Introductions to InverseJacobiND

Introduction to the inverse Jacobi elliptic functions

General

The inverses of the Jacobi elliptic functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ can be represented through elliptic integrals. They first appeared in a paper by N. H. Abel (1826) who studied the so-called hyperelliptic and Abelian integrals. Later A. G. Greenhill (1892) paid some attention to these functions. This interest was continued by L. M. Milne–Thompson (1948).

Definitions of the inverse Jacobi functions

The inverses of the twelve Jacobi elliptic functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ are defined by the following formulas:

$$z = \mathrm{cd}(w \mid m) /; w = \mathrm{cd}^{-1}(z \mid m) \quad \mathrm{cd}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt /; -1 < z < 1 \land m < 1$$

$$z = \mathrm{cn}(w \mid m) /; w = \mathrm{cn}^{-1}(z \mid m) \quad \mathrm{cn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{m\,t^2-m+1}}\,dt /; -1 < z < 1 \land m\left(z^2-1\right) > -1$$

$$z = \mathrm{cs}(w \mid m) /; w = \mathrm{cs}^{-1}(z \mid m) \quad \mathrm{cs}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt /; z \in \mathbb{R} \land z^2 - m > -1$$

$$z = \mathrm{dc}(w \mid m) /; w = \mathrm{dc}^{-1}(z \mid m) \quad \mathrm{dc}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\,dt /; z \in \mathbb{R} \land z^2 > 1 \land z^2 - m > 0 \land m < 1$$

$$z = \mathrm{dn}(w \mid m) /; w = \mathrm{dn}^{-1}(z \mid m) \quad \mathrm{dn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\,dt /; -1 < z < 1 \land z^2 + m > 1$$

$$z = \mathrm{ds}(w \mid m) /; w = \mathrm{ds}^{-1}(z \mid m) \quad \mathrm{ds}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt /; z \in \mathbb{R} \land z^2 + m > 1$$

$$z = \mathrm{nc}(w \mid m) /; w = \mathrm{nc}^{-1}(z \mid m) \quad \mathrm{nc}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{(1-m)\,t^2+m}}\,dt /; z \in \mathbb{R} \land z^2 > 1 \land (1-m)\,z^2 + m > 0$$

$$z = \mathrm{nd}(w \mid m) /; w = \mathrm{nd}^{-1}(z \mid m) \quad \mathrm{nd}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\,dt /; z \in \mathbb{R} \land z^2 > 1 \land (1-m)\,z^2 < 1 \land m > 0$$

$$z = \mathrm{ns}(w \mid m) /; w = \mathrm{ns}^{-1}(z \mid m) \quad \mathrm{ns}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\,dt /; z \in \mathbb{R} \land z^2 > 1 \land z^2 > m$$

$$z = \mathrm{sc}(w \mid m) /; w = \mathrm{sc}^{-1}(z \mid m) \quad \mathrm{sc}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{t^2+1}\,\sqrt{(1-m)\,t^2+1}}\,dt /; z \in \mathbb{R} \land (1-m)\,z^2 > -1$$

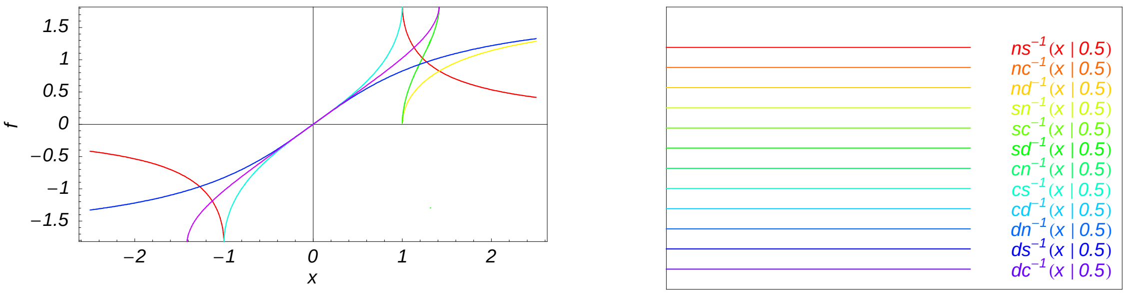
$$z = \mathrm{sd}(w \mid m) /; w = \mathrm{sd}^{-1}(z \mid m) \quad \mathrm{sd}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{m\,t^2+1}\,\sqrt{1-(1-m)\,t^2}}\,dt /; z \in \mathbb{R} \land m\,z^2 > -1 \land (1-m)\,z^2 < 1$$

$$z = \mathrm{sn}(w \mid m) /; w = \mathrm{sn}^{-1}(z \mid m) \quad \mathrm{sn}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt /; -1 < z < 1 \land m\,z^2 < 1 .$$

It is obvious that the inverses of the twelve Jacobi elliptic functions are actually the definite elliptic integrals and can be expressed through the Legendre elliptic integrals.

A quick look at the inverse Jacobi functions

Here is a quick look at the graphics for the inverse Jacobi functions along the real axis.



