Introductions to EllipticE2

Introduction to the incomplete elliptic integrals

General

Elliptic integrals were encountered in the work of J. Wallis (1655–1659) who investigated the integral in modern notation:

$$E(z) = \int_0^{\pi/2} \sqrt{1 - m \sin^2(t)}\, dt\, /;\, 0 < m < 1.$$

Broader interest in such integrals was stimulated by the appearance and development of integral calculus in the 18th and 19th centuries. Mathematicians found that integrals, containing quadratic polynomials under a square root of the form:

$$\int \left(a\, t^2 + b\, t + c\right)^{\pm\frac{1}{2}} dt$$

can be evaluated through elementary functions by formulas such as:

$$\int \left(a\, t^2 + b\, t + c\right)^{\pm\frac{1}{2}} dt =$$

$$\frac{(1 \pm 1)(b + 2\, a\, t)\sqrt{a\, t^2 + b\, t + c}}{8\, a} - \frac{1}{\sqrt{a}} \left(\frac{\left(b^2 - 4\, a\, c\right)}{16\, a} (1 \pm 1) + \frac{(1 \mp 1)}{2}\right) \log\left(\frac{b + 2\, a\, t}{\sqrt{a}} + 2\sqrt{a\, t^2 + b\, t + c}\right).$$

But these early mathematicians could not find simple formulas for similar integrals containing higher-degree polynomials:

$$\int \left(a\, t^3 + b\, t^2 + c\, t + d\right)^{\pm\frac{1}{2}} dt$$

$$\int \left(a\, t^4 + b\, t^3 + c\, t^2 + d\, t + e\right)^{\pm\frac{1}{2}} dt.$$

Many important applications of these integrals were found at that time. The problem of evaluating such integrals was converted into the problem of evaluating only three basic integrals. They were later denoted by their special notation and named the incomplete elliptic integrals of the first, second, and third kinds—$F(z \mid m)$, $E(z \mid m)$, and $\Pi(n; z \mid m)$ (A. M. Legendre):

$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m \sin^2(t)}}\, dt\, /;\, 0 < m < 1$$

$$E(z \mid m) = \int_0^z \sqrt{1 - m \sin^2(t)}\, dt\, /;\, 0 < m < 1$$

$$\Pi(n; z \mid m) = \int_0^z \frac{1}{\left(1 - n\sin^2(t)\right)\sqrt{1 - m\sin^2(t)}}\, dt.$$

The corresponding definite integrals (for $z = \frac{\pi}{2}$) were named the complete elliptic integrals of the first, second, and third kinds denoted by the symbols $K(z)$, $E(z)$, and $\Pi(n \mid m)$:

$$K(z) = F\left(\frac{\pi}{2} \,\middle|\, z\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m\sin^2(t)}}\, dt$$

$$E(z) = E\left(\frac{\pi}{2} \,\middle|\, z\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m\sin^2(t)}\, dt$$

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \,\middle|\, m\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\left(1 - n\sin^2(t)\right)\sqrt{1 - m\sin^2(t)}}\, dt.$$

These integrals were extensively studied for another important reason—development of the theory of the double periodic functions. These functions were called elliptic functions. The elliptic integrals and elliptic functions were studied simultaneously on several occasions throughout history and a deep connection exists between these two areas of mathematics. The following chronology reflects the main steps in building the theory of elliptic integrals.

L. Euler (1733, 1757, 1763, 1766) derived the addition theorem for the incomplete elliptic integrals $F(z \mid m)$, and $E(z \mid m)$.

J.-L. Lagrange (1783) and especially A. M. Legendre (1793, 1811, 1825–1828) devoted a lot of attention to the study of the different properties of those two elliptic integrals. C. F. Gauss (1799, 1818) also used these integrals during his research.

Simultaneously, A. M. Legendre (1811) introduced the incomplete elliptic integral of the third kind and the complete versions of all three elliptic integrals.

C. G. J. Jacobi (1827–1829) introduced inverse functons of the elliptic integrals $F(z \mid m)$ and $E(z \mid m)$, which led him to build the theory of elliptic functions. In 1829, C. G. J. Jacobi defined the following function:

$$Z(z \mid m) = E(z \mid m) - \frac{E(m)}{K(m)} F(z \mid m),$$

which was later called the Jacobi zeta function. J. Liouville (1840) also studied elliptic integrals $F(z \mid m)$ and $E(z \mid m)$.

N. H. Abel, independently from C. G. J. Jacobi, got some of his results and studied the so-called hyperelliptic and Abelian integrals.

Definitions of incomplete elliptic integrals

The incomplete elliptic integral of the first kind $F(z \mid m)$, incomplete elliptic integral of the second kind $E(z \mid m)$, incomplete elliptic integral of the third kind $\Pi(n; z \mid m)$, and Jacobi zeta function $Z(z \mid m)$ are defined by the following formulas:

$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m \sin^2(t)}} \, dt$$

$$E(z \mid m) = \int_0^z \sqrt{1 - m \sin^2(t)} \, dt$$

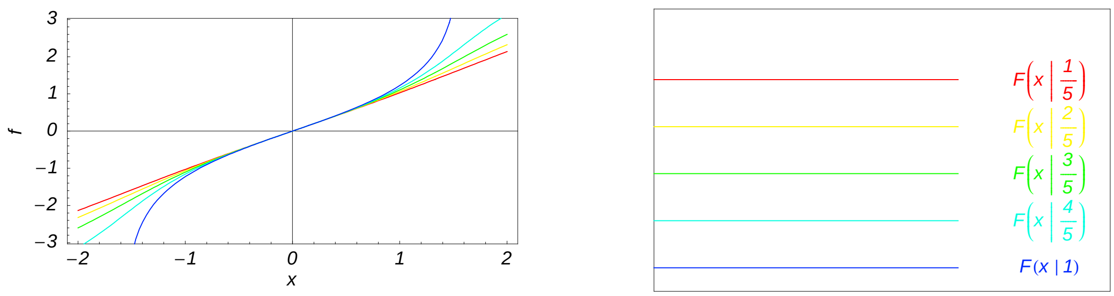
$$\Pi(n; z \mid m) = \int_0^z \frac{1}{\left(1 - n \sin^2(t)\right) \sqrt{1 - m \sin^2(t)}} \, dt$$

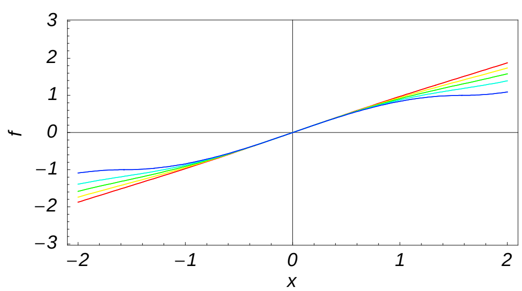
$$Z(z \mid m) = E(z \mid m) - \frac{E(m)}{K(m)} F(z \mid m).$$

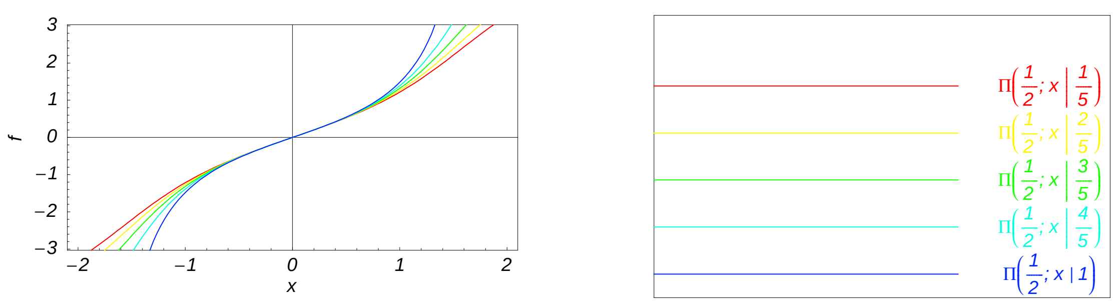
The previous functions are called incomplete elliptic integrals.

