac{2\left(i z+\sqrt{1-z^2}\right)}{1+\left(i z+\sqrt{1-z^2}\right)^2}\right\}$$

$$\left\{\text{Sin}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Cos}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\right.$$
$$\text{Tan}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Cot}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],$$
$$\left.\text{Csc}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Sec}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \text{FullSimplify}$$

$$\left\{z, \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{z}, \frac{1}{\sqrt{1-z^2}}\right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sin[z], {z, 0, 5}]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]
```

$$\left\{z-\frac{z^3}{6}+O[z]^4, 1-\frac{z^2}{2}+O[z]^4, z+\frac{z^3}{3}+O[z]^4, \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}+O[z]^4, \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+O[z]^4, 1+\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Cot[z], {z, 0, 100}] // Timing
```

$$\left\{1.442\,\text{Second}, \frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2z^5}{945}-\frac{z^7}{4725}-\frac{2z^9}{93\,555}-\frac{1382z^{11}}{638\,512\,875}-\right.$$
$$\frac{4z^{13}}{18\,243\,225}-\frac{3617z^{15}}{162\,820\,783\,125}-\frac{87\,734z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222z^{19}}{1\,531\,329\,465\,290\,625}-$$
$$\frac{310\,732z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182z^{23}}{201\,919\,571\,963\,756\,521\,875}-\frac{2\,631\,724z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$
$$\frac{13\,571\,120\,588z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}-\frac{13\,785\,346\,041\,608z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$
$$\frac{7\,709\,321\,041\,217z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}-\frac{303\,257\,395\,102z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$
$$\frac{52\,630\,543\,106\,106\,954\,746z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}-\frac{616\,840\,823\,966\,644z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$
$$\frac{522\,165\,436\,992\,898\,244\,102z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}-$$
$$\frac{6\,080\,390\,575\,672\,283\,210\,764z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375}-$$
$$\frac{10\,121\,188\,937\,927\,645\,176\,372z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875}-$$

$$\frac{207461256206578143748856\,z^{45}}{7670102214448301053033358480610212529462890625}-$$

$$\frac{11218806737995635372498255094\,z^{47}}{4093648603384274996519698921478879580162286669921875}-$$

$$\frac{79209152838572743713996404\,z^{49}}{285258771457546764463363635252374414183254365234375}-$$

$$\frac{246512528657073833030130766724\,z^{51}}{8761982491474419367550817114626909562924278968505859375}-$$

$$\frac{233199709079078899371344990501528\,z^{53}}{81807125729900063867074959072425603825198823017351806640625}-$$

$$\frac{1416795959607558144963094708378988\,z^{55}}{4905352087939496310826487207538302184255342959123162841796875}-$$

$$\frac{23305824372104839134357731308699592\,z^{57}}{796392368980577121745974726570063253238310542073919837646484375}-$$

$$\frac{9721865123870044576322439952638561968331928\,z^{59}}{3278777586273629598615520165380455583231003564645636125000418914794921875}-$$

$$\frac{6348689256302894731330601216724328336\,z^{61}}{21132271510899613925529439369536628424678570233931462891949462890625}-$$

$$\frac{106783830147866529886385444979142647942017\,z^{63}}{3508062732166890409707514582539928001638766051683792497378070587158203125}-$$

$$\left(267745458568424664373021714282169516771254382\,z^{65}\right)/86812790293146213360651966604262937105495141563588806888204273501373291015625-$$

$$\left(250471004320250327955196022920428000776938\,z^{67}\right)/801528196428242695121010267455843804062822357897831858125102407684326171875-$$

$$\left(172043582552384800434637321986040823829878646884\,z^{69}\right)/5433748964547053581149916185708338218048392402830337634114958370880742156982421875-$$

$$\left(11655909923339888220876554489282134730564976603688520858\,z^{71}\right)/3633348205269879230856840004304821536968049780112803650817771432558560793458452606201171875-$$

$$\left(3692153220456342488035683646645690290452790030604\,z^{73}\right)/11359005221796317918049302062760294302183889391189419445133951612582060536346435546875-$$

$$\left(5190545015986394254249936008544252611445319542919116\,z^{75}\right)/157606197452423911112934066120799083442801465302753194801233578624576089941806793212890625-$$

$$\left(255290071123323586643187098799718199072122692536861835992\,z^{77}\right)/76505736228426953173738238352183101801688392812244485181277127930109049138257655704498291015625-$$

$$\left(9207568598958915293871149938038093699588515745502577839313734\,z^{79}\right)/27233582984369795892070228410001578355986013571390071723225259349721067988068852863296604156494140625-$$

$$\left(163611136505867886519332147296221453678803514884902772183572\,z^{81}\right)/4776089171877348057451105924101750653118402745283825543113171217116857704024700607798175811767578125-$$

$$\left(8098304783741161440924524640446924039959669564792363509124335729908\,z^{83}\right)/$$

```
2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393
  530 725 143 500 144 033 328 342 437 744 140 625 -
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^85) /
 349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231
  069 047 713 981 709 496 784 210 205 078 125 -
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^87) /
 13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114
  528 962 242 087 062 453 152 690 410 614 013 671 875 -
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
   z^89) /
 522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471
  400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 -
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
   z^91) /
 1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743
  269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 -
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
   z^93) /
 11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254
  692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 -
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336
   699 974 518 z^95) /
 112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237
  591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 -
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229
   244 z^97) /
 14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225
  810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 -
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581
   660 788 990 844 z^99) /
 9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359
  080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd trigonometric functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]
```

$$\left\{\frac{i^{-1+n}\,\text{KroneckerDelta}[\text{Mod}[-1+n,2]]\,\text{UnitStep}[-1+n]}{\text{Gamma}[1+n]},\ \text{If}\left[\text{Odd}[n],\ \frac{i^{-1+n}\,2^{1+n}\,\left(-1+2^{1+n}\right)\text{BernoulliB}[1+n]}{(1+n)\,!},\ 0\right],\ \frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}[1+n]}{(1+n)\,!},\ \frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}\left[1+n,\frac{1}{2}\right]}{(1+n)\,!},\ \frac{i^{n}\,\text{KroneckerDelta}[\text{Mod}[n,2]]}{\text{Gamma}[1+n]},\ \frac{i^{n}\,\text{EulerE}[n]}{n\,!}\right\}$$

Differentiation

Mathematica can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

```
D[Sin[z], z]
```

```
Cos[z]
```

```
Sin[z] // D[#, z] &
```

```
Cos[z]
```

∂_z **{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

```
{Cos[z], -Sin[z], Sec[z]^2, -Csc[z]^2, -Cot[z] Csc[z], Sec[z] Tan[z]}
```