Connections within the group of inverse Jacobi functions and with other related function groups

Representations through more general functions

The inverses of the Jacobi elliptic functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ can be represented through hypergeometric functions of two variables (including the Appell F_1 function) by the following formulas:

$$\mathrm{cd}^{-1}(z \mid m) = K(m) - z\,F^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\ m\,z^2,\ z^2\right)$$

$$\mathrm{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}}\,z\,F^{1}_{1}\ldots$$

$$\mathrm{cs}^{-1}(z \mid m) = \frac{1}{z}\,F^{2\times0\times1}_{1\times0\times1}\left(\begin{matrix}\frac{1}{2},\ 1;;\ \frac{1}{2};\\ \frac{3}{2};;\ 1;\end{matrix}\ -\frac{1}{z^2},\ \frac{m}{z^2}\right)$$

$$\mathrm{dc}^{-1}(z \mid m) = K(m) - \frac{1}{z}\,F^{1\times1\times1}_{1\times0\times0}\left(\ldots\right.$$

$$\mathrm{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}}\,K\!\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}}\,F^{1\times1\times1}_{0\times1\times0}\left(\begin{matrix}1;\ \frac{1}{2};\ \frac{1}{2};\\ ;\ \frac{3}{2};\ ;\end{matrix}\ z^2,\ m\right)$$

$$\mathrm{ds}^{-1}(z \mid m) = \frac{1}{z}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2 z^{-2k}}{\left(\frac{3}{2}\right)_k}\,F^{1\times1}_{1\times0}\ldots$$

$$\mathrm{nc}^{-1}(z \mid m) = K(m) - \frac{1}{z}\left(F^{1\times2\times2}_{2\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2},\ \frac{1}{2};\ \frac{1}{2},\ 1;\\ \frac{3}{2},\ 1;;;\end{matrix}\ -\frac{m}{z^2},\ \frac{1}{z^2}\right) + \frac{m}{2}\,F^{2\times1\times1}_{1\times1\times1}\left(\begin{matrix}\frac{3}{2},\ \frac{3}{2};\ \frac{1}{2};\ 1;\\ 2;\ \frac{3}{2};\ \frac{3}{2};\end{matrix}\ -\frac{m}{z^2},\ m\right)\right)$$

$$\mathrm{nd}^{-1}(z \mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}\ \ldots}{\sqrt{z^2-1}}\ldots$$

$$\mathrm{ns}^{-1}(z \mid m) = \frac{1}{z}\,F^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\ \frac{m}{z^2},\ \frac{1}{z^2}\right)$$

$$\mathrm{sc}^{-1}(z \mid m) = z\,F^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\ \ldots\right.$$

$$\mathrm{sd}^{-1}(z \mid m) = z\,F^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\ (1-m)\,z^2,\ -m\,z^2\right)$$

$$\mathrm{sn}^{-1}(z \mid m) = z\,F^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix}\frac{1}{2};\ \frac{1}{2};\ \frac{1}{2};\\ \frac{3}{2};;;\end{matrix}\ \ldots\right.$$

$$\mathrm{cd}^{-1}(z \mid m) = K(m) - z\,F_1\!\left(\frac{1}{2};\ \frac{1}{2},\ \frac{1}{2};\ \frac{3}{2};\ z^2,\ m\,z^2\right) /; -1 < z < 1 \land m < 1$$

$$\text{cn}^{-1}(z \mid m) = \sqrt{1-z^2}\; F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1-z^2, m\left(1-z^2\right)\right) /; -1 < z < 1 \land -1 < m < 1$$

$$\text{cs}^{-1}(z \mid m) = \frac{1}{z} F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{z^2}, \frac{m-1}{z^2}\right) /; z \in \mathbb{R} \bigwedge m - z^2 < 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\!\left(\frac{1}{m}\right) - z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{m}\right)\right) /; z > 1 \land m < 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\!\left(\frac{1}{1-m}\right) - z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{1-m}\right)\right) /; -1 < z < 1 \bigwedge z^2 + m > 1 \bigwedge m > 0$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{z} F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right) /; z^2 + m > 1$$

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} K\!\left(1 - \frac{1}{m}\right) - \frac{i\, z}{\sqrt{m}} F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \left(1 - \frac{1}{m}\right) z^2\right) /; -1 < z < 1 \land m \in \mathbb{R}$$

$$\text{nd}^{-1}(z \mid m) = i \left(K(1-m) - z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, (1-m)\, z^2\right)\right) /; -1 < z < 1 \land -1 < m < 1$$

$$\text{ns}^{-1}(z \mid m) = \frac{1}{z} F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) /; (z > 1 \lor z < -1) \bigwedge z^2 > m$$

$$\text{sc}^{-1}(z \mid m) = z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2, (m-1)\, z^2\right) /; z \in \mathbb{R} \bigwedge (m-1)\, z^2 < 1$$

$$\text{sd}^{-1}(z \mid m) = z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -m\, z^2, (1-m)\, z^2\right) /; z \in \mathbb{R} \bigwedge (1-m)\, z^2 < 1$$

$$\text{sn}^{-1}(z \mid m) = z\, F_1\!\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, m\, z^2\right) /; -1 < z < 1 \bigwedge m\, z^2 < 1.$$