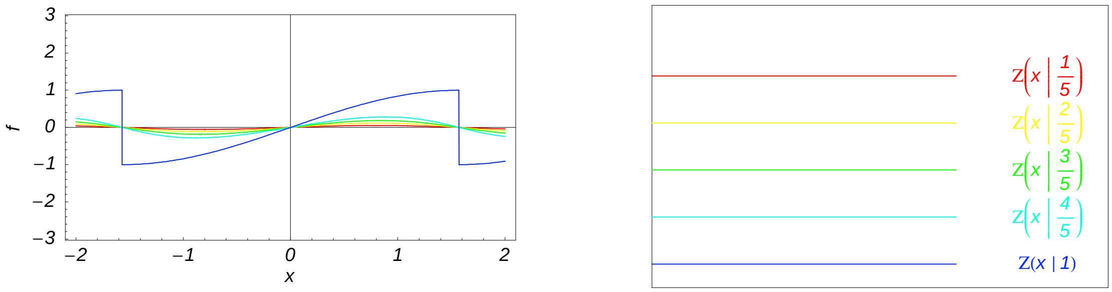
A quick look at the incomplete elliptic integrals

Here is a quick look at the graphics for the incomplete elliptic integrals along the real axis.









Connections within the group of incomplete elliptic integrals and with other function groups

Representations through more general functions

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $\mathrm{Z}(z \mid m)$ can be represented through more general functions. Through the hypergeometric Appell F_1 function of two variables:

$$F(z \mid m) = \sin(z)\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) /;\ |\mathrm{Re}(z)| < \frac{\pi}{2}$$

$$E(z \mid m) = \sin(z)\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) /;\ |\mathrm{Re}(z)| < \frac{\pi}{2}$$

$$\Pi(n; z \mid m) = \sum_{k=0}^{\infty} \frac{\sin^{2k+1}(z)}{2k+1}\, F_1\!\left(k + \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k + \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) n^k$$

$$\Pi(n; z \mid m) = \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{(2k+1)\, k!}\, F_1\!\left(k + \frac{1}{2}; \frac{1}{2}, 1; k + \frac{3}{2}; \sin^2(z), n \sin^2(z)\right) m^k$$

$$\mathrm{Z}(z \mid m) = \sin(z) \left(F_1\!\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) - \frac{E(m)}{K(m)}\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right)\right) /;\ |\mathrm{Re}(z)| < \frac{\pi}{2}.$$

Through the generalized hypergeometric function of two variables:

$$F(z \mid m) = \sin(z)\, F_{1,0,0}^{1,1,1}\!\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\ m \sin^2(z), \sin^2(z)\right)$$

$$E(z \mid m) = \sin(z)\, F_{1,0,0}^{1,1,1}\!\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; -\frac{1}{2}; \\ \frac{3}{2};;; \end{matrix}\ \sin^2(z), m \sin^2(z)\right)$$

$$\Pi(n; z \mid m) = \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{(2k+1)\, k!}\, F_{1,0,0}^{1,1,1}\!\left(\begin{matrix} k + \frac{1}{2}; \frac{1}{2}; 1; \\ k + \frac{3}{2};;; \end{matrix}\ \sin^2(z), n \sin^2(z)\right) m^k$$

$$\Pi(n; z \mid m) = \sin(z)\, F_D^{(3)}\!\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; \frac{3}{2}; m \sin^2(z), \sin^2(z), n \sin^2(z)\right)$$

$$\mathrm{Z}(z \mid m) = \sin(z)\, F_{1,0,0}^{1,1,1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; -\frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \sin^2(z),\, m\sin^2(z)\right) - \frac{E(m)}{K(m)} \sin(z)\, F_{1,0,0}^{1,1,1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} m\sin^2(z),\, \sin^2(z)\right).$$

Through elliptic theta functions, for example:

$$E(z \mid m) = \frac{E(m)}{K(m)} F(z \mid m) + \frac{\pi}{2\,K(m)\,\vartheta_4\left(\frac{\pi F(z|m)}{2\,K(m)}, q(m)\right)} \vartheta_4'\left(\frac{\pi F(z \mid m)}{2\,K(m)}, q(m)\right)$$

$$\mathrm{Z}(z \mid m) = \sqrt{1 - m\sin^2(z)}\ \frac{\partial \log(\vartheta_4(\pi F(z \mid m)/(2\,K(m)),\, q(m)))}{\partial z}.$$

Through inverse Jacobi elliptic functions, for example:

$$F\left(\sin^{-1}(z) \,\middle|\, m\right) = \mathrm{sn}^{-1}(z \mid m) /; -1 < z < 1 \land m < 1$$

$$F\left(\cos^{-1}(z) \,\middle|\, m\right) = \mathrm{cn}^{-1}(z \mid m) /; -1 < z < 1 \land m \in \mathbb{R}$$

$$F\left(i \sinh^{-1}(z) \,\middle|\, m\right) = i\, \mathrm{sc}^{-1}(z \mid 1-m) /; m > 0 \land m \in \mathbb{R}.$$

Through Weierstrass elliptic functions and inverse elliptic nome $q^{-1}(t)$, for example:

$$E(z \mid m) = \frac{1}{K(m)}\left(E(m)\,F(z \mid m) + \eta_3\,\omega_1 + \omega_1\,\zeta\left(\frac{\omega_1\,F(z \mid m)}{K(m)} - \omega_3; g_2, g_3\right)\right) - \frac{\omega_1\ \eta_1}{K(m)^2} F(z \mid m) /;$$
$$m = q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \{\eta_1, \eta_3\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\}$$

$$\mathrm{Z}(z \mid m) = \frac{\omega_1}{K(m)^2}\left(K(m)\left(\eta_3 + \zeta\left(\frac{\omega_1\,F(z \mid m)}{K(m)} - \omega_3; g_2, g_3\right)\right) - F(z \mid m)\,\eta_1\right) /;$$
$$m = q^{-1}\left(\exp\left(\frac{i\,\pi\,\omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \{\eta_1, \eta_3\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\}.$$

Through some elliptic-type functions, for example:

$$F\left(\cot^{-1}\left(\sqrt{\frac{2\,z_1}{a + \sqrt{a^2 - 4\,b}}}\right) \,\middle|\, \frac{2\sqrt{a^2 - 4\,b}}{a + \sqrt{a^2 - 4\,b}}\right) = -2\,\frac{1}{\sqrt{\frac{2\left(a - \sqrt{a^2 - 4\,b}\right)}{b}}}\, \mathrm{elog}\left(z_1, \sqrt{z_1^3 + a\,z_1^2 + b\,z_1}\,; a, b\right).$$

Representations through related functions

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $\mathrm{Z}(z \mid m)$ can be represented through some related functions, for example:

$$F(\phi \mid m) = \frac{\left(\sqrt{1-n}\ \cot(\phi) + E(\phi \mid m)\right) K(m) - \sqrt{1-n}\ \cot(\phi)\,\Pi(n \mid m)}{E(m)} /; \phi = \sin^{-1}\left(\sqrt{\frac{n}{m}}\right) \bigwedge 0 < n < 1 \bigwedge 0 < m < 1$$

$$E(\phi \mid m) = \frac{E(m)\,F(\phi \mid m) + \sqrt{1-n}\ \cot(\phi)\,(\Pi(n \mid m) - K(m))}{K(m)} /; \phi = \sin^{-1}\left(\sqrt{\frac{n}{m}}\right) \bigwedge 0 < n < 1 \bigwedge 0 < m < 1$$

$$E(\operatorname{am}(z \mid m) \mid m) = Z(\operatorname{am}(z \mid m) \mid m) + \frac{z\, E(m)}{K(m)}$$

$$Z(\operatorname{am}(z \mid m) \mid m) = E(\operatorname{am}(z \mid m) \mid m) - \frac{z\, E(m)}{K(m)}$$

$$Z(\operatorname{am}(z \mid m) \mid m) = (1 - m)\, \Pi(m; \operatorname{am}(z \mid m) \mid m) - \frac{z\, E(m)}{K(m)} + \frac{m \sin(2 \operatorname{am}(z \mid m))}{2 \sqrt{1 - m \sin^2(\operatorname{am}(z \mid m))}}.$$

$$E(\operatorname{am}(z \mid m) \mid m) = \int_0^z \operatorname{dn}(t \mid m)^2 \, dt$$

$$E\left(\sin^{-1}(\operatorname{sn}(z \mid m)) \,\middle|\, m\right) = \int_0^z \operatorname{dn}(t \mid m)^2 \, dt.$$