$\partial_{\{z,2\}}$ **{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

```
{-Sin[z], -Cos[z], 2 Sec[z]^2 Tan[z], 2 Cot[z] Csc[z]^2,
 Cot[z]^2 Csc[z] + Csc[z]^3, Sec[z]^3 + Sec[z] Tan[z]^2}
```

```
Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]
```

```
{{Cos[z], -Sin[z], Sec[z]^2, -Csc[z]^2, -Cot[z] Csc[z], Sec[z] Tan[z]}, {-Sin[z], -Cos[z],
  2 Sec[z]^2 Tan[z], 2 Cot[z] Csc[z]^2, Cot[z]^2 Csc[z] + Csc[z]^3, Sec[z]^3 + Sec[z] Tan[z]^2},
 {-Cos[z], Sin[z], 2 Sec[z]^4 + 4 Sec[z]^2 Tan[z]^2, -4 Cot[z]^2 Csc[z]^2 - 2 Csc[z]^4,
  -Cot[z]^3 Csc[z] - 5 Cot[z] Csc[z]^3, 5 Sec[z]^3 Tan[z] + Sec[z] Tan[z]^3},
 {Sin[z], Cos[z], 16 Sec[z]^4 Tan[z] + 8 Sec[z]^2 Tan[z]^3,
  8 Cot[z]^3 Csc[z]^2 + 16 Cot[z] Csc[z]^4, Cot[z]^4 Csc[z] + 18 Cot[z]^2 Csc[z]^3 + 5 Csc[z]^5,
  5 Sec[z]^5 + 18 Sec[z]^3 Tan[z]^2 + Sec[z] Tan[z]^4}}
```

Finite summation

Mathematica can calculate finite sums that contain trigonometric functions. Here are two examples.

```
Sum[Sin[a k], {k, 0, n}]
```

$$\frac{1}{2}\left(\text{Cos}\left[\frac{a}{2}\right]-\text{Cos}\left[\frac{a}{2}+a\,n\right]\right)\text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^{n}(-1)^{k}\,\textbf{Sin}[a\,k]$$

$$\frac{1}{2}\,\mathrm{Sec}\left[\frac{a}{2}\right]\left(-\mathrm{Sin}\left[\frac{a}{2}\right]+\mathrm{Sin}\left[\frac{a}{2}+a\,n+n\,\pi\right]\right)$$

Infinite summation

Mathematica can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\,\mathrm{Sin}[k\,x]$$

$$\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(e^{i\,x}-z\right)\left(-1+e^{i\,x}\,z\right)}$$

$$\sum_{k=1}^{\infty}\frac{\mathrm{Sin}[k\,x]}{k\,!}$$

$$\frac{1}{2}\,i\left(e^{e^{-i\,x}}-e^{e^{i\,x}}\right)$$

$$\sum_{k=1}^{\infty}\frac{\mathrm{Cos}[k\,x]}{k}$$

$$\frac{1}{2}\left(-\mathrm{Log}\left[1-e^{-i\,x}\right]-\mathrm{Log}\left[1-e^{i\,x}\right]\right)$$

Finite products

Mathematica can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\mathrm{Product}\left[\mathrm{Sin}\left[\frac{\pi\,k}{n}\right],\,\{k,\,1,\,n-1\}\right]$$

$$2^{1-n}\,n$$

$$\prod_{k=1}^{n-1}\mathrm{Cos}\left[z+\frac{\pi\,k}{n}\right]$$

$$-(-1)^n\,2^{1-n}\,\mathrm{Sec}[z]\,\mathrm{Sin}\left[\frac{1}{2}\,n\,(\pi-2\,z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain trigonometric functions. Here are some examples.

In[2]:= $$\prod_{k=1}^{\infty}\mathrm{Exp}\left[z^k\,\mathrm{Sin}[k\,x]\right]$$

Out[2]= $$e^{\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(z+e^{2\,i\,x}\,z-e^{i\,x}\left(1+z^2\right)\right)}}$$

In[3]:= $\prod_{k=1}^{\infty} \mathrm{Exp}\left[\frac{\mathrm{Cos}[k\,x]}{k\,!}\right]$

Out[3]= $e^{\frac{1}{2}\left(-2+e^{e^{-i\,x}}+e^{e^{i\,x}}\right)}$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \mathrm{Sin}[7\,z]\,dz$$

$$-\frac{1}{7}\,\mathrm{Cos}[7\,z]$$

$$\int \{\{\mathrm{Sin}[z],\,\mathrm{Sin}[z]^a\},\,\{\mathrm{Cos}[z],\,\mathrm{Cos}[z]^a\},\,\{\mathrm{Tan}[z],\,\mathrm{Tan}[z]^a\},\,\{\mathrm{Cot}[z],\,\mathrm{Cot}[z]^a\},\,\{\mathrm{Csc}[z],\,\mathrm{Csc}[z]^a\},\,\{\mathrm{Sec}[z],\,\mathrm{Sec}[z]^a\}\}\,dz$$

$$\Big\{\Big\{-\mathrm{Cos}[z],\,-\mathrm{Cos}[z]\,\mathrm{Hypergeometric2F1}\Big[\frac{1}{2},\,\frac{1-a}{2},\,\frac{3}{2},\,\mathrm{Cos}[z]^2\Big]\,\mathrm{Sin}[z]^{1+a}\,\left(\mathrm{Sin}[z]^2\right)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\mathrm{Sin}[z],\,-\frac{\mathrm{Cos}[z]^{1+a}\,\mathrm{Hypergeometric2F1}\left[\frac{1+a}{2},\,\frac{1}{2},\,\frac{3+a}{2},\,\mathrm{Cos}[z]^2\right]\,\mathrm{Sin}[z]}{(1+a)\,\sqrt{\mathrm{Sin}[z]^2}}\Big\},$$

$$\Big\{-\mathrm{Log}[\mathrm{Cos}[z]],\,\frac{\mathrm{Hypergeometric2F1}\left[\frac{1+a}{2},\,1,\,1+\frac{1+a}{2},\,-\mathrm{Tan}[z]^2\right]\,\mathrm{Tan}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\mathrm{Log}[\mathrm{Sin}[z]],\,-\frac{\mathrm{Cot}[z]^{1+a}\,\mathrm{Hypergeometric2F1}\left[\frac{1+a}{2},\,1,\,1+\frac{1+a}{2},\,-\mathrm{Cot}[z]^2\right]}{1+a}\Big\},$$

$$\Big\{-\mathrm{Log}\Big[\mathrm{Cos}\Big[\frac{z}{2}\Big]\Big]+\mathrm{Log}\Big[\mathrm{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\mathrm{Cos}[z]\,\mathrm{Csc}[z]^{-1+a}\,\mathrm{Hypergeometric2F1}\Big[\frac{1}{2},\,\frac{1+a}{2},\,\frac{3}{2},\,\mathrm{Cos}[z]^2\Big]\,\left(\mathrm{Sin}[z]^2\right)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{-\mathrm{Log}\Big[\mathrm{Cos}\Big[\frac{z}{2}\Big]-\mathrm{Sin}\Big[\frac{z}{2}\Big]\Big]+\mathrm{Log}\Big[\mathrm{Cos}\Big[\frac{z}{2}\Big]+\mathrm{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\frac{\mathrm{Hypergeometric2F1}\left[\frac{1-a}{2},\,\frac{1}{2},\,\frac{3-a}{2},\,\mathrm{Cos}[z]^2\right]\,\mathrm{Sec}[z]^{-1+a}\,\mathrm{Sin}[z]}{(1-a)\,\sqrt{\mathrm{Sin}[z]^2}}\Big\}\Big\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\mathrm{Sin}[z]}\,dz$$

$$\frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{3}\right]}{2\ \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2}\left\{\sqrt{\text{Sin}[z]},\ \sqrt{\text{Cos}[z]},\ \sqrt{\text{Tan}[z]},\ \sqrt{\text{Cot}[z]},\ \sqrt{\text{Csc}[z]},\ \sqrt{\text{Sec}[z]}\right\} dz$$

$$\left\{2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}\right\}$$

$$\int_0^{\frac{\pi}{2}}\{\{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\}\} dz$$

$$\left\{\left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\}, \left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\},\right.$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Tan}[z]\, dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\ \text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Tan}[z]^a\, dz\right]\right\},$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Cot}[z]\, dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\ \text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\, dz\right]\right\},$$

$$\left.\left\{\int_0^{\frac{\pi}{2}}\text{Csc}[z]\, dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}, \left\{\int_0^{\frac{\pi}{2}}\text{Sec}[z]\, dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}\right\}$$