Representations through related equivalent functions

The twelve inverses of the Jacobi elliptic functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ can be represented through incomplete and complete elliptic integrals $F\!\left(\sin^{-1}(z) \,\middle|\, m\right)$ and $K(m)$ by the following formulas:

$$\text{cd}^{-1}(z \mid m) = K(m) - F\!\left(\sin^{-1}(z) \,\middle|\, m\right) /; m \notin (1, \infty)$$

$$\text{cn}^{-1}(z \mid m) = F\!\left(\cos^{-1}(z) \,\middle|\, m\right) /; -1 < z < 1 \land m \in \mathbb{R}$$

$$\text{cs}^{-1}(z \mid m) = -i\, F\!\left(i \sinh^{-1}\!\left(\frac{1}{z}\right) \,\middle|\, 1-m\right) /; z > 0 \land m \in \mathbb{R}$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\!\left(\frac{1}{m}\right) - F\!\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right)\right) /; -1 < z < 1 \land m > 1 \lor z > 1 \land m < 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\!\left(\frac{1}{1-m}\right) - F\!\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)\right) /; z < 1 \land m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(\frac{\sqrt{1-m}}{z}\right) \middle| \frac{m}{m-1}\right) /; z > 1 \land m > 1$$

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \left(F\left(\sin^{-1}(z) \middle| \frac{m-1}{m}\right) - K\left(\frac{m-1}{m}\right)\right) /; z > 1 \land m > 0$$

$$\text{nd}^{-1}(z \mid m) = i\left(F\left(\sin^{-1}(z) \mid 1-m\right) - K(1-m)\right) /; z > 1 \land m > 1$$

$$\text{ns}^{-1}(z \mid m) = F\left(\sin^{-1}\left(\frac{1}{z}\right) \middle| m\right) /; (z < -1 \lor z > 1) \land m < 1$$

$$\text{sc}^{-1}(z \mid m) = -i\, F\left(i \sinh^{-1}(z) \mid 1-m\right) /; |z| < 1$$

$$\text{sd}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\sqrt{m}\, z\right) \middle| \frac{m-1}{m}\right) /; |z| < 1 \land |m| < 1$$

$$\text{sn}^{-1}(z \mid m) = F\left(\sin^{-1}(z) \mid m\right) /; |z| < 1 \land |m| < 1.$$

The twelve inverse Jacobi functions can also be expressed through the elliptic logarithm $\text{elog}(z_1, z_2; a, b)$ and the complete elliptic integral $K(m)$ by the following formulas:

$$\text{cd}^{-1}(z \mid m) = K(m) + \frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /;$$
$$\{a, b, z_1\} = \left\{-m-1, m, \frac{1}{z^2}\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge (0 < z < 1 \land m \in \mathbb{R}) \bigvee z < -1 \land m > 1$$

$$\text{cn}^{-1}(z \mid m) = -\frac{i\sqrt{z_2^2}}{z_2 \sqrt{m}} \left(K\left(1 - \frac{1}{m}\right) + \text{elog}(z_1, z_2; a, b)\right) /;$$
$$\{a, b, z_1\} = \left\{\frac{1}{m} - 2, 1 - \frac{1}{m}, z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge 0 < z < 1 \bigwedge 0 < m < 1$$

$$\text{cs}^{-1}(z \mid m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{2-m, 1-m, z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m < 1$$

$$\text{dc}^{-1}(z \mid m) = K(m) + \frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{-m-1, m, z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 1 \bigwedge m < 1$$

$$\text{dn}^{-1}(z \mid m) = -\frac{i\sqrt{z_2^2}}{z_2} (K(1-m) + \text{elog}(z_1, z_2; a, b)) /;$$
$$\{a, b, z_1\} = \left\{m-2, 1-m, z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge 0 < z < 1 \bigwedge m > 1$$

$$\text{ds}^{-1}(z \mid m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{2m-1, m(m-1), z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m > 1$$

$$\text{nc}^{-1}(z \mid m) == \frac{1}{\sqrt{1-m}} \left(\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b) - K\left(\frac{m}{m-1}\right) \right) /;$$

$$\{a, b, z_1\} == \left\{ \frac{2m-1}{1-m}, \frac{m}{m-1}, z^2 \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge 0 < z < 1 \bigwedge m > 1$$

$$\text{nd}^{-1}(z \mid m) == -\frac{i \sqrt{z_2^2}}{z_2} (K(1-m) + \operatorname{elog}(z_1, z_2; a, b)) /;$$

$$\{a, b, z_1\} == \left\{ m-2, 1-m, \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge z > 1 \bigwedge m > 1$$

$$\text{ns}^{-1}(z \mid m) == -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} == \left\{ -m-1, m, z^2 \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge z > 0 \bigwedge m < 1$$

$$\text{sc}^{-1}(z \mid m) == -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} == \left\{ 2-m, 1-m, \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge z > 0 \bigwedge m < 1$$

$$\text{sd}^{-1}(z \mid m) == -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} == \left\{ 2m-1, m(m-1), \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge z > 0 \bigwedge m > 1$$

$$\text{sn}^{-1}(z \mid m) == -z \operatorname{elog}\left(1, \sqrt{a+b+1}; a, b\right) /; \{a, b\} == \left\{ -z^2 (m+1), m z^4 \right\} \bigwedge |z| < 1.$$

Relations to inverse functions

The twelve inverses of the Jacobi elliptic functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ are connected with the corresponding direct Jacobi functions by the following formulas:

$$\text{cd}\left(\text{cd}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{cn}\left(\text{cn}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{cs}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right) == z$$

$$\text{dc}\left(\text{dc}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{dn}\left(\text{dn}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{ds}\left(\text{ds}^{-1}(z \mid m) \,\middle|\, m\right) == z$$

$$\text{nc}\left(\text{nc}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{nd}\left(\text{nd}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{ns}\left(\text{ns}^{-1}(z \mid m) \,\middle|\, m\right) == z$$

$$\text{sc}\left(\text{sc}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{sd}\left(\text{sd}^{-1}(z \mid m) \,\middle|\, m\right) == z \quad \text{sn}\left(\text{sn}^{-1}(z \mid m) \,\middle|\, m\right) == z.$$