Relations to inverse functions

The incomplete elliptic integral $F(z \mid m)$ is related to the Jacobi amplitude by the following formulas, which demonstrate that the Jacobi amplitude is within a restricted domain, the inverse function of elliptic integral $F(z \mid m)$:

$$F(\operatorname{am}(z \mid m) \mid m) = z \;/;\; m \leq 1 \land -2 \leq z \leq 2$$

$$\operatorname{am}(F(z \mid m) \mid m) = z \;/;\; m < 1 \bigvee -\frac{3}{2} \leq z \leq \frac{3}{2}.$$

Representations through other incomplete elliptic integrals

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, and $Z(z \mid m)$ can be represented through incomplete elliptic integral $\Pi(n; z \mid m)$ by the following formulas:

$$F(z \mid m) = \Pi(0; z \mid m)$$

$$E(z \mid m) = (1 - m)\, \Pi(m; z \mid m) + \frac{m \sin(2z)}{2 \sqrt{1 - m \sin^2(z)}}$$

$$Z(z \mid m) = -\frac{E(m)\, \Pi(0; z \mid m)}{K(m)} + (1 - m)\, \Pi(m; z \mid m) + \frac{m \sin(2z)}{2 \sqrt{1 - m \sin^2(z)}}.$$

The best-known properties and formulas for incomplete elliptic integrals

Simple values at zero

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ are equal to 0 at the origin points:

$$F(0 \mid 0) = 0 \quad E(0 \mid 0) = 0 \quad \Pi(0; 0 \mid 0) = 0 \quad Z(0 \mid 0) = 0.$$

Specific values

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ for particular argument values can be evaluated in closed forms, for example:

$$F(z\mid 0) == z \quad F(z\mid 1) == \tanh^{-1}(\sin(z)) \text{ /; } |\operatorname{Re} z| < \tfrac{\pi}{2}$$

$$F(0\mid m) == 0 \quad F\left(\tfrac{\pi}{2} \,\middle|\, m\right) == K(m) \quad F\left(\tfrac{k\pi}{2} \,\middle|\, m\right) == k\,K(m) \text{ /; } k \in \mathbb{Z}$$

$$E(z\mid 0) == z \quad E(z\mid 1) == \sin(z) \text{ /; } |\operatorname{Re}(z)| \leq \tfrac{\pi}{2}$$

$$E(0\mid m) == 0 \quad E\left(\tfrac{\pi}{2} \,\middle|\, m\right) == E(m) \quad E\left(\tfrac{k\pi}{2} \,\middle|\, m\right) == k\,E(m) \text{ /; } k \in \mathbb{Z}$$

$$Z(z\mid 0) == 0 \quad Z(z\mid 1) == \sin(z) \text{ /; } |\operatorname{Re}(z)| \leq \tfrac{\pi}{2}$$

$$Z(0\mid m) == 0 \quad Z\left(\tfrac{\pi}{2} \,\middle|\, m\right) == 0 \quad Z\left(\tfrac{k\pi}{2} \,\middle|\, m\right) == 0 \text{ /; } k \in \mathbb{Z}$$

$$\Pi(n; z\mid 0) == \frac{\tanh^{-1}\left(\sqrt{n-1}\,\tan(z)\right)}{\sqrt{n-1}}$$

$$\Pi(n; z\mid 1) == \frac{\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right) - \tanh^{-1}(\sin(z))}{n-1} \text{ /; } |\operatorname{Re}(z)| < \frac{\pi}{2}$$

$$\Pi(n; z\mid n) == \frac{1}{1-n}\left(E(z\mid n) - \frac{n\sin(2z)}{2\sqrt{1-n\sin^2(z)}}\right)$$

$$\Pi\left(n; \frac{\pi}{2} \,\middle|\, m\right) == \Pi(n\mid m)$$

$$\Pi\left(n; \frac{k\pi}{2} \,\middle|\, m\right) == k\,\Pi(n\mid m) \text{ /; } k \in \mathbb{Z}$$

$$\Pi\left(n; \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) \,\middle|\, m\right) == \frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)$$

$$\Pi(0; z\mid m) == F(z\mid m)$$

$$\Pi(1; z\mid m) == \frac{\sqrt{1-m\sin^2(z)}\,\tan(z) - E(z\mid m)}{1-m} + F(z\mid m).$$

At infinities, the incomplete elliptic integrals $F(z\mid m)$, $E(z\mid m)$, $\Pi(n; z\mid m)$, and $Z(z\mid m)$ have the following values:

$$F(z\mid\infty) == 0 \quad F(z\mid -\infty) == 0 \quad F(i\infty\mid m) == K(m) - \tfrac{1}{\sqrt{m}}\,K\left(\tfrac{1}{m}\right) \text{ /; } 0 < m < 1 \quad F(-i\infty\mid m) == \tfrac{1}{\sqrt{m}}\,K\left(\tfrac{1}{m}\right) - K(m) \text{ /; } 0 < m < 1$$

$$E(z\mid\infty) == \tilde{\infty} \quad E(z\mid -\infty) == \tilde{\infty}$$

$$\Pi(\infty\mid m) == 0 \quad \Pi(-\infty\mid m) == 0 \quad \Pi(n\mid\infty) == 0 \quad \Pi(n\mid -\infty) == 0$$

$$\Pi(\infty; z\mid m) == 0 \quad \Pi(-\infty; z\mid m) == 0 \quad \Pi(n; z\mid\infty) == 0 \quad \Pi(n; z\mid -\infty) == 0$$

$$Z(z\mid\infty) == \tilde{\infty} \quad Z(z\mid -\infty) == \tilde{\infty}\,.$$

Analyticity

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, and $Z(z \mid m)$ are analytical functions of z and m, which are defined over $\mathbb{C}^2$.

The incomplete elliptic integral $\Pi(n; z \mid m)$ is an analytical function of n, z, and m, which is defined over $\mathbb{C}^3$.

Poles and essential singularities

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ do not have poles and essential singularities with respect to their variables.

Branch points and branch cuts

For fixed m, the functions $F(z \mid m)$, $E(z \mid m)$, and $Z(z \mid m)$ have an infinite number of branch points at $z = \pm \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2 \pi k /; k \in \mathbb{Z}$ and $z = \tilde{\infty}$.

They have complicated branch cut.

For fixed n, z, the function $\Pi(n; z \mid m)$ has two branch points at $m = \csc^2(z)$ and $m = \tilde{\infty}$. For fixed n, m, the function $\Pi(n; z \mid m)$ has an infinite number of branch points at $z = \pm \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2 \pi k /; k \in \mathbb{Z}$, $z = \pm \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + 2 \pi k /; k \in \mathbb{Z}$ and $z = \tilde{\infty}$. For fixed z, m, the function $\Pi(n; z \mid m)$ does not have branch points.