Limit operation

Mathematica can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \to 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \to 0\right]$$

$e^{1/3}$

Solving equations

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3\ \text{Sin}\left[z + \text{Pi}/6\right] == 4, z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 1]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 2]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 3]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 4]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 5]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 6]]}}
```

Complete solutions can be obtained by using the function `Reduce`.

```
Reduce[Sin[x] == a, x] // TraditionalForm
```

```
// InputForm =
 C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

```
Reduce[Cos[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

```
Reduce[Tan[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

```
Reduce[Cot[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

```
Reduce[Csc[x] == a, x] // TraditionalForm
```

$$c_1 \in \mathbb{Z} \bigwedge a \neq 0 \bigwedge \left(x == -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \bigvee x == \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

```
Reduce[Sec[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sin(z) and cos(z).

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

```
dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]
```

```
{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ-calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
```

$\{\{w \to \text{Function}[\{z\}, C[3]\,\text{Cos}[z] + C[1]\,\text{Cos}[\sqrt{2}\,z] + C[4]\,\text{Sin}[z] + C[2]\,\text{Sin}[\sqrt{2}\,z]]\}\}$

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ζ] /. dsol1
```

$\{w'[\zeta]\}$

```
w'[ζ] /. dsol2
```

$\{C[4]\,\text{Cos}[\zeta] + \sqrt{2}\,C[2]\,\text{Cos}[\sqrt{2}\,\zeta] - C[3]\,\text{Sin}[\zeta] - \sqrt{2}\,C[1]\,\text{Sin}[\sqrt{2}\,\zeta]\}$

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

$\text{DSolve}[\{w'[z] == \sqrt{1 - w[z]^2}, w[0] == 0\}, w[z], z]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Sin}[z]\}\}$

$\text{DSolve}[\{w'[z] == \sqrt{1 - w[z]^2}, w[0] == 1\}, w[z], z]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Cos}[z]\}\}$

$\text{DSolve}[\{w'[z] - w[z]^2 - 1 == 0, w[0] == 0\}, w[z], z]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Tan}[z]\}\}$

$\text{DSolve}[\{w'[z] + w[z]^2 + 1 == 0, w[\frac{\pi}{2}] == 0\}, w[z], z]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Cot}[z]\}\}$

$\text{DSolve}[\{w'[z] == \sqrt{w[z]^4 - w[z]^2}, 1/w[0] == 0\}, w[z], z]$ // Simplify[#, 0 < z < Pi/2] &

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Csc[z]}, {w[z] → Csc[z]}}
```

$$\text{DSolve}\left[\left\{w'[z] == \sqrt{w[z]^4 - w[z]^2},\ 1 \Big/ w\left[\frac{\pi}{2}\right] == 0\right\},\ w[z],\ z\right]\ //\ \text{Simplify}\left[\#,\ 0 < z < \text{Pi}/2\right]\ \&$$

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Sec[z]}, {w[z] → Sec[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

```
LaplaceTransform[Sin[t], t, s]
```

$$\frac{1}{1+s^2}$$

```
FourierTransform[Sin[t], t, s]
```

$$i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

```
FourierSinTransform[Sin[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - \sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

```
FourierCosTransform[Sin[t], t, s]
```

$$-\frac{1}{\sqrt{2\pi}\ (-1+s)} + \frac{1}{\sqrt{2\pi}\ (1+s)}$$

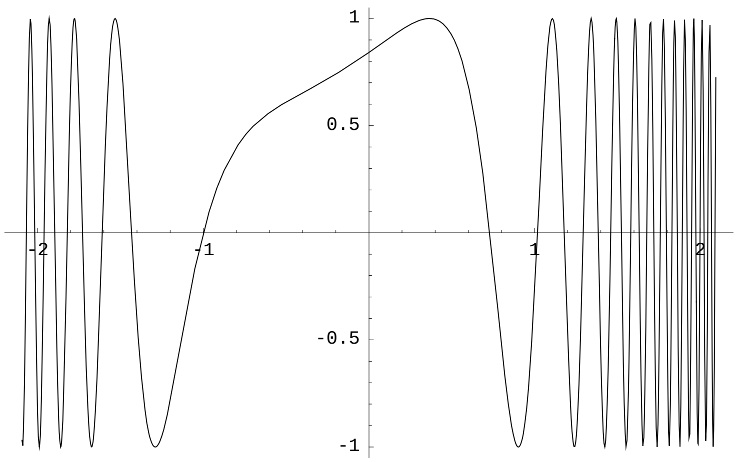
```
ZTransform[Sin[π t], t, s]
```

```
0
```

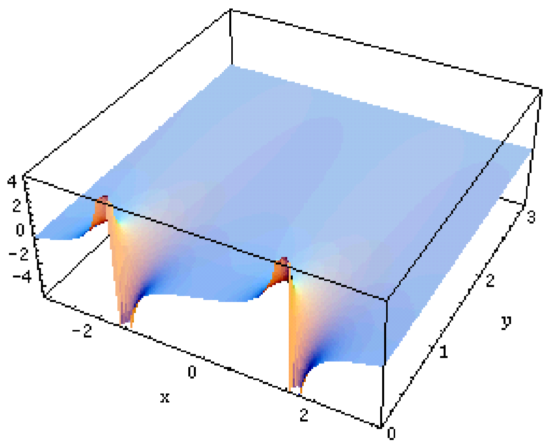
Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

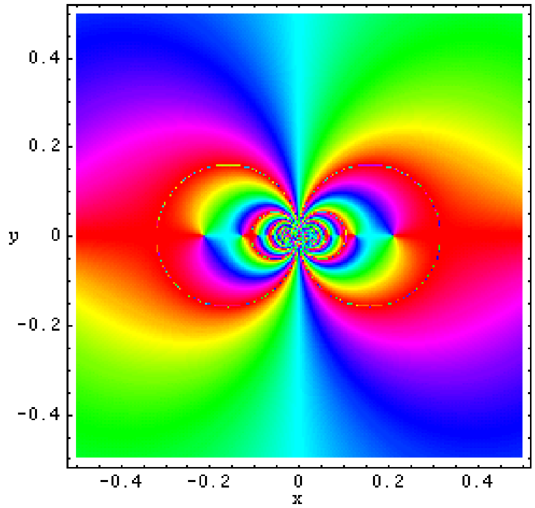
$$\text{Plot}\left[\text{Sin}\left[\sum_{k=0}^{5} z^k\right],\ \left\{z,\ -\frac{2\pi}{3},\ \frac{2\pi}{3}\right\}\right];$$




```
Plot3D[Re[Tan[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```




```
ContourPlot[Arg[Sec[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```



Introduction to the Cosine Function in *Mathematica*

Overview

The following shows how the cosine function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the cosine function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the cosine function in `StandardForm`.