Representations through other inverse Jacobi functions

The twelve inverses of the Jacobi elliptic functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ are interconnected by formulas that include the complete elliptic integral $K(m)$, rational functions, simple powers, and arithmetical operations of the arguments of other inverse Jacobi functions. These formulas can be divided into the following eleven groups:

Representations of $\text{cd}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{cd}^{-1}(z \mid m) == K(m) - \text{cn}^{-1}\left(\sqrt{1-z^2} \,\middle|\, m\right) /; 0 < z < 1 \land m < 1$$

$$\text{cd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + i\, \text{cs}^{-1}\left(-i\, z \,\middle|\, 1 - \frac{1}{m}\right)\right) /;\ 0 < z < 1 \land 0 < m < 1$$

$$\text{cd}^{-1}(z \mid m) = \text{dc}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /;\ z < 0 \land m < 0 \lor z < 1 \land m < 1$$

$$\text{cd}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}}\, \text{dn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right) /;\ -1 < z < 1 \land m > 1$$

$$\text{cd}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}}\, \text{ds}^{-1}\left(\frac{1}{\sqrt{1-m}\, z} \,\middle|\, \frac{m}{m-1}\right) /;\ z > 0 \land m \in \mathbb{R}$$

$$\text{cd}^{-1}(z \mid m) = -i\, \text{nd}^{-1}(z \mid 1 - m) /;\ z \in \mathbb{R} \land m < 0$$

$$\text{cd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(\text{ns}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) - K\left(\frac{1}{m}\right)\right) /;\ -1 < z < 1 \land m < 0$$

$$\text{cd}^{-1}(z \mid m) = K(m) - i\, \text{sc}^{-1}(-i\, z \mid 1 - m) /;\ z \in \mathbb{R} \land m \in \mathbb{R}$$

$$\text{cd}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}}\, \text{sd}^{-1}\left(z \sqrt{1-m} \,\middle|\, \frac{m}{m-1}\right) /;\ -1 < z < 1 \land m < 1$$

$$\text{cd}^{-1}(z \mid m) = K(m) - \text{sn}^{-1}(z \mid m) /;\ z \in \mathbb{R} \land m \in \mathbb{R}.$$

Representations of $\text{cn}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{cn}^{-1}(z \mid m) = K(m) - \text{cd}^{-1}\left(\sqrt{1 - z^2} \,\middle|\, m\right) /;\ 0 < z < 1 \land m < 1$$

$$\text{cn}^{-1}(z \mid m) = i\, \text{cs}^{-1}\left(\frac{1}{\sqrt{z^2 - 1}} \,\middle|\, 1 - m\right) /;\ z > 1 \land m > 1$$

$$\text{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}}\, \text{dc}^{-1}\left(\sqrt{1 - z^2} \,\middle|\, \frac{1}{m}\right) /;\ 0 < z < 1 \land 0 < m < 1$$

$$\text{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{dn}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /;\ -1 < z < 1 \land m < 1$$

$$\text{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\, \text{ds}^{-1}\left(\frac{1}{\sqrt{(z^2 - 1)(m - 1)}} \,\middle|\, \frac{m}{m-1}\right) /;\ z > 1 \land m > 1$$

$$\text{cn}^{-1}(z \mid m) = -i\, \text{nc}^{-1}(z \mid 1 - m) /;\ -1 < z < 1 \land m \in \mathbb{R}$$

$$\text{cn}^{-1}(z \mid m) = K(m) + i\, \text{nd}^{-1}\left(\sqrt{1 - z^2} \,\middle|\, 1 - m\right) /;\ z > 0 \land m \in \mathbb{R}$$

$$\text{cn}^{-1}(z \mid m) = \text{ns}^{-1}\left(\frac{1}{\sqrt{1 - z^2}} \,\middle|\, m\right) /;\ 0 < z < 1 \land m \in \mathbb{R}$$

$$\text{cn}^{-1}(z \mid m) = -i\,\text{sc}^{-1}\left(i\sqrt{1-z^2} \;\middle|\; 1-m\right) /;\ 0 < z < 1 \land m \in \mathbb{R}$$

$$\text{cn}^{-1}(z \mid m) = -\frac{i}{\sqrt{1-m}}\,\text{sd}^{-1}\left(\sqrt{(m-1)\left(1-z^2\right)} \;\middle|\; \frac{1}{1-m}\right) /;\ 0 < z < 1 \land 0 < m < 1$$

$$\text{cn}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\left(K\left(\frac{m}{m-1}\right) - \text{sn}^{-1}\left(z \;\middle|\; \frac{m}{m-1}\right)\right) /;\ -1 < z < 1 \land m < 1.$$