Periodicity

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, and $\Pi(n; z \mid m)$ are quasi-periodic functions with respect to z:

$$F(z + \pi k \mid m) = F(z \mid m) + 2 k K(m) /; k \in \mathbb{Z}$$

$$E(z + \pi k \mid m) = E(z \mid m) + 2 k E(m) /; k \in \mathbb{Z}$$

$$\Pi(n; z + \pi k \mid m) = \Pi(n; z \mid m) + 2 k \Pi(n \mid m) /; k \in \mathbb{Z} \land -1 \leq n \leq 1.$$

The incomplete elliptic integral $Z(z \mid m)$ is a periodic function with respect to z with period π:

$$Z(z + \pi \mid m) = Z(z \mid m)$$

$$Z(z + \pi k \mid m) = Z(z \mid m) /; k \in \mathbb{Z}.$$

Parity and symmetry

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ have mirror symmetry:

$$F(\bar{z} \mid \bar{m}) = \overline{F(z \mid m)} \quad E(\bar{z} \mid \bar{m}) = \overline{E(z \mid m)} \quad \Pi(\bar{n}; \bar{z} \mid \bar{m}) = \overline{\Pi(n; z \mid m)} \quad Z(\bar{z} \mid \bar{m}) = \overline{Z(z \mid m)}.$$

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ are odd functions with respect to z.

$$F(-z \mid m) = -F(z \mid m) \quad E(-z \mid m) = -E(z \mid m) \quad \Pi(n; -z \mid m) = -\Pi(n; z \mid m) \quad Z(-z \mid m) = -Z(z \mid m).$$

Series representations

The incomplete elliptic integral $\Pi(n; z \mid m)$ has the following series expansions at the point $n = 0$:

$$\Pi(n;z\mid m)\propto \sin(z)\,F_1\left(\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{3}{2};\sin^2(z),m\sin^2(z)\right)+$$
$$\frac{\sin^3(z)}{3}\,F_1\left(\frac{3}{2};\frac{1}{2},\frac{1}{2};\frac{5}{2};\sin^2(z),m\sin^2(z)\right)n+\frac{\sin^5(z)}{5}\,F_1\left(\frac{5}{2};\frac{1}{2},\frac{1}{2};\frac{7}{2};\sin^2(z),m\sin^2(z)\right)n^2+\ldots\;/;\;(n\to 0)$$

$$\Pi(n;z\mid m)=\sum_{k=0}^{\infty}\frac{\sin^{2k+1}(z)}{2k+1}\,F_1\left(k+\frac{1}{2};\frac{1}{2},\frac{1}{2};k+\frac{3}{2};\sin^2(z),m\sin^2(z)\right)n^k.$$

The incomplete elliptic integrals $F(z\mid m)$, $E(z\mid m)$, $\Pi(n;z\mid m)$, and $Z(z\mid m)$ have the following series expansions at the point $z = 0$:

$$F(z\mid m)\propto z+\frac{m z^3}{6}+\frac{m(-4+9m)z^5}{120}+\ldots\;/;\;(z\to 0)$$

$$F(z\mid m)=z+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{\left(\frac{1}{2}\right)_k\binom{2k}{j}(-1)^{i-j+k}\,2^{2i-2k+1}\,(j-k)^{2i}\,m^k}{k!\,(2i+1)!}\,z^{2i+1}\;/;\;|z|<1$$

$$E(z\mid m)\propto z-\frac{m z^3}{6}-\frac{m(3m-4)}{120}\,z^5+\ldots\;/;\;(z\to 0)$$

$$E(z\mid m)=z+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{\left(-\frac{1}{2}\right)_k\binom{2k}{j}(-1)^{i-j+k}\,2^{2i-2k+1}\,(j-k)^{2i}\,m^k}{k!\,(2i+1)!}\,z^{2i+1}\;/;\;|z|<1$$

$$\Pi(n;z\mid m)\propto z+\frac{m+2n}{6}\,z^3+\frac{1}{120}\left(9m^2+4(m+2n)(3n-1)\right)z^5+\ldots\;/;\;(z\to 0)$$

$$\Pi(n;z\mid m)\propto -\frac{2^{-\frac{5}{2}}\sqrt{\pi}\;n}{\left(\frac{m-2\left(\sqrt{1-m}+1\right)}{m}\right)^{3/2}\sqrt{1-n}}\sqrt{1+\frac{1}{\sqrt{1-m}}}$$
$$\sum_{q=0}^{\infty}\frac{(-1)^q\,2^{2q+1}}{(2q+1)!}\sum_{k=0}^{2q}\mathcal{S}_{2q}^{(k)}\sum_{j=0}^{k}\frac{2^{-k}(-1)^j\,j!\,n^j}{\Gamma\left(j-k+\frac{1}{2}\right)}\binom{k}{k-j}\left(\frac{m}{1-\sqrt{1-m}}\right)^{k-j}\left(\left(\sqrt{1-n}+1\right)^{-j-1}-\left(1-\sqrt{1-n}\right)^{-j-1}\right)$$
$$F_1\left(\frac{1}{2};\frac{1}{2},-\frac{3}{2};j-k+\frac{1}{2};\frac{1}{2}-\frac{1}{2\sqrt{1-m}},\frac{2\left(\sqrt{1-m}+1\right)}{m}\right)z^{2q+1}\;/;\;|z|<1$$

$$Z(z\mid m)\propto\left(1-\frac{E(m)}{K(m)}\right)z-\frac{m}{6}\left(\frac{E(m)}{K(m)}+1\right)z^3+\frac{m}{120}\left(4-3m-\frac{(9m-4)E(m)}{K(m)}\right)z^5+\ldots\;/;\;(z\to 0)$$

$$Z(z\mid m)=z-\frac{E(m)\,z}{K(m)}+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\binom{2k}{j}\frac{(-1)^{i-j+k}\,2^{2i-2k+1}\,(j-k)^{2i}\,m^k}{k!\,(2i+1)!}\left(\left(-\frac{1}{2}\right)_k-\frac{E(m)}{K(m)}\left(\frac{1}{2}\right)_k\right)z^{2i+1}\;/;\;|z|<1.$$

The incomplete elliptic integrals $F(z\mid m)$, $E(z\mid m)$, $\Pi(n;z\mid m)$, and $Z(z\mid m)$ have the following series expansions at the point $m = 0$:

$$F(z \mid m) \propto z + \frac{2z - \sin(2z)}{8} m + \frac{3}{256} (12z - 8\sin(2z) + \sin(4z)) m^2 + \ldots /; (m \to 0)$$

$$F(z \mid m) == \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!} \left(\frac{1}{2}\right)_k \left(z \binom{2k}{k} + \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{2k}{k-j} \sin(2jz)\right) m^k /; |m| < 1 \bigwedge -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$E(z \mid m) \propto z - \frac{2z - \sin(2z)}{8} m - \frac{1}{256} (12z - 8\sin(2z) + \sin(4z)) m^2 + \ldots /; (m \to 0)$$

$$E(z \mid m) == \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!} \left(-\frac{1}{2}\right)_k \left(z \binom{2k}{k} + \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{2k}{k-j} \sin(2jz)\right) m^k /; |m| < 1 \bigwedge -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$\Pi(n; z \mid m) \propto \frac{\tanh^{-1}\left(\sqrt{n-1} \tan(z)\right)}{\sqrt{n-1}} + \frac{\sin^3(z)}{6} F_1\left(\frac{3}{2}; \frac{1}{2}, 1; \frac{5}{2}; \sin^2(z), n\sin^2(z)\right) m + \frac{3\sin^5(z)}{40} F_1\left(\frac{5}{2}; \frac{1}{2}, 1; \frac{7}{2}; \sin^2(z), n\sin^2(z)\right) m^2 + \ldots /; (m \to 0)$$

$$\Pi(n; z \mid m) == \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{(2k+1)\, k!} F_1\left(k + \frac{1}{2}; \frac{1}{2}, 1; k + \frac{3}{2}; \sin^2(z), n\sin^2(z)\right) m^k /; |m| < 1 \bigwedge -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$Z(z \mid m) \propto \frac{1}{4} \sin(2z)\, m + \left(\frac{1}{16} \sin(2z) - \frac{1}{64} \sin(4z)\right) m^2 + \ldots /; (m \to 0).$$