```
Cos[z]
```

```
Cos[z]
```

This shows the cosine function in `TraditionalForm`.

```
% // TraditionalForm
```

cos(z)

Additional forms of notations

Mathematica also knows the most popular forms of notations for the cosine function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Cos[2 π z]], FortranForm[Cos[2 π z]], TeXForm[Cos[2 π z]]}
```

```
{Cos(2*Pi*z), Cos(2*Pi*z), \cos(2\, \pi \, z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For the exact argument $z = \pi/4$, *Mathematica* returns an exact result.

```
Cos[π/4]
```

$$\frac{1}{\sqrt{2}}$$

```
Cos[z] /. z → π/4
```

$$\frac{1}{\sqrt{2}}$$

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

```
Cos[5.]
```

```
0.283662
```

```
Cos[z] /. z → 3.
```

```
-0.989992
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Cos[z] /. z → 1, 100]
```

```
0.540302305868139717400936607442976603732310420617922227670097255381100394774471764 5
  179518560871830893
```

```
N[Cos[2], 100]
```

```
-0.4161468365471423869975682295007621897660007710755448907551499737819649361240791690
   745317778601691404
```

```
Cos[2] // N[#, 100] &
```

```
-0.4161468365471423869975682295007621897660007710755448907551499737819649361240791690
   745317778601691404
```

Within a second, it is possible to calculate thousands of digits for the cosine function. The next input calculates 10000 digits for cos(1) and analyzes the frequency of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
  N[{Cos[z]} /. z → 1, 10 000]]
```

```
{{{0, 998}, {1, 1034}, {2, 982}, {3, 1015},
   {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}}}
```

Here is a 50-digit approximation of the cosine function at the complex argument $z = 3 + 2i$.

```
N[Cos[3 + 2 ⅈ], 50]
```

```
-3.7245455049153225654739707032559725286749657732153-
 0.51182256998738460883446384980187563424555660949074 ⅈ
```

```
{N[Cos[z] /. z → 3 + 2 ⅈ, 50], Cos[3 + 2 ⅈ] // N[#, 50] &}
```

```
{-3.7245455049153225654739707032559725286749657732153-
  0.51182256998738460883446384980187563424555660949074 ⅈ,
 -3.7245455049153225654739707032559725286749657732153-
  0.51182256998738460883446384980187563424555660949074 ⅈ}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Cos[2.], N[Cos[2]], N[Cos[2], 16], N[Cos[2], 5], N[Cos[2], 20]}
```

```
{-0.416147, -0.416147, -0.416147, -0.416147, -0.41614683654714238700}
```

```
% // InputForm
```

```
{-0.4161468365471424, -0.4161468365471424, -0.4161468365471424, -0.4161468365471424,
 -0.416146836547142386997568229500762`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the cosine function. Here are some examples:

```
Cos[-3]
```

```
Cos[3]
```

```
{Cos[-z], Cos[z + π], Cos[z + 2 π], Cos[z + 342 π], Cos[-z + 21 π]}
```

```
{Cos[z], -Cos[z], Cos[z], Cos[z], -Cos[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse cosine functions into its argument.

```
Cos[ArcCos[z]]
```

```
z
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

In[1]:=
```
{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
 Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}
```

Out[1]= $\left\{\sqrt{1-z^2},\ z,\ \frac{1}{\sqrt{1+z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{z}\right\}$

If the argument has the structure $\pi k/2 + z$ or $\pi k/2 - z$, and $\pi k/2 + i z$ or $\pi k/2 - i z$ with integer k, the cosine function can be automatically transformed into trigonometric or hyperbolic sine or cosine functions.

$$\text{Cos}\left[\frac{\pi}{2} - 4\right]$$

Sin[4]

$$\left\{\text{Cos}\left[\frac{\pi}{2} - z\right], \text{Cos}\left[\frac{\pi}{2} + z\right], \text{Cos}\left[-\frac{\pi}{2} - z\right], \text{Cos}\left[-\frac{\pi}{2} + z\right], \text{Cos}[\pi - z], \text{Cos}[\pi + z]\right\}$$

{Sin[z], -Sin[z], -Sin[z], Sin[z], -Cos[z], -Cos[z]}

Cos[ⅈ 5]

Cosh[5]

$$\left\{\text{Cos}[i\, z], \text{Cos}\left[\frac{\pi}{2} - i\, z\right], \text{Cos}\left[\frac{\pi}{2} + i\, z\right], \text{Cos}[\pi - i\, z], \text{Cos}[\pi + i\, z]\right\}$$

{Cosh[z], ⅈ Sinh[z], -ⅈ Sinh[z], -Cosh[z], -Cosh[z]}

Simplification of simple expressions containing the cosine function

Sometimes simple arithmetic operations containing the cosine function can automatically produce other trigonometric functions.

1 / Cos[4]

Sec[4]

{1 / Cos[z], 1 / Cos[π / 2 - z], Cos[π / 2 - z] / Cos[z],
Cos[z] / Cos[π / 2 - z], 1 / Cos[π / 2 - z], Cos[π / 2 - z] / Cos[z] ^2}

{Sec[z], Csc[z], Tan[z], Cot[z], Csc[z], Sec[z] Tan[z]}

The cosine function arising as special cases from more general functions

The cosine function can be treated as a particular case of some more general special functions. For example, cos(z) can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\left\{\text{BesselJ}\left[-\frac{1}{2}, z\right], \text{MathieuC}[1, 0, z], \text{JacobiCD}[z, 0], \text{Hypergeometric0F1}\left[\frac{1}{2}, -\frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, \frac{z^2}{4}\right]\right\}$$

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \text{Cos}[z]}{\sqrt{z}}, \text{Cos}[z], \text{Cos}[z], \text{Cos}\left[\sqrt{z^2}\right], \frac{\text{Cos}[z]}{\sqrt{\pi}}\right\}$$

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\cos(z)/2$ instead of $\sin(\pi/2 - z)\cos(\pi/3)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\cos(2z)\cos(\pi/16)$ should not be automatically expanded into the more complicated expression $\left(\cos^2(z) - \frac{1}{2}\right)\left(2 + \left(2 + 2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in arguments of trigonometric and hyperbolic functions, and then expands out products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Cos[x - y]]
```

```
Cos[x] Cos[y] + Sin[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

$$\text{Cos}[z]^4 - 6\,\text{Cos}[z]^2\,\text{Sin}[z]^2 + \text{Sin}[z]^4$$

```
Cos[2 z]^2 // TrigExpand
```

$$\frac{1}{2} + \frac{\text{Cos}[z]^4}{2} - 3\,\text{Cos}[z]^2\,\text{Sin}[z]^2 + \frac{\text{Sin}[z]^4}{2}$$