Representations of $\text{cs}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{cs}^{-1}(z \mid m) = i\,K(1-m) - \frac{i}{\sqrt{1-m}}\,\text{cd}^{-1}\left(i\,z \;\middle|\; \frac{1}{1-m}\right)$$

$$\text{cs}^{-1}(z \mid m) = i\,\text{cn}^{-1}\left(\frac{\sqrt{z^2+1}}{z} \;\middle|\; 1-m\right) /;\ z > 0 \land m > 0$$

$$\text{cs}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{1-m}}\,\text{dc}^{-1}\left(\frac{i}{z} \;\middle|\; \frac{1}{1-m}\right) - K(1-m)\right) /;\ 0 < z < 1 \land 0 < m < 1$$

$$\text{cs}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{m-1}}\,\text{dn}^{-1}\left(\frac{i}{z} \;\middle|\; \frac{m}{m-1}\right) - K(1-m)\right) /;\ z > 0 \land m < 0$$

$$\text{cs}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\,\text{ds}^{-1}\left(\frac{i\,z}{\sqrt{m}} \;\middle|\; \frac{m-1}{m}\right) /;\ z \in \mathbb{R} \land m < 0$$

$$\text{cs}^{-1}(z \mid m) = i\,K(1-m) - \frac{i}{\sqrt{m}}\,\text{nc}^{-1}\left(i\,z \;\middle|\; 1-\frac{1}{m}\right) /;\ z \in \mathbb{R} \land m < 1$$

$$\text{cs}^{-1}(z \mid m) = \text{nd}^{-1}\left(\frac{i}{z} \;\middle|\; m\right) - i\,K(1-m) /;\ z > 0 \land m \in \mathbb{R}$$

$$\text{cs}^{-1}(z \mid m) = -i\,\text{ns}^{-1}(-i\,z \mid 1-m)$$

$$\text{cs}^{-1}(z \mid m) = \text{sc}^{-1}\left(\frac{1}{z} \;\middle|\; m\right)$$

$$\text{cs}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\,\text{sd}^{-1}\left(\frac{\sqrt{m}}{z} \;\middle|\; \frac{1}{m}\right) /;\ z > 0 \land m \in \mathbb{R}$$

$$\text{cs}^{-1}(z \mid m) = -i\,\text{sn}^{-1}\left(\frac{i}{z} \;\middle|\; 1-m\right) /;\ z > 0 \land m \in \mathbb{R}.$$

Representations of $\text{dc}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{dc}^{-1}(z \mid m) = \text{cd}^{-1}\left(\frac{1}{z} \;\middle|\; m\right)$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\left(K\left(\frac{1}{m}\right) - \text{cn}^{-1}\left(\sqrt{1-z^2} \;\middle|\; \frac{1}{m}\right)\right) /;\ 0 < z < 1 \land m > 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - i \, \text{cs}^{-1}\left(\frac{i}{z} \,\middle|\, 1 - \frac{1}{m}\right)\right) /; z < 1 \land m > 1$$

$$\text{dc}^{-1}(z \mid m) = 2 \, i \, K(1 - m) - i \, \text{dn}^{-1}(z \mid 1 - m)$$

$$\text{dc}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{1 - m}} \, \text{ds}^{-1}\left(-\frac{i \, z}{\sqrt{1 - m}} \,\middle|\, \frac{1}{1 - m}\right) /; 0 < m < 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{1 - m}} \, \text{nc}^{-1}\left(z \,\middle|\, \frac{m}{m - 1}\right) /; z > 0 \land m > 1$$

$$\text{dc}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} \, \text{nd}^{-1}\left(z \,\middle|\, 1 - \frac{1}{m}\right) /; z > 0 \land m > 0$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - \text{ns}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{m}\right)\right) /; z < 1 \land m > 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + i \, \text{sc}^{-1}\left(i \, z \,\middle|\, \frac{m - 1}{m}\right)\right) /; m > 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \, K\left(\frac{1}{m}\right) - \frac{1}{\sqrt{1 - m}} \, \text{sd}^{-1}\left(\frac{z \sqrt{1 - m}}{\sqrt{m}} \,\middle|\, \frac{m}{m - 1}\right) /; -1 < z < 1 \land m > 1$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - \text{sn}^{-1}\left(z \,\middle|\, \frac{1}{m}\right)\right) /; -1 < z < 1 \land m > 1 \lor z > 1 \land m < 1.$$

Representations of $\text{dn}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m - 1}} \, \text{cd}^{-1}\left(z \,\middle|\, \frac{1}{1 - m}\right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \, \text{cn}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{1 - m}} \left(i \, K\left(\frac{1}{1 - m}\right) + \text{cs}^{-1}\left(\frac{i}{z} \,\middle|\, \frac{m}{m - 1}\right)\right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = 2 \, K(m) + i \, \text{dc}^{-1}(z \mid 1 - m) /; -1 < z < 1 \land m < 0 \lor z \in \mathbb{R} \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m - 1}} \, K\left(\frac{1}{1 - m}\right) - \frac{1}{\sqrt{m}} \, \text{ds}^{-1}\left(\frac{\sqrt{m - 1}}{\sqrt{m} \, z} \,\middle|\, \frac{1}{m}\right) /; 0 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}} \, \text{nc}^{-1}\left(z \,\middle|\, 1 - \frac{1}{m}\right) /; z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \text{nd}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; -1 < z < 0 \land m < 0 \lor z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - \text{ns}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{1-m}\right) \right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - i\,\text{sc}^{-1}\left(-i\,z \,\middle|\, \frac{m}{m-1}\right) \right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + \frac{i}{\sqrt{m}} \text{sd}^{-1}\left(\frac{i\sqrt{m}\,z}{\sqrt{m-1}} \,\middle|\, 1 - \frac{1}{m} \right) /; -1 < z < 1 \land m > 1$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(\text{sn}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) - K\left(\frac{1}{1-m}\right) \right) /; z \in \mathbb{R} \bigwedge z^2 + m < 1 \bigwedge m > 0.$$