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, and $\Pi(n; z \mid m)$ have the following series expansions at the point $m == 1$:

$$F(z \mid m) \propto \log(\sec(z) + \tan(z)) + \frac{1}{4} (\sec(z)\tan(z) - \log(\sec(z) + \tan(z))) (m-1) + \frac{3}{256} \left(-5\sin(3z)\sec^4(z) + 3\tan(z)\sec^3(z) + 12\log(\sec(z) + \tan(z))\right) (m-1)^2 + \ldots /; (m \to 1) \bigwedge |\mathrm{Re}\, z| < \frac{\pi}{2}$$

$$F(z \mid m) == \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{(2k+1)\, k!}\, {}_2F_1\left(k + \frac{1}{2}, k+1; k + \frac{3}{2}; \sin^2(z)\right) (m-1)^k$$

$$E(z \mid m) \propto \sqrt{\cos^2(z)} \tan(z) + \frac{1}{4} \left(\sec(z) \left(\sqrt{\cos^2(z)} (\cos(2z) + 3)\sec(z) - 2\right) \tan(z) - 2\tanh^{-1}(\sin(z))\right) (m-1) + \frac{1}{32} \left(3(\cos(2z) - 3)\tan(z)\sec^3(z) + \frac{2(\cos(2z) + 7)\tan^3(z)}{\sqrt{\cos^2(z)}} + 6\tanh^{-1}(\sin(z))\right) (m-1)^2 + \ldots /; (m \to 1) \bigwedge |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

$$E(z \mid m) == \sqrt{\cos^2(z)} \tan(z) + \sum_{j=0}^{\infty} \left(\frac{(4j + \cos(2z) + 3) \left(\frac{1}{2}\right)_j \tan^{2j+1}(z)}{4\sqrt{\cos^2(z)}\, (j+1)!} - \frac{\Gamma\left(j + \frac{1}{2}\right) \sin^{2j+1}(z)}{\sqrt{\pi}\; j!}\, {}_2F_1\left(j + \frac{1}{2}, j+2; j + \frac{3}{2}; \sin^2(z)\right)\right) (m-1)^{j+1}$$

$$\Pi(n; z \mid m) \propto \frac{\sqrt{n} \tanh^{-1}\left(\sqrt{n} \sin(z)\right) - \tanh^{-1}(\sin(z))}{n-1} - \frac{1}{4(n-1)^2}\left((n+1)\tanh^{-1}(\sin(z)) - 2\sqrt{n} \tanh^{-1}\left(\sqrt{n} \sin(z)\right) + (n-1)\sec(z)\tan(z)\right)(m-1) + \frac{3}{64(n-1)^3}\left(\frac{1}{2}(n-1)(-3n+(n-5)\cos(2z)-1)\tan(z)\sec^3(z) + \left(n^2-6n-3\right)\tanh^{-1}(\sin(z)) + 8\sqrt{n} \tanh^{-1}\left(\sqrt{n} \sin(z)\right)\right)(m-1)^2 + \ldots /; (m \to 1) \bigwedge |\text{Re}(z)| \leq \frac{\pi}{2}$$

$$\Pi(n; z \mid m) = \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{(2k+1)k!} F_1\left(k+\frac{1}{2}; k+1, 1; k+\frac{3}{2}; \sin^2(z), n\sin^2(z)\right)(m-1)^k.$$

Other series representations

The incomplete elliptic integrals $F\left(\sin^{-1}(z) \mid m\right)$ and $E\left(\sin^{-1}(z) \mid m\right)$ have the following series expansions at the point $z = 0$:

$$F\left(\sin^{-1}(z) \mid m\right) \propto z + \frac{m+1}{6} z^3 + \frac{3+2m+3m^2}{40} z^5 + \ldots /; (z \to 0)$$

$$F\left(\sin^{-1}(z) \mid m\right) = \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{m}\right) z^{2k+1} /; |z| < 1$$

$$E\left(\sin^{-1}(z) \mid m\right) \propto z + \frac{1-m}{6} z^3 + \frac{3-2m-m^2}{40} z^5 + \ldots /; (z \to 0)$$

$$E\left(\sin^{-1}(z) \mid m\right) = \sum_{k=0}^{\infty} \frac{m^k \left(-\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{m}\right) z^{2k+1} /; |z| < 1.$$

The incomplete elliptic integrals $F\left(\sin^{-1}(z) \mid m\right)$ and $E\left(\sin^{-1}(z) \mid m\right)$ have the following series expansions at the point $m = 0$:

$$F\left(\sin^{-1}(z) \mid m\right) \propto \sin^{-1}(z) - \frac{1}{4}\left(z\sqrt{1-z^2} - \sin^{-1}(z)\right)m - \frac{3}{64}\left(z\left(2z^2+3\right)\sqrt{1-z^2} - 3\sin^{-1}(z)\right)m^2 + \ldots /; (m \to 0)$$

$$F\left(\sin^{-1}(z) \mid m\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; z^2\right) m^k /; |m| < 1$$

$$E\left(\sin^{-1}(z) \mid m\right) \propto \sin^{-1}(z) + \frac{1}{4}\left(z\sqrt{1-z^2} - \sin^{-1}(z)\right)m + \frac{1}{64}\left(z\sqrt{1-z^2}\left(2z^2+3\right) - 3\sin^{-1}(z)\right)m^2 + \ldots /; (m \to 0)$$

$$E\left(\sin^{-1}(z) \mid m\right) = \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; z^2\right) m^k /; |m| < 1.$$

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $\mathrm{Z}(z \mid m)$ can be represented through different kinds of series, for example:

$$F(z \mid m) = \frac{\sin(z)}{\sqrt{\sin^2(z)}} \left(K(m) - \cos(z) \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{k!} \, {}_2F_1\!\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; \cos^2(z)\right) \right) /; |\cos(z)| < 1$$

$$F(z \mid m) = \frac{2z}{\pi} K(m) + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left(\frac{m}{4}\right)^k \sum_{j=0}^{k-1} \frac{(-1)^{k-j} \binom{2k}{j}}{j-k} \sin(2(j-k)z)$$

$$E(z \mid m) = E(m) - \frac{\pi}{2} + z - \cos(z) \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{\left(\frac{3}{2}\right)_j j!} \left({}_2F_1\!\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2} - j; m\right) - 1 \right) \cos^{2j}(z) /; |\cos(z)| < 1$$

$$E(z \mid m) = \frac{2z}{\pi} E(m) + \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k \left(\frac{m}{4}\right)^k}{k!} \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{j-k} \binom{2k}{j} \sin(2(j-k)z)$$

$$\Pi(n; z \mid m) = \sum_{k=0}^{\infty} n^k \left(\sqrt{1 + \frac{m}{n}} - \frac{\sqrt{\pi}}{2\,\Gamma\!\left(\frac{1}{2} - k\right)(k+1)!} \left(\frac{m}{n}\right)^{k+1} {}_2F_1\!\left(1, k + \frac{1}{2}; k+2; -\frac{m}{n}\right) \right)$$
$$\left(\frac{\sqrt{\pi}\, \Gamma\!\left(k + \frac{1}{2}\right)}{2\,k!} - \cos(z)\, {}_2F_1\!\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; \cos^2(z)\right) \right) /; 0 \leq m < 1 \bigwedge 0 \leq n < 1 \bigwedge 0 \leq z \leq \frac{\pi}{2}$$

$$\Pi(n; z \mid m) \propto \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{n^k \left(k + \frac{1}{2}\right)_{i+j} \left(\frac{1}{2}\right)_i \left(\frac{1}{2}\right)_j m^j \sin^{2i+2j+2k+1}(z)}{(2k+1) \left(k + \frac{3}{2}\right)_{i+j} i!\, j!}$$

$$\Pi(n; z \mid m) \propto \sum_{k=0}^{\infty} \frac{n^k \sin^{2k+1}(z)}{2k+1} F_1\!\left(k + \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k + \frac{3}{2}; \sin^2(z), m \sin^2(z)\right)$$

$$\Pi(n; z \mid m) \propto \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k m^k}{k!} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{j+k} n^j}{(j+k)!} \left(z - \frac{1}{2} \sin(2z) \sum_{i=0}^{j+k-1} \frac{i! \sin^{2i}(z)}{\left(\frac{3}{2}\right)_i} \right)$$

$$\mathrm{Z}(z \mid m) = \frac{2\pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^k}{1 - q(m)^{2k}} \sin\!\left(\frac{(k\pi) F(z \mid m)}{K(m)}\right),$$

where $q(m)$ is an elliptic nome and $K(m)$ is a complete elliptic integral.