```
TrigExpand[{Cos[x + y + z], Cos[3 z]}]
```

$$\{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z] - \text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y] - \text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z] - \text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z],\ \text{Cos}[z]^3 - 3\,\text{Cos}[z]\,\text{Sin}[z]^2\}$$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Cos[x] + Cos[y]]
```

$$2\,\text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right]\text{Cos}\left[\frac{x}{2} + \frac{y}{2}\right]$$

```
Cos[x] + Sin[y] // TrigFactor
```

$$\left(\text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] - \text{Sin}\left[\frac{x}{2} - \frac{y}{2}\right]\right)\left(\text{Cos}\left[\frac{x}{2} + \frac{y}{2}\right] + \text{Sin}\left[\frac{x}{2} + \frac{y}{2}\right]\right)$$

TrigReduce

The function TrigReduce rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. TrigReduce is approximately inverse to TrigExpand and TrigFactor. Here are some examples.

```
TrigReduce[Cos[x] Cos[y]]
```

$$\frac{1}{2}(\text{Cos}[x-y]+\text{Cos}[x+y])$$

```
Sin[x] Cos[y] // TrigReduce
```

$$\frac{1}{2}(\text{Sin}[x-y]+\text{Sin}[x+y])$$

```
Table[TrigReduce[Cos[z]^n], {n, 2, 5}]
```

$$\left\{\frac{1}{2}(1+\text{Cos}[2z]),\ \frac{1}{4}(3\,\text{Cos}[z]+\text{Cos}[3z]),\ \frac{1}{8}(3+4\,\text{Cos}[2z]+\text{Cos}[4z]),\ \frac{1}{16}(10\,\text{Cos}[z]+5\,\text{Cos}[3z]+\text{Cos}[5z])\right\}$$

```
TrigReduce[TrigExpand[{Cos[x + y + z], Cos[3 z], Cos[x] Cos[y]}]]
```

$$\left\{\text{Cos}[x+y+z],\ \text{Cos}[3z],\ \frac{1}{2}(\text{Cos}[x-y]+\text{Cos}[x+y])\right\}$$

```
TrigFactor[Cos[x] + Cos[y]] // TrigReduce
```

$$\text{Cos}[x]+\text{Cos}[y]$$

TrigToExp

The function TrigToExp converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Cos[z]]
```

$$\frac{e^{-iz}}{2}+\frac{e^{iz}}{2}$$

```
Cos[a z] + Cos[b z] // TrigToExp
```

$$\frac{1}{2}e^{-iaz}+\frac{1}{2}e^{iaz}+\frac{1}{2}e^{-ibz}+\frac{1}{2}e^{ibz}$$

ExpToTrig

The function ExpToTrig converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to TrigToExp. Here are some examples.

```
ExpToTrig[TrigToExp[Cos[z]]]
```

$$\text{Cos}[z]$$

```
{α ⅇ^(-ⅈ x β) + α ⅇ^(ⅈ x β), α ⅇ^(-ⅈ x β) + γ ⅇ^(ⅈ x β)} // ExpToTrig
```

```
{2 α Cos[x β], α Cos[x β] + γ Cos[x β] - ⅈ α Sin[x β] + ⅈ γ Sin[x β]}
```

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Cos[x + ⅈ y]]
```

```
Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y]
```

```
Cos[x + ⅈ y] + Cos[x - ⅈ y] // ComplexExpand
```

```
2 Cos[x] Cosh[y]
```

```
ComplexExpand[Re[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
Cos[x] Cosh[y]
```

```
ComplexExpand[Im[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
-Sin[x] Sinh[y]
```

```
ComplexExpand[Abs[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\sqrt{\text{Cos}[x]^2\,\text{Cosh}[y]^2 + \text{Sin}[x]^2\,\text{Sinh}[y]^2}$$

```
ComplexExpand[Abs[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}] //
  Simplify[#, {x, y} ∈ Reals] &
```

$$\frac{\sqrt{\text{Cos}[2\,x] + \text{Cosh}[2\,y]}}{\sqrt{2}}$$

```
ComplexExpand[Re[Cos[x + ⅈ y]] + Im[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
Cos[x] Cosh[y] - Sin[x] Sinh[y]
```

```
ComplexExpand[Arg[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

```
ArcTan[Cos[x] Cosh[y], -Sin[x] Sinh[y]]
```

```
ComplexExpand[Arg[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}] //
  Simplify[#, {x, y} ∈ Reals] &
```

```
ArcTan[Cos[x] Cosh[y], -Sin[x] Sinh[y]]
```

```
ComplexExpand[Conjugate[Cos[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify
```

```
Cos[x] Cosh[y] + ⅈ Sin[x] Sinh[y]
```

Simplify

The function Simplify performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are some examples.

```
Cos[2 ArcCos[z]] / (-1 + 2 z^2) // Simplify
```

```
1
```

```
{Simplify[Cos[2 z] - Cos[z]^2], Cos[2 z] + Sin[z]^2 // Simplify}
```

```
{-Sin[z]^2, Cos[z]^2}
```

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#]\ \&\ /@\ \Big(\cos(z)^2+\sin(z)^2==1\ \land$$

$$\cos(z)^2==\frac{1+\cos(2z)}{2}\ \land$$

$$\cos(2z)==\cos(z)^2-\sin(z)^2==2\cos(z)^2-1\ \land$$

$$\cos(a+b)==\cos(a)\cos(b)-\sin(a)\sin(b)\ \land$$

$$\cos(a-b)==\cos(a)\cos(b)+\sin(a)\sin(b)\ \land$$

$$\cos(a)+\cos(b)==2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)\land\cos(a)-\cos(b)==2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)\land$$

$$A\sin(z)+B\cos(z)==A\sqrt{1+\frac{B^2}{A^2}}\,\sin\left(z+\tan^{-1}\left(\frac{B}{A}\right)\right)\land$$

$$\cos(a)\cos(b)==\frac{\cos(a-b)+\cos(a+b)}{2}\ \land$$

$$\sin(a)\cos(b)==\frac{\sin(a+b)+\sin(a-b)}{2}\land\cos\left(\frac{z}{2}\right)^2==\frac{1+\cos(z)}{2}\Big)$$

```
True
```

The function Simplify has the Assumption option. For example, *Mathematica* knows that $-1 \leq \cos(x) \leq 1$ for all real x, and knows about the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\pi$.

```
Simplify[Abs[Cos[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Cos[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Cos[z + 2 k π], Cos[z + k π] / Cos[z]}, k ∈ Integers]
```

```
{Cos[z], (-1)^k}
```

Mathematica also knows that the composition of the inverse and direct trigonometric functions produces the value of the internal argument under the corresponding restriction.

```
ArcCos[Cos[z]]
```

```
ArcCos[Cos[z]]
```

```
Simplify[ArcCos[Cos[z]], 0 < Re[z] < π]
```

```
z
```

FunctionExpand (and Together)

While the cosine function auto-evaluates for simple fractions of π, for more complicated cases it stays as a cosine function to avoid the build up of large expressions. Using the function `FunctionExpand`, the cosine function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\cos\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\cos\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{60}\right]\right\}$$

```
FunctionExpand[%]
```

$$\left\{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, -\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}-\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}\right\}$$

```
Together[%]
```

$$\left\{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right)\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\cos\left[\frac{\pi}{9}\right]\right\}$$

$$\left\{\cos\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\left\{\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+\right.\right.$$
$$\left.\left. i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\right\}$$