Representations of ds$^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - \text{cd}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{m}{m-1} \right) \right) /; z > 0 \land m > 0$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{cn}^{-1}\left(\frac{\sqrt{z^2 + m - 1}}{z} \,\middle|\, \frac{m}{m-1} \right) /; z > 1 \land m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}} \text{cs}^{-1}\left(\frac{i\,z}{\sqrt{1-m}} \,\middle|\, \frac{1}{1-m} \right) /; z \in \mathbb{R} \land m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - \text{dc}^{-1}\left(\frac{i\,z}{\sqrt{m}} \,\middle|\, \frac{m-1}{m} \right) \right) /; m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{1}{\sqrt{m}} \text{dn}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{1}{m} \right) /; z > 0 \land m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + i\,\text{nc}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, 1 - m \right) /; z > 0 \land m < 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) + i\,\text{nd}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{1}{1-m} \right) \right) /; z > 0 \land m > 0$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{ns}^{-1}\left(\frac{z}{\sqrt{1-m}} \,\middle|\, \frac{m}{m-1} \right) /; z > 0 \land m > 0$$

$$\text{ds}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}} \text{sc}^{-1}\left(-\frac{i\sqrt{1-m}}{z} \,\middle|\, \frac{1}{1-m} \right) /; z > 0 \land m > 0$$

$$\text{ds}^{-1}(z \mid m) = \text{sd}^{-1}\left(\frac{1}{z} \,\middle|\, m \right) /; z > 0 \land m > 1$$

$$\text{ds}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{sn}^{-1}\left(\frac{\sqrt{1-m}}{z} \,\middle|\, \frac{m}{m-1} \right) /; z > 0 \land m > 0$$

Representations of $\text{nc}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\, \text{cd}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right) /;\ -1 < z < 1 \land m > 0$$

$$\text{nc}^{-1}(z \mid m) = i\, \text{cn}^{-1}(z \mid 1-m) /;\ -1 < z < 1 \land m \in \mathbb{R}$$

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - i\, \text{cs}^{-1}\left(i\, z \,\middle|\, \frac{1}{1-m}\right)\right) /;\ z > 1 \land m < 1$$

$$\text{nc}^{-1}(z \mid m) = -\frac{1}{\sqrt{1-m}}\, \text{dc}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right) /;\ 0 < z < 1 \land m > 1$$

$$\text{nc}^{-1}(z \mid m) = -\frac{1}{\sqrt{m-1}}\, \text{dn}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /;\ -1 < z < 1 \land 0 < m < 1$$

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\, K\left(\frac{m-1}{m}\right) - i\, \text{ds}^{-1}\left(\frac{\sqrt{m}}{z} \,\middle|\, 1-m\right) /;\ z > 0 \land m > 0$$

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{nd}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /;\ -1 < z < 1 \land m > 0$$

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - \text{ns}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right)\right) /;\ z > 1 \land m < 1$$

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(i\, K\left(\frac{m-1}{m}\right) - \text{sc}^{-1}\left(-i\, z \,\middle|\, \frac{1}{m}\right)\right) /;\ -1 < z < 1 \land m > 0$$

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\, K\left(\frac{m-1}{m}\right) - \text{sd}^{-1}\left(\frac{i\, z}{\sqrt{m}} \,\middle|\, m\right) /;\ -1 < z < 1 \land m > 0$$

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - \text{sn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right)\right) /;\ 0 < z < 1 \land m > 0.$$

Representations of $\text{nd}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{nd}^{-1}(z \mid m) = i\, \text{cd}^{-1}(z \mid 1-m)$$

$$\text{nd}^{-1}(z \mid m) = -i \left(\text{cn}^{-1}\left(\sqrt{1-z^2} \,\middle|\, 1-m\right) - K(1-m)\right) /;\ 0 < z < 1 \land m > 0$$

$$\text{nd}^{-1}(z \mid m) = i\, K(1-m) + \text{cs}^{-1}\left(\frac{i}{z} \,\middle|\, m\right) /;\ -1 < z < 1 \land m > 0$$

$$\text{nd}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}}\, \text{dc}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /;\ -1 < z < 1 \land 0 < m < 1$$

$$\text{nd}^{-1}(z \mid m) = \text{dn}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /;\ z < -1 \land m < 0 \lor z > 1 \land m > 1$$

$$\text{nd}^{-1}(z \mid m) = i\left(K(1-m) - \frac{1}{\sqrt{m}}\,\text{ds}^{-1}\left(\frac{1}{\sqrt{m}\,z} \;\middle|\; \frac{m-1}{m}\right)\right) /;\ 0 < z < 1 \land m \in \mathbb{R}$$

$$\text{nd}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\,\text{nc}^{-1}\left(z \;\middle|\; \frac{1}{m}\right) /;\ -1 < z < 1 \land m > 1$$

$$\text{nd}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}}\left(K\left(\frac{1}{1-m}\right) - \text{ns}^{-1}\left(z \;\middle|\; \frac{1}{1-m}\right)\right) /;\ -1 < z < 1$$

$$\text{nd}^{-1}(z \mid m) = i\,K(1-m) + \text{sc}^{-1}(-i\,z \mid m) /;\ -1 < z < 1 \land m \in \mathbb{R}$$

$$\text{nd}^{-1}(z \mid m) = i\,K(1-m) - \frac{1}{\sqrt{m}}\,\text{sd}^{-1}\left(i\,z\,\sqrt{m} \;\middle|\; \frac{1}{m}\right)$$

$$\text{nd}^{-1}(z \mid m) = -i\left(\text{sn}^{-1}(z \mid 1-m) - K(1-m)\right) /;\ z > 1 \land m > 1.$$