Integral representations

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $\mathrm{Z}(z \mid m)$ have the following integral representations:

$$F(z \mid m) = \int_0^z \frac{1}{\sqrt{1 - m \sin^2(t)}} \, dt$$

$$F(z \mid m) = \int_0^{\sin(z)} \frac{1}{\sqrt{1-t^2}\sqrt{1-m\,t^2}}\,dt \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

$$E(z \mid m) = \int_0^{z} \sqrt{1-m\sin^2(t)}\,dt$$

$$E(z \mid m) = \int_0^{\sin(z)} \frac{\sqrt{1-m\,t^2}}{\sqrt{1-t^2}}\,dt \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

$$\Pi(n; z \mid m) = \int_0^{z} \frac{1}{\left(1-n\sin^2(t)\right)\sqrt{1-m\sin^2(t)}}\,dt$$

$$\Pi(n; z \mid m) = \int_0^{\sin(z)} \frac{1}{\left(1-n\,t^2\right)\sqrt{1-t^2}\sqrt{1-m\,t^2}}\,dt \;/;\; -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$\Pi(n; z \mid m) = \int_0^{F(z|m)} \frac{1}{1-n\,\mathrm{sn}(t \mid m)^2}\,dt$$

$$Z(z \mid m) = \int_0^{z} \left(\sqrt{1-m\sin^2(t)} - \frac{E(m)}{K(m)\sqrt{1-m\sin^2(t)}}\right) dt$$

$$Z(z \mid m) = \int_0^{\sin(z)} \frac{1}{\sqrt{1-t^2}} \left(\sqrt{1-m\,t^2} - \frac{E(m)}{K(m)\sqrt{1-m\,t^2}}\right) dt \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}.$$

Transformations

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ with linear arguments can sometimes be simplified, for example:

$$F(-z \mid m) = -F(z \mid m) \quad E(-z \mid m) = -E(z \mid m) \quad \Pi(n; -z \mid m) = -\Pi(n; z \mid m) \quad Z(-z \mid m) = -Z(z \mid m)$$

$$F(z + \pi k \mid m) = F(z \mid m) + 2\,k\,K(m) \;/;\; k \in \mathbb{Z}$$

$$E(z + \pi k \mid m) = E(z \mid m) + 2\,k\,E(m) \;/;\; k \in \mathbb{Z}$$

$$\Pi(n; z + \pi k \mid m) = \Pi(n; z \mid m) + 2\,k\,\Pi(n \mid m) \;/;\; k \in \mathbb{Z} \bigwedge -1 \leq n \leq 1$$

$$Z(z + \pi k \mid m) = Z(z \mid m) \;/;\; k \in \mathbb{Z}.$$

In some cases, simplification can be realized for more complicated arguments, for example:

$$F\left(\sin^{-1}\left(\sqrt{m}\,\sin(z)\right) \,\middle|\, \frac{1}{m}\right) = \sqrt{m}\,F(z \mid m) \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

$$F\left(\frac{1}{2}\sin^{-1}\left(\frac{\sqrt{m}\,\tan(z) + \sqrt{(1-m)\tan^4(z) + \tan^2(z)}}{\tan^2(z)+1}\right) \,\middle|\, \frac{4\sqrt{m}}{\left(\sqrt{m}+1\right)^2}\right) = \frac{\sqrt{m}+1}{2}\,F(z \mid m) \;/;\; 0 \leq m < 1 \land 0 \leq z < 1.$$

Sums of the incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ with different values $z == z_1$ and $z == z_2$ can be evaluated by the following summation formulas:

$$F(z_1 \mid m) + F(z_2 \mid m) == F\left(\sin^{-1}\left(\frac{\cos(z_2)\sqrt{1 - m\sin^2(z_2)}\,\sin(z_1) + \cos(z_1)\sqrt{1 - m\sin^2(z_1)}\,\sin(z_2)}{1 - m\sin^2(z_1)\sin^2(z_2)}\right) \,\middle|\, m\right) /; \; 0 \leq m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1$$

$$E(z_1 \mid m) + E(z_2 \mid m) == E\left(\sin^{-1}(w) \,\middle|\, m\right) + m\, w \sin(z_1)\sin(z_2) /; \; w == \frac{\cos(z_2)\sqrt{1 - m\sin^2(z_2)}\,\sin(z_1) + \cos(z_1)\sqrt{1 - m\sin^2(z_1)}\,\sin(z_2)}{1 - m\sin^2(z_1)\sin^2(z_2)} \bigwedge 0 \leq m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1$$

$$\Pi(n; z_1 \mid m) + \Pi(n; z_2 \mid m) == \Pi(n; z \mid m) - \sqrt{\frac{n}{(1-n)(n-m)}}\,\tan^{-1}\left(\frac{\sqrt{(1-n)\,n\,(n-m)}\,\sin(z)\sin(z_1)\sin(z_2)}{-n\sin^2(z) + n\cos(z)\sqrt{1 - m\sin^2(z)}\,\sin(z_1)\sin(z_2) + 1}\right) /;$$

$$z == \cos^{-1}\left(\frac{\cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)\sqrt{\left(1 - m\sin^2(z_1)\right)\left(1 - m\sin^2(z_2)\right)}}{1 - m\sin^2(z_1)\sin^2(z_2)}\right) \bigwedge 0 < m < n < 1 \bigwedge 0 < z_1 < 1 \bigwedge 0 < z_2 < 1$$

$$Z(z_1 \mid m) + Z(z_2 \mid m) == Z(z \mid m) - m\sin(z_1)\sin(z_2)\sin(z) /; \; z == 2\tan^{-1}\left(\frac{\sin(z_1)\sqrt{1 - m\sin^2(z_2)} - \sin(z_2)\sqrt{1 - m\sin^2(z_1)}}{\cos(z_1) - \cos(z_2)}\right) \bigwedge 0 < m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1.$$

Identities

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ satisfy numerous identities, for example:

$$F(z \mid m) == \frac{1}{\sqrt{m}}\, F\left(\sin^{-1}\left(\sqrt{m}\,\sin(z)\right) \,\middle|\, \frac{1}{m}\right) /; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

$$F(z \mid m) == \frac{2}{\sqrt{m} + 1}\, F\left(\frac{1}{2}\sin^{-1}\left(\frac{\sqrt{m}\,\tan(z) + \sqrt{(1-m)\tan^4(z) + \tan^2(z)}}{\tan^2(z) + 1}\right) \,\middle|\, \frac{4\sqrt{m}}{\left(\sqrt{m} + 1\right)^2}\right) /; 0 \leq m < 1 \land 0 \leq z < 1$$

$$\Pi(n; z \mid m) == \frac{1}{\sqrt{m}}\, \Pi\left(\frac{n}{m}; \sin^{-1}\left(\sqrt{m}\,\sin(z)\right) \,\middle|\, \frac{1}{m}\right) /; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

$$\Pi\left(n; i\sinh^{-1}(\tan(z)) \,\middle|\, 1 - m\right) == \frac{i}{1-n}\left(F(z \mid m) - n\,\Pi(1-n; z \mid m)\right)$$

$$E(\phi \mid m)\, F(\theta \mid m) + \cot(\theta)\left(\Pi\left(m\sin^2(\theta); \phi \mid m\right) - F(\phi \mid m)\right)\sqrt{1 - m\sin^2(\theta)} == \cot(\phi)\sqrt{1 - m\sin^2(\phi)}\left(\Pi\left(m\sin^2(\phi); \theta \mid m\right) - F(\theta \mid m)\right) + E(\theta \mid m)\, F(\phi \mid m)$$

$$Z(z \mid m) == i\,\mathrm{dn}(-i\, F(z \mid m) \mid 1 - m)\,\mathrm{sc}(-i\, F(z \mid m) \mid 1 - m) - i\, Z(\mathrm{am}(-i\, F(z \mid m) \mid 1 - m) \mid 1 - m) - \frac{\pi\, F(z \mid m)}{2\, K(m)\, K(1 - m)}.$$