The function `RootReduce` allows for writing the last algebraic numbers as roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$\{\text{Root}[-1 - 6 \#1 + 8 \#1^3 \&, 3]\}$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including inverse trigonometric functions, to simpler ones. Here are some examples.

$$\left\{\mathbf{Cos}\left[\sqrt{\mathbf{z}^2}\right], \mathbf{Cos}\left[\frac{\mathbf{ArcCos[z]}}{\mathbf{2}}\right], \mathbf{Cos[3\,ArcCos[z]]}\right\} \mathbf{//\ FunctionExpand}$$

$$\left\{\text{Cos}[z], \frac{\sqrt{1+z}}{\sqrt{2}}, z^3 - 3 z (1 - z^2)\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

Simplify[%]

$$\left\{\text{Cos}[z], \frac{\sqrt{1+z}}{\sqrt{2}}, z(-3 + 4 z^2)\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

$$\mathbf{set1} = \left\{\text{Cos}\left[-i\,\text{Log}\left[i\,z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[\frac{\pi}{2} + i\,\text{Log}\left[i\,z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[\frac{1}{2} i\,\text{Log}[1 - i\,z] - \frac{1}{2} i\,\text{Log}[1 + i\,z]\right], \text{Cos}\left[\frac{1}{2} i\,\text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2} i\,\text{Log}\left[1 + \frac{i}{z}\right]\right], \text{Cos}\left[-i\,\text{Log}\left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right]\right], \text{Cos}\left[\frac{\pi}{2} + i\,\text{Log}\left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right]\right]\right\}$$

$$\left\{\frac{1 + \left(i\,z + \sqrt{1 - z^2}\right)^2}{2\left(i\,z + \sqrt{1 - z^2}\right)}, -\frac{i\left(-1 + \left(i\,z + \sqrt{1 - z^2}\right)^2\right)}{2\left(i\,z + \sqrt{1 - z^2}\right)}, \text{Cosh}\left[\frac{1}{2}\text{Log}[1 - i\,z] - \frac{1}{2}\text{Log}[1 + i\,z]\right], \text{Cosh}\left[\frac{1}{2}\text{Log}\left[1 - \frac{i}{z}\right] - \frac{1}{2}\text{Log}\left[1 + \frac{i}{z}\right]\right], \frac{1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)^2}{2\left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)}, -\frac{i\left(-1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)^2\right)}{2\left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z}\right)}\right\}$$

set1 // Simplify

$$\left\{\frac{1-z^2+i\,z\sqrt{1-z^2}}{i\,z+\sqrt{1-z^2}},\ z,\ \mathrm{Cosh}\left[\frac{1}{2}\left(\mathrm{Log}[1-i\,z]-\mathrm{Log}[1+i\,z]\right)\right],\right.$$

$$\left.\mathrm{Cosh}\left[\frac{1}{2}\left(\mathrm{Log}\left[\frac{-i+z}{z}\right]-\mathrm{Log}\left[\frac{i+z}{z}\right]\right)\right],\ \frac{-1+i\sqrt{1-\frac{1}{z^2}}\ z+z^2}{z\left(i+\sqrt{1-\frac{1}{z^2}}\ z\right)},\ \frac{1}{z}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\sqrt{1-z^2},\ z,\ \frac{1}{\sqrt{1+z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{z}\right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of a cosine function to hundreds of terms can be done in seconds.

```
Series[Cos[z], {z, 0, 3}]
```

$$1-\frac{z^2}{2}+O[z]^4$$

```
Normal[%]
```

$$1-\frac{z^2}{2}$$

```
Series[Cos[z], {z, 0, 100}] // Timing
```

$$\left\{8.53484\times 10^{-16}\ \mathrm{Second},\ 1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+\frac{z^8}{40\,320}-\frac{z^{10}}{3\,628\,800}+\right.$$
$$\frac{z^{12}}{479\,001\,600}-\frac{z^{14}}{87\,178\,291\,200}+\frac{z^{16}}{20\,922\,789\,888\,000}-\frac{z^{18}}{6\,402\,373\,705\,728\,000}+$$
$$\frac{z^{20}}{2\,432\,902\,008\,176\,640\,000}-\frac{z^{22}}{1\,124\,000\,727\,777\,607\,680\,000}+\frac{z^{24}}{620\,448\,401\,733\,239\,439\,360\,000}-$$
$$\frac{z^{26}}{403\,291\,461\,126\,605\,635\,584\,000\,000}+\frac{z^{28}}{304\,888\,344\,611\,713\,860\,501\,504\,000\,000}-$$
$$\frac{z^{30}}{265\,252\,859\,812\,191\,058\,636\,308\,480\,000\,000}+\frac{z^{32}}{263\,130\,836\,933\,693\,530\,167\,218\,012\,160\,000\,000}-$$
$$\frac{z^{34}}{295\,232\,799\,039\,604\,140\,847\,618\,609\,643\,520\,000\,000}+$$
$$\frac{z^{36}}{371\,993\,326\,789\,901\,217\,467\,999\,448\,150\,835\,200\,000\,000}-$$

$$\frac{z^{38}}{523 022 617 466 601 111 760 007 224 100 074 291 200 000 000} +$$

$$\frac{z^{40}}{815 915 283 247 897 734 345 611 269 596 115 894 272 000 000 000} -$$

$$\frac{z^{42}}{1 405 006 117 752 879 898 543 142 606 244 511 569 936 384 000 000 000} +$$

$$\frac{z^{44}}{2 658 271 574 788 448 768 043 625 811 014 615 890 319 638 528 000 000 000} -$$

$$\frac{z^{46}}{5 502 622 159 812 088 949 850 305 428 800 254 892 961 651 752 960 000 000 000} +$$

$$\frac{z^{48}}{12 413 915 592 536 072 670 862 289 047 373 375 038 521 486 354 677 760 000 000 000} -$$

$$\frac{z^{50}}{30 414 093 201 713 378 043 612 608 166 064 768 844 377 641 568 960 512 000 000 000 000} +$$

$$\frac{z^{52}}{80 658 175 170 943 878 571 660 636 856 403 766 975 289 505 440 883 277 824 000 000 000 000} -$$

$$\frac{z^{54}}{230 843 697 339 241 380 472 092 742 683 027 581 083 278 564 571 807 941 132 288 000 000 000 000} +$$

$$z^{56} / 710 998 587 804 863 451 854 045 647 463 724 949 736 497 978 881 168 458 687 447 040 000 000 000 000 -$$

$$z^{58} / 2 350 561 331 282 878 571 829 474 910 515 074 683 828 862 318 181 142 924 420 699 914 240 000 000 000 000 +$$

$$z^{60} / 8 320 987 112 741 390 144 276 341 183 223 364 380 754 172 606 361 245 952 449 277 696 409 600 000 000 000 000 -$$

$$z^{62} / 31 469 973 260 387 937 525 653 122 354 950 764 088 012 280 797 258 232 192 163 168 247 821 107 200 000 000 000 000 +$$

$$z^{64} / 126 886 932 185 884 164 103 433 389 335 161 480 802 865 516 174 545 192 198 801 894 375 214 704 230 400 000 000 000 000 -$$