Representations of $\text{ns}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{ns}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}}\,\text{cd}^{-1}\left(z \;\middle|\; \frac{1}{m}\right) /;\ -1 < z < 1 \land m < 0$$

$$\text{ns}^{-1}(z \mid m) = \text{cn}^{-1}\left(\frac{\sqrt{z^2-1}}{z} \;\middle|\; m\right) /;\ z > 1 \land m < 1$$

$$\text{ns}^{-1}(i\,z \mid m) = i\,\text{cs}^{-1}(-z \mid 1-m)$$

$$\text{ns}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}}\,\text{dc}^{-1}\left(\frac{1}{z} \;\middle|\; \frac{1}{m}\right) /;\ -1 < z < 1 \land m < 0$$

$$\text{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{m}}\,\text{dn}^{-1}\left(\frac{1}{z} \;\middle|\; 1 - \frac{1}{m}\right) /;\ z > -1 \land m > 1$$

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\,\text{ds}^{-1}\left(\frac{z}{\sqrt{1-m}} \;\middle|\; \frac{m}{m-1}\right) /;\ 0 < z < 1 \land m < 1$$

$$\text{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{1-m}}\,\text{nc}^{-1}\left(\frac{1}{z} \;\middle|\; \frac{1}{1-m}\right) /;\ z > 1 \land m \in \mathbb{R}$$

$$\text{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{m}}\,\text{nd}^{-1}\left(z \;\middle|\; 1 - \frac{1}{m}\right) /;\ -1 < z < 1 \land m < 0$$

$$\text{ns}^{-1}(z \mid m) = -i\,\text{sc}^{-1}\left(\frac{i}{z} \;\middle|\; 1-m\right) /;\ z > 1 \land m \in \mathbb{R}$$

$$\text{ns}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}}\,\text{sd}^{-1}\left(\frac{\sqrt{m-1}}{z} \;\middle|\; \frac{1}{1-m}\right) /;\ z > 0 \land m > 1$$

$$\text{ns}^{-1}(z \mid m) = \text{sn}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /;\ z \in \mathbb{R} \land m < 1.$$

Representations of $\text{sc}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{sc}^{-1}(z \mid m) = i\left(\text{cd}^{-1}(i z \mid 1-m) - K(1-m)\right) /; z \in \mathbb{R} \land m \in \mathbb{R}$$

$$\text{sc}^{-1}(z \mid m) = i\, \text{cn}^{-1}\left(\sqrt{z^2+1} \;\middle|\; 1-m\right) /; 0 < z < 1 \land m > 1$$

$$\text{sc}^{-1}(z \mid m) = \text{cs}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /; z > 0 \land m < 1$$

$$\text{sc}^{-1}(z \mid m) = i\left(K(1-m) - \frac{1}{\sqrt{1-m}}\, \text{dc}^{-1}\left(-i z \;\middle|\; \frac{1}{1-m}\right)\right) /; z \in \mathbb{R} \land 0 < m < 1$$

$$\text{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\, \text{dn}^{-1}\left(i z \;\middle|\; 1 - \frac{1}{1-m}\right) - i\, K(1-m) /; z < 1 \land m < 0$$

$$\text{sc}^{-1}(z \mid m) = -\frac{i}{\sqrt{m}}\, \text{ds}^{-1}\left(-\frac{i}{z\sqrt{m}} \;\middle|\; \frac{m-1}{m}\right) /; z > 0 \land m > 1$$

$$\text{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{nc}^{-1}\left(i z \;\middle|\; \frac{1}{m}\right) - i\, K(1-m) /; -1 < z < 1 \land m > 0$$

$$\text{sc}^{-1}(z \mid m) = \text{nd}^{-1}(i z \mid m) - i\, K(1-m) /; z \in \mathbb{R} \land m \in \mathbb{R}$$

$$\text{sc}^{-1}(z \mid m) = -i\, \text{ns}^{-1}\left(-\frac{i}{z} \;\middle|\; 1-m\right) /; 0 < z < 1 \land m > 0$$

$$\text{sc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{sd}^{-1}\left(z\sqrt{m} \;\middle|\; \frac{1}{m}\right) /; z < 1 \land m < 0$$

$$\text{sc}^{-1}(z \mid m) = -i\, \text{sn}^{-1}(i z \mid 1-m).$$

Representations of $\text{sd}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{sd}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}}\left(K\left(\frac{m}{m-1}\right) - \text{cd}^{-1}\left(\sqrt{1-m}\, z \;\middle|\; \frac{m}{m-1}\right)\right) /; z \in \mathbb{R} \land m > 1$$

$$\text{sc}^{-1}(z \mid m) = -i\, \text{cn}^{-1}\left(\sqrt{z^2+1} \;\middle|\; 1-m\right) /; 0 < z < 1 \land 0 < m < 1$$

$$\text{sc}^{-1}(z \mid m) = \text{cs}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /; z > 0 \land m \in \mathbb{R}$$

$$\text{sc}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{1-m}}\, \text{dc}^{-1}\left(i z \;\middle|\; \frac{1}{1-m}\right) - K(1-m)\right) /; z \in \mathbb{R} \land 0 < m < 1$$