Representations of derivatives

The first derivative of the incomplete elliptic integral $\Pi(n; z \mid m)$ with respect to variable n has the following representation:

$$\frac{\partial \Pi(n; z \mid m)}{\partial n} = \frac{1}{2(m-n)(n-1)} \left(E(z \mid m) + \frac{m-n}{n} F(z \mid m) + \frac{n^2-m}{n} \Pi(n; z \mid m) - \frac{n\sqrt{1-m\sin^2(z)}\,\sin(2z)}{2\left(1-n\sin^2(z)\right)} \right).$$

The previous formula can be generalized to the p^{th} derivative:

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial n^p} = n^{-p} \sin(z) \sum_{k=0}^{\infty} \frac{k!\left(n\sin^2(z)\right)^k}{(2k+1)\,\Gamma(k-p+1)} F_1\left(k+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m\sin^2(z)\right) /; p \in \mathbb{N}.$$

The first derivatives of the incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ with respect to variable z have the following simple representations:

$$\frac{\partial F(z \mid m)}{\partial z} = \frac{1}{\sqrt{1-m\sin^2(z)}}$$

$$\frac{\partial E(z \mid m)}{\partial z} = \sqrt{1-m\sin^2(z)}$$

$$\frac{\partial \Pi(n; z \mid m)}{\partial z} = \frac{1}{\sqrt{1-m\sin^2(z)}\left(1-n\sin^2(z)\right)}$$

$$\frac{\partial Z(z \mid m)}{\partial z} = \sqrt{1-m\sin^2(z)} - \frac{E(m)}{K(m)\sqrt{1-m\sin^2(z)}}.$$

The previous formulas can be generalized to the arbitrary-order symbolic derivatives:

$$\frac{\partial^n F(z \mid m)}{\partial z^n} = F(z \mid m)\,\delta_n + \sum_{j=1}^{n} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(z)\,(k_1+2k_2-j)^n\, e^{-\frac{1}{2} i\,(\pi(j+n-k_1-2k_2)+2(-j+k_1+2k_2)z)} \binom{j-k_1}{k_2}$$
$$\sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z))^{j-2i-1}} \sum_{i_1=0}^{i} \binom{i}{i_1} \left(\frac{1}{2}\right)_{i_1} \left(\frac{1}{2}\right)_{i-i_1} m^{i-i_1} \cos^{-2i_1-1}(z) \left(1-m\sin^2(z)\right)^{-i+i_1-\frac{1}{2}} /; n \in \mathbb{N}$$

$$\frac{\partial^n E(z \mid m)}{\partial z^n} = E(z \mid m)\,\delta_n + \sum_{j=1}^{n} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(z)\,(k_1+2k_2-j)^n\, e^{-\frac{1}{2} i\,(\pi(j+n-k_1-2k_2)+2(-j+k_1+2k_2)z)} \binom{j-k_1}{k_2}$$
$$\sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z))^{j-2i-1}} \sum_{i_1=0}^{i} \binom{i}{i_1} \left(\frac{1}{2}\right)_{i_1} \left(-\frac{1}{2}\right)_{i-i_1} m^{i-i_1} \cos^{-2i_1-1}(z) \left(1-m\sin^2(z)\right)^{-i+i_1+\frac{1}{2}} /; n \in \mathbb{N}$$

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial z^p} = \Pi(n; z \mid m)\,\delta_p + \sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(z) (k_1 + 2k_2 - j)^p e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2)+2(-j+k_1+2k_2) z)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z))^{j-2i-1}} \sum_{i_1=0}^{i} \sum_{i_2=0}^{i} \sum_{i_3=0}^{i} (-1)^{i_1} \delta_{-i+i_1+i_2+i_3} (i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n\sin^2(z)\right)^{-i_1-1} \cos^{-2i_2-1}(z) \left(1 - m\sin^2(z)\right)^{-i_3-\frac{1}{2}} \;/;\; p \in \mathbb{N}$$

$$\frac{\partial^n Z(z \mid m)}{\partial z^n} = \delta_n Z(z \mid m) - \sum_{j=1}^{n} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} \sin^{k_1}(z) (k_1 + 2k_2 - j)^n e^{-\frac{1}{2} i (\pi (j+n-k_1-2k_2)+2(-j+k_1+2k_2) z)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{2^{2i-2j+k_1} (1-j)_{2(j-i)-2}}{(j-i-1)!\,\sin^{j-2i-1}(z)} \sum_{i_1=0}^{i} \binom{i}{i_1} \left(\frac{1}{2}\right)_{i_1} m^{i-i_1} \cos^{-2i_1-1}(z) \left(\frac{1}{2}\right)_{i-i_1-1} \left(1 - m\sin^2(z)\right)^{-i+i_1-\frac{1}{2}} \left(\left(1 - m\sin^2(z)\right) - \frac{E(m)}{K(m)} (2i_1 - 2i + 1)\right) \;/;\; n \in \mathbb{N}.$$

The first derivatives of the incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$ and $Z(z \mid m)$ with respect to variable m have the following representations:

$$\frac{\partial F(z \mid m)}{\partial m} = \frac{E(z \mid m)}{2(1-m)m} - \frac{F(z \mid m)}{2m} - \frac{\sin(2z)}{4(1-m)\sqrt{1 - m\sin^2(z)}}$$

$$\frac{\partial E(z \mid m)}{\partial m} = \frac{E(z \mid m) - F(z \mid m)}{2m}$$

$$\frac{\partial \Pi(n; z \mid m)}{\partial m} = \frac{1}{2(n-m)} \left(\frac{1}{m-1} E(z \mid m) + \Pi(n; z \mid m) - \frac{m\sin(2z)}{2(m-1)\sqrt{1 - m\sin^2(z)}}\right)$$

$$\frac{\partial Z(z \mid m)}{\partial m} = \frac{1}{2m}\left(1 + \frac{E(m)}{(m-1)K(m)}\right) Z(z \mid m) - \frac{E(m)\sin(2z)}{4(m-1)\sqrt{1 - m\sin^2(z)}\;K(m)}.$$

The previous formulas can be generalized to the arbitrary-order symbolic derivatives:

$$\frac{\partial^n F(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, \sin^{2n+1}(z)}{(2n+1)\,\Gamma\left(\frac{1}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; \sin^2(z), m\sin^2(z)\right) \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^n E(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, \sin^{2n+1}(z)}{2(2n+1)\,\Gamma\left(\frac{3}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n - \frac{1}{2}; n + \frac{3}{2}; \sin^2(z), m\sin^2(z)\right) \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial m^p} = m^{-p} \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(m\sin^2(z)\right)^k}{(2k+1)\,\Gamma(k-p+1)} F_1\left(k + \frac{1}{2}; \frac{1}{2}, 1; k + \frac{3}{2}; \sin^2(z), n\sin^2(z)\right) \;/;\; p \in \mathbb{N}.$$

Integration

The indefinite integral of the incomplete elliptic integral $\Pi(n; z \mid m)$ with respect to variable n has the following representation through the infinite series that includes the Appell F_1 hypergeometric function of two variables:

$$\int \Pi(n; z \mid m)\, dn = \sin(z)\, n \sum_{k=0}^{\infty} \frac{(n \sin^2(z))^k}{(2k+1)(k+1)} F_1\left(k+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m \sin^2(z)\right).$$