$$z^{66} / 544 344 939 077 443 064 003 729 240 247 842 752 644 293 064 388 798 874 532 860 126 869 671 081 148 416 000 000 000 000 000 +$$

$$z^{68} / 2 480 035 542 436 830 599 600 990 418 569 171 581 047 399 201 355 367 672 371 710 738 018 221 445 712 183 296 000 000 000 000 000 -$$

$$z^{70} / 11 978 571 669 969 891 796 072 783 721 689 098 736 458 938 142 546 425 857 555 362 864 628 009 582 789 845 319 680 000 000 000 000 000 +$$

$$z^{72} / 61 234 458 376 886 086 861 524 070 385 274 672 740 778 091 784 697 328 983 823 014 963 978 384 987 221 689 274 204 160 000 000 000 000 000 -$$

$$z^{74} / 330 788 544 151 938 641 225 953 028 221 253 782 145 683 251 820 934 971 170 611 926 835 411 235 700 971 565 459 250 872 320 000 000 000 000 000 +$$

$$z^{76} / 1 885 494 701 666 050 254 987 932 260 861 146 558 230 394 535 379 329 335 672 487 982 961 844 043 495 537 923 117 729 972 224 000 000 000 000 000 000 -$$

$$z^{78} / 11 324 281 178 206 297 831 457 521 158 732 046 228 731 749 579 488 251 990 048 962 825 668 835 325$$

```
   234 200 766 245 086 213 177 344 000 000 000 000 000 000 + z^80 /
  71 569 457 046 263 802 294 811 533 723 186 532 165 584 657 342 365 752 577 109 445 058 227 039 255 \
   480 148 842 668 944 867 280 814 080 000 000 000 000 000 000 - z^82 /
  475 364 333 701 284 174 842 138 206 989 404 946 643 813 294 067 993 328 617 160 934 076 743 994 \
   734 899 148 613 007 131 808 479 167 119 360 000 000 000 000 000 000 + z^84 /
  3 314 240 134 565 353 266 999 387 579 130 131 288 000 666 286 242 049 487 118 846 032 383 059 131 \
   291 716 864 129 885 722 968 716 753 156 177 920 000 000 000 000 000 000 - z^86 /
  24 227 095 383 672 732 381 765 523 203 441 259 715 284 870 552 429 381 750 838 764 496 720 162 249 \
   742 450 276 789 464 634 901 319 465 571 660 595 200 000 000 000 000 000 000 + z^88 /
  185 482 642 257 398 439 114 796 845 645 546 284 380 220 968 949 399 346 684 421 580 986 889 562 \
   184 028 199 319 100 141 244 804 501 828 416 633 516 851 200 000 000 000 000 000 000 - z^90 /
  1 485 715 964 481 761 497 309 522 733 620 825 737 885 569 961 284 688 766 942 216 863 704 985 393 \
   094 065 876 545 992 131 370 884 059 645 617 234 469 978 112 000 000 000 000 000 000 000 + z^92 /
  12 438 414 054 641 307 255 475 324 325 873 553 077 577 991 715 875 414 356 840 239 582 938 137 710 \
   983 519 518 443 046 123 837 041 347 353 107 486 982 656 753 664 000 000 000 000 000 000 000 -
 z^94 /
  108 736 615 665 674 308 027 365 285 256 786 601 004 186 803 580 182 872 307 497 374 434 045 199 \
   869 417 927 630 229 109 214 583 415 458 560 865 651 202 385 340 530 688 000 000 000 000 000 000 \
   000 + z^96 /
  991 677 934 870 949 689 209 571 401 541 893 801 158 183 648 651 267 795 444 376 054 838 492 222 \
   809 091 499 987 689 476 037 000 748 982 075 094 738 965 754 305 639 874 560 000 000 000 000 000 \
   000 000 - z^98 /
  9 426 890 448 883 247 745 626 185 743 057 242 473 809 693 764 078 951 663 494 238 777 294 707 070 \
   023 223 798 882 976 159 207 729 119 823 605 850 588 608 460 429 412 647 567 360 000 000 000 000 \
   000 000 000 + z^100 /
  93 326 215 443 944 152 681 699 238 856 266 700 490 715 968 264 381 621 468 592 963 895 217 599 993 \
   229 915 608 941 463 976 156 518 286 253 697 920 827 223 758 251 185 210 916 864 000 000 000 000 \
   000 000 000 000 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of the series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of $\cos(z)$ can be evaluated.

In[2]:=
```
<< DiscreteMath`RSolve`
```

In[3]:=
```
SeriesTerm[Cos[z], {z, 0, n}] z^n
```

Out[3]=
$$\frac{i^n z^n \, \text{KroneckerDelta}[\text{Mod}[n, 2]]}{\text{Gamma}[1 + n]}$$

This result can be checked by the following process.

In[4]:=
$$\text{FunctionExpand}\left[\sum_{n=0}^{\infty} \text{Evaluate}[\%]\right]$$

Out[4]=
```
Cos[z]
```

Differentiation

Mathematica can evaluate derivatives of the cosine function of an arbitrary positive integer order.

$\partial_z \mathrm{Cos}[z]$

$-\mathrm{Sin}[z]$

$\partial_{\{z,2\}} \mathrm{Cos}[z]$

$-\mathrm{Cos}[z]$