The indefinite integrals of all incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ with respect to variable z have the following representations:

$$\int F(z \mid m)\, dz = \frac{z^2}{\pi} K(m) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k (1-\cos(2kz))}{k!\, k^2} \left(\frac{1}{2}\right)_k {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) \left(\frac{m}{4}\right)^k$$

$$\int E(z \mid m)\, dz = \frac{z^2}{\pi} E(m) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k (1-\cos(2kz))}{k!\, k^2} \left(-\frac{1}{2}\right)_k {}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) \left(\frac{m}{4}\right)^k$$

$$\int \Pi(n; z \mid m)\, dz =$$
$$\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k+l-1} m^l n^k 2^{-2j-2k-2l}}{(2j+2k+2l+1)\, j!\, l!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_l \sum_{p=0}^{j+k+l} \frac{(-1)^p \cos((2j+2k+2l-2p+1)z)}{2j+2k+2l-2p+1} \binom{2j+2k+2l+1}{p}$$

$$\int Z(z \mid m)\, dz =$$
$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k (1-\cos(2kz))}{k!\, k^2} \left(-\frac{1}{2}\right)_k \left({}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) + \frac{(2k-1)E(m)}{K(m)} {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right)\right) \left(\frac{m}{4}\right)^k.$$

The indefinite integrals of incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, and $\Pi(n; z \mid m)$ with respect to variable m can be expressed in closed forms by the following formulas:

$$\int F(z \mid m)\, dm = \sqrt{2\cos(2z)\, m - 2m + 4}\, \cot(z) + 2E(z \mid m) + 2(m-1)F(z \mid m)$$

$$\int E(z \mid m)\, dm = \frac{2}{3}\left((m+1)E(z \mid m) - (1-m)F(z \mid m) - \cot(z)\left(1 - \sqrt{1 - m\sin^2(z)}\right)\right)$$

$$\int \Pi(n; z \mid m)\, dm = \sqrt{2\cos(2z)\, m - 2m + 4}\, \cot(z) + 2E(z \mid m) - 2F(z \mid m) + 2(m-n)\Pi(n; z \mid m).$$

Differential equations

The incomplete elliptic integrals $F(z \mid m)$, $E(z \mid m)$, $\Pi(n; z \mid m)$, and $Z(z \mid m)$ satisfy the following differential equations:

$$(1-m)\, m \frac{\partial^2 w(m)}{\partial m^2} + (1-2m) \frac{\partial w(m)}{\partial m} - \frac{w(m)}{4} = -\frac{\sin(2z)}{8\left(1 - m\sin^2(z)\right)^{3/2}} \;/;\; w(m) = F(z \mid m)$$

$$\left(\frac{\partial w(z)}{\partial z} - 1\right)\left(\frac{\partial w(z)}{\partial z}\right)^2 \left(\frac{\partial w(z)}{\partial z} + 1\right)\left((m-1)\left(\frac{\partial w(z)}{\partial z}\right)^2 + 1\right) - \left(\frac{\partial^2 w(z)}{\partial z^2}\right)^2 = 0 \;/;\; w(z) = F(z \mid m)$$

$$(1-m)\,m\,\frac{\partial^2 w(m)}{\partial m^2}+(1-m)\,\frac{\partial w(m)}{\partial m}+\frac{w(m)}{4}=\frac{\sin(2z)}{8\sqrt{1-m\sin^2(z)}}\;/;\;w(m)=E(z\,|\,m)$$

$$\left(\frac{\partial w(z)}{\partial z}\right)^4+\left(\left(\frac{\partial^2 w(z)}{\partial z^2}\right)^2+m-2\right)\left(\frac{\partial w(z)}{\partial z}\right)^2=m-1\;/;\;w(z)=E(z\,|\,m)$$

$$8\,(m-1)\,m\,(m-n)\,\frac{\partial^3 w(m)}{\partial m^3}+4\left(11\,m^2-6\,n\,m-7\,m+2\,n\right)\frac{\partial^2 w(m)}{\partial m^2}+6\,(7\,m-n-2)\,\frac{\partial w(m)}{\partial m}+3\,w(m)=\frac{3\sin(2z)}{2\sqrt{\left(1-m\sin^2(z)\right)^5}}\;/;$$

$$w(m)=\Pi(n;z\,|\,m)$$

$$2\,(n-1)\,(m-n)\,n\,\frac{\partial^3 w(n)}{\partial n^3}+\left(-13\,n^2+8\,m\,n+8\,n-3\,m\right)\frac{\partial^2 w(n)}{\partial n^2}+4\,(m-4\,n+1)\,\frac{\partial w(n)}{\partial n}-2\,w(n)=\frac{\sqrt{1-m\sin^2(z)}\,\sin(2z)}{\left(n\sin^2(z)-1\right)^3}\;/;$$

$$w(n)=\Pi(n;z\,|\,m)$$

$$\begin{aligned}&\left(w'(z)^2-1\right)\left(4\,w'(z)^2\,(m-n)^3+27\,m^2\,n\right)^2\left((m-1)\,(n-1)^2\,w'(z)^2+1\right)w'(z)^6+\left(8\,(m-1)\,(m-n)^5\,(n-1)\,n\,w'(z)^6-\right.\\&\quad 2\,(m-n)^2\left(4\,(n-1)\,(3\,n-2)\,m^4+n\,(3\,n\,(5-11\,n)+8)\,m^3-3\,(n-9)\,n^2\,(2\,n-1)\,m^2+2\,(n-10)\,n^3\,m+4\,n^4\right)w'(z)^4-\\&\quad \left.9\,m^2\,n\left((6\,n\,(3\,n-5)+16)\,m^3+3\,n\,(n\,(3\,n-16)+4)\,m^2-3\,(n-12)\,n^2\,m-10\,n^3\right)w'(z)^2-243\,m^4\,n^2\right)w''(z)^2\,w'(z)^4+\\&\quad n\left(-27\,n\,m^4-(m-n)^2\,(m\,(9\,n-8)-10\,n)\,w'(z)^2\,m^2+(m-1)\,(m-n)^4\,n\,w'(z)^4\right)w''(z)^4\,w'(z)^2-\\&\quad m^4\,n^2\,w''(z)^6=0\;/;\;w(z)=\Pi(n;z\,|\,m)\end{aligned}$$

$$\begin{aligned}&m^2\cos^2(z)\sin^2(z)\,w^{(1,0)}(z,m)^2+\left(m\sin^2(z)-1\right)\\&\quad\left(4\,m^2\cos^2(z)\sin^2(z)+\left(m\sin^2(z)-1\right)w^{(2,0)}(z,m)^2+m\sin(2z)\,w^{(1,0)}(z,m)\,w^{(2,0)}(z,m)\right)=0\;/;\;w(z,m)=\mathrm{Z}(z\,|\,m)\end{aligned}$$

$$\begin{aligned}&16\,E(m)^6+8\left(w'(z)^2+w''(z)^2+4\,m-8\right)K(m)\,E(m)^5+\\&\quad\left(w'(z)^4+2\left(w''(z)^2+16\,(m-2)\right)w'(z)^2+\left(w''(z)^2+4\,m\right)^2-16\left(w''(z)^2+6\,m-6\right)\right)K(m)^2\,E(m)^4+\\&\quad 2\left(5\,(m-2)\,w'(z)^4+\left(3\,(m-2)\,w''(z)^2+4\,((m-14)\,m+14)\right)w'(z)^2-4\,(m-1)\left(w''(z)^2+4\,(m-2)\right)\right)K(m)^3\,E(m)^3+\\&\quad\left((m-2)\,w'(z)^6+\left((m-2)\,w''(z)^2+(m-54)\,m+54\right)w'(z)^4-2\,(m-1)\left(9\,w''(z)^2+16\,(m-2)\right)w'(z)^2+16\,(m-1)^2\right)\\&\qquad K(m)^4\,E(m)^2-2\,(m-1)\,K(m)^5\,w'(z)^2\left(6\,w'(z)^4+\left(4\,w''(z)^2+5\,m-10\right)w'(z)^2-4\,m+4\right)E(m)-\\&\quad(m-1)\,K(m)^6\,w'(z)^4\left(w'(z)^4+\left(w''(z)^2+m-2\right)w'(z)^2-m+1\right)=0\;/;\;w(z)=\mathrm{Z}(z\,|\,m).\end{aligned}$$

Applications of incomplete elliptic integrals

Applications of incomplete elliptic integrals include geometry, physics, mechanics, electrodynamics, statistical mechanics, astronomy, geodesy, geodesics on conics, and magnetic field calculations.

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