```
Table[D[Cos[z], {z, n}], {n, 10}]
```

```
{-Sin[z], -Cos[z], Sin[z], Cos[z], -Sin[z], -Cos[z], Sin[z], Cos[z], -Sin[z], -Cos[z]}
```

Finite summation

Mathematica can calculate finite symbolic sums that contain the cosine function. Here are some examples.

$$\sum_{k=1}^{n} \mathrm{Cos}[a\,k]$$

$$-1 + \mathrm{Cos}\left[\frac{a\,n}{2}\right] \mathrm{Csc}\left[\frac{a}{2}\right] \mathrm{Sin}\left[\frac{a}{2} + \frac{a\,n}{2}\right]$$

$$\sum_{k=1}^{n} (-1)^k \mathrm{Cos}[a\,k]$$

$$-1 + \mathrm{Cos}\left[\frac{a\,n}{2} + \frac{n\,\pi}{2}\right] \mathrm{Cos}\left[\frac{a}{2} + \frac{a\,n}{2} + \frac{n\,\pi}{2}\right] \mathrm{Sec}\left[\frac{a}{2}\right]$$

Infinite summation

Mathematica can calculate infinite sums including the cosine function. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \mathrm{Cos}[k\,x]$$

$$-\frac{z\left(1 + e^{2\,i\,x} - 2\,e^{i\,x}\,z\right)}{2\left(e^{i\,x} - z\right)\left(-1 + e^{i\,x}\,z\right)}$$

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cos}[k\,x]}{k\,!}$$

$$\frac{1}{2}\left(-2 + e^{e^{-i\,x}} + e^{e^{i\,x}}\right)$$

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cos}[k\,x]}{k}$$

$$\frac{1}{2}\left(-\mathrm{Log}\left[1 - e^{-i\,x}\right] - \mathrm{Log}\left[1 - e^{i\,x}\right]\right)$$

Finite products

Mathematica can calculate some finite symbolic products that contain the cosine function. Here is an example.

$$\prod_{k=1}^{n-1} \mathrm{Cos}\left[z + \frac{\pi k}{n}\right]$$

$$-(-1)^n\, 2^{1-n}\, \mathrm{Sec}[z]\, \mathrm{Sin}\left[\frac{1}{2} n (\pi - 2 z)\right]$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain the cosine function. Here are some examples.

$$\int \mathrm{Cos}[z]\, dz$$

$$\mathrm{Sin}[z]$$

$$\int \mathrm{Cos}[z]^a\, dz$$

$$-\frac{\mathrm{Cos}[z]^{1+a}\, \mathrm{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \mathrm{Cos}[z]^2\right] \mathrm{Sin}[z]}{(1+a)\sqrt{\mathrm{Sin}[z]^2}}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the cosine function. Here are some examples.

$$\int_0^{\pi/2} \sqrt{\mathrm{Cos}[z]}\, dz$$

$$2\, \mathrm{EllipticE}\left[\frac{\pi}{4}, 2\right]$$

$$\int_0^{\frac{\pi}{2}} \mathrm{Cos}[z]^a\, dz$$

$$\frac{\sqrt{\pi}\, \mathrm{Gamma}\left[\frac{1+a}{2}\right]}{a\, \mathrm{Gamma}\left[\frac{a}{2}\right]}$$

Limit operation

Mathematica can calculate limits that contain the cosine function. Here are some examples.

$$\mathrm{Limit}\left[\frac{\mathrm{Cos}[2 z] - 1}{z^2}, z \to 0\right]$$

$$-2$$

$$\text{Limit}\left[\frac{\text{Cos}\left[\sqrt{z^2}\right]-1}{z^2},\ z\to 0\right]$$

$$-\frac{1}{2}$$

In[573]:= $$\text{Limit}\left[\frac{\text{Cos}\left[\left(z^2\right)^{1/4}\right]-1}{z},\ z\to 0,\ \text{Direction}\to 1\right]$$

Out[573]= $$\frac{1}{2}$$

In[574]:= $$\text{Limit}\left[\frac{\text{Cos}\left[\left(z^2\right)^{1/4}\right]-1}{z},\ z\to 0,\ \text{Direction}\to -1\right]$$

Out[574]= $$-\frac{1}{2}$$

Solving equations

The next inputs solve two equations that contain the cosine function. Because of the multivalued nature of the inverse cosine function, a printed message indicates that only some of the possible solutions are returned.

$$\text{Solve}\left[\text{Cos}[z]^2+4\,\text{Cos}\left[z-\text{Pi}/6\right]==4,\ z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$$\Bigg\{\Bigg\{z\to\text{ArcCos}\Bigg[-\sqrt{3}-\frac{1}{2}\sqrt{\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}\,2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}}-$$
$$\frac{1}{2}\sqrt{\left(\frac{40}{3}-\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}-\frac{2}{3}\,2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}-\frac{16}{\sqrt{\frac{1}{3}\left(\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}\,2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}\right)}}\right)}\Bigg]\Bigg\},$$

$$\Bigg\{z\to\text{ArcCos}\Bigg[-\sqrt{3}-\frac{1}{2}\sqrt{\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}\,2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}}+$$
$$\frac{1}{2}\sqrt{\left(\frac{40}{3}-\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}-\frac{2}{3}\,2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}-\right.}$$

$$\left.\left.\frac{16}{\sqrt{\frac{1}{3}\left(\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}\right)}}\right)\right]\Bigg\},$$

$$\Bigg\{z \to \text{ArcCos}\Bigg[-\sqrt{3}+\frac{1}{2}\sqrt{\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}} - \frac{1}{2}\sqrt{\left(\frac{40}{3}-\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}-\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}+\frac{16}{\sqrt{\frac{1}{3}\left(\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}\right)}}\right)}\Bigg]\Bigg\},$$

$$\Bigg\{z \to -\text{ArcCos}\Bigg[-\sqrt{3}+\frac{1}{2}\sqrt{\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}} + \frac{1}{2}\sqrt{\left(\frac{40}{3}-\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}-\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}+\frac{16}{\sqrt{\frac{1}{3}\left(\frac{20}{3}+\frac{1}{3}\left(22112-288\sqrt{85}\right)^{1/3}+\frac{2}{3}2^{2/3}\left(691+9\sqrt{85}\right)^{1/3}\right)}}\right)}\Bigg]\Bigg\}\Bigg\}$$

```
Solve[Cos[x] == a, x]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{x → -ArcCos[a]}, {x → ArcCos[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Cos[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear independent solutions include the cosine function. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented using $\sin(z)$ and $\cos(z)$.

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}

In[9]:= `dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]`

Out[9]= $\{\{w[z] \to C[3]\,\mathrm{Cos}[z] + C[1]\,\mathrm{Cos}[\sqrt{2}\,z] + C[4]\,\mathrm{Sin}[z] + C[2]\,\mathrm{Sin}[\sqrt{2}\,z]\}\}$

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ-calculus).

In[10]:= `dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]`

Out[10]= $\{\{w \to \mathrm{Function}[\{z\}, C[3]\,\mathrm{Cos}[z] + C[1]\,\mathrm{Cos}[\sqrt{2}\,z] + C[4]\,\mathrm{Sin}[z] + C[2]\,\mathrm{Sin}[\sqrt{2}\,z]]\}\}$

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

In[11]:= `w'[ζ] /. dsol1`

Out[11]= {w'[ζ]}

In[12]:= `w'[ζ] /. dsol2`

Out[12]= $\{C[4]\,\mathrm{Cos}[\zeta] + \sqrt{2}\,C[2]\,\mathrm{Cos}[\sqrt{2}\,\zeta] - C[3]\,\mathrm{Sin}[\zeta] - \sqrt{2}\,C[1]\,\mathrm{Sin}[\sqrt{2}\,\zeta]\}$

In carrying out the algorithm to solve the following nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

In[13]:= $\mathrm{DSolve}[\{w'[z] == \sqrt{1 - w[z]^2},\ w[0] == 1\},\ w[z],\ z]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[13]= {{w[z] → Cos[z]}}

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions.

```
LaplaceTransform[Cos[t], t, s]
```

$$\frac{s}{1 + s^2}$$

```
FourierTransform[Cos[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \mathrm{DiracDelta}[-1 + s] + \sqrt{\frac{\pi}{2}}\ \mathrm{DiracDelta}[1 + s]$$

```
FourierSinTransform[Cos[t], t, s]
```

$$\frac{1}{\sqrt{2\pi}\,(-1+s)}+\frac{1}{\sqrt{2\pi}\,(1+s)}$$

```
FourierCosTransform[Cos[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s]+\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

```
ZTransform[Cos[π t], t, s]
```

$$\frac{s}{1+s}$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

$$\text{Plot}\left[\text{Cos}\left[\sum_{k=0}^{5} z^k\right],\ \left\{z,\ -\frac{2\pi}{3},\ \frac{2\pi}{3}\right\}\right];$$

```
Plot3D[Re[Cos[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\text{ContourPlot}\left[\text{Arg}\left[\text{Cos}\left[\frac{1}{x+i\,y}\right]\right],\ \left\{x,\ -\frac{1}{2},\ \frac{1}{2}\right\},\ \left\{y,\ -\frac{1}{2},\ \frac{1}{2}\right\},\right.$$

```
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

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