$$\text{sc}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{m-1}}\, \text{dn}^{-1}\left(i z \;\middle|\; \frac{m}{m-1}\right) - K(1-m)\right) /; -1 < z < 1 \land m > 1$$

$$\text{sd}^{-1}(z \mid m) = \text{ds}^{-1}\left(\frac{1}{z} \;\middle|\; m\right) /; z > 0 \land m > 1$$

$$\text{sc}^{-1}(z \mid m) == i\,K(1-m) - \frac{1}{\sqrt{m}}\,\text{nc}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /;\ -1 < z < 1 \land m > 1$$

$$\text{sd}^{-1}(z \mid m) == \frac{1}{\sqrt{m}}\left(\text{nd}^{-1}\left(i\,z\sqrt{m} \,\middle|\, \frac{1}{m}\right) - i\,K\left(1-\frac{1}{m}\right)\right) /;\ z \in \mathbb{R} \land m > 1$$

$$\text{sc}^{-1}(z \mid m) == -i\,\text{ns}^{-1}\left(-\frac{i}{z} \,\middle|\, 1-m\right) /;\ z > 0 \land m \in \mathbb{R}$$

$$\text{sc}^{-1}(z \mid m) == -i\,\text{sn}^{-1}(i\,z \mid 1-m)$$

$$\text{sd}^{-1}(z \mid m) == -\frac{i}{\sqrt{m}}\,\text{sn}^{-1}\left(\sqrt{-m}\,z \,\middle|\, \frac{m-1}{m}\right) /;\ -1 < z < 1 \land m > 0.$$

Representations of $\text{sn}^{-1}(z \mid m)$ through other inverse Jacobi functions are:

$$\text{sn}^{-1}(z \mid m) == K(m) - \text{cd}^{-1}(z \mid m) /;\ z \in \mathbb{R} \land m \in \mathbb{R}$$

$$\text{sn}^{-1}(z \mid m) == K(m) - \frac{1}{\sqrt{1-m}}\,\text{cn}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right) /;\ -1 < z < 1 \land m < 1$$

$$\text{sn}^{-1}(z \mid m) == K(m) + \frac{i}{\sqrt{m}}\,\text{dn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right) /;\ z < 0 \land m > 1$$

$$\text{sn}^{-1}(z \mid m) == K(m) - \frac{1}{\sqrt{m}}\,\text{dc}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /;\ z > 1 \land m > 1$$

$$\text{sn}^{-1}(z \mid m) == K(m) + \frac{i}{\sqrt{m}}\,\text{dn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right) /;\ z < 0 \land m > 1$$

$$\text{sn}^{-1}(z \mid m) == \text{dn}^{-1}\left(\sqrt{1-m\,z^2} \,\middle|\, m\right) /;\ z > 1 \land m < 0$$

$$\text{sn}^{-1}(z \mid m) == \frac{1}{\sqrt{1-m}}\,\text{ds}^{-1}\left(\frac{1}{\sqrt{1-m}\,z} \,\middle|\, \frac{m}{m-1}\right) /;\ z > 0 \land m > 1$$

$$\text{sn}^{-1}(z \mid m) == K(m) + \frac{i}{\sqrt{1-m}}\,\text{nc}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /;\ -1 < z < 1 \land m < 1$$

$$\text{sn}^{-1}(z \mid m) == K(m) + i\,\text{nd}^{-1}(z \mid 1-m) /;\ z \in \mathbb{R} \land m \in \mathbb{R}$$

$$\text{sn}^{-1}(z \mid m) == \text{ns}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /;\ -1 < z < 0 \land m < 0 \lor z > 0 \land m < 0$$

$$\text{sn}^{-1}(z \mid m) == -i\,\text{sc}^{-1}(i\,z \mid 1-m)$$

$$\text{sn}^{-1}(z \mid m) == -\frac{i}{\sqrt{1-m}}\,\text{sd}^{-1}\left(\sqrt{m-1}\,z \,\middle|\, \frac{1}{1-m}\right) /;\ -1 < z < 1 \land m < 1.$$

The best-known properties and formulas for inverse Jacobi functions

Simple values at zero

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ have the following simple values at the origin:

$$\text{cd}^{-1}(0 \mid 0) == \frac{\pi}{2} \quad \text{cn}^{-1}(0 \mid 0) == \frac{\pi}{2} \quad \text{cs}^{-1}(0 \mid 0) == \frac{\pi}{2}$$
$$\text{dc}^{-1}(0 \mid 0) == \tilde{\infty} \quad \text{dn}^{-1}(0 \mid 0) \quad \text{ds}^{-1}(0 \mid 0) == \tilde{\infty}$$
$$\text{nc}^{-1}(0 \mid 0) == \tilde{\infty} \quad \text{nd}^{-1}(0 \mid 0) \quad \text{ns}^{-1}(0 \mid 0) == \tilde{\infty}$$
$$\text{sc}^{-1}(0 \mid 0) == 0 \quad \text{sd}^{-1}(0 \mid 0) == 0 \quad \text{sn}^{-1}(0 \mid 0) == 0.$$

Specific values for specialized parameter values

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ can be represented through elementary functions when $m == 0$ or $m == 1$. In these cases they degenerate into inverse trigonometric and inverse hyperbolic functions. If $m == \frac{1}{2}$, they can be represented through the elliptic integrals $F(w(z) \mid n)$ and $K(n)$:

$$\text{cd}^{-1}(z \mid 0) == \cos^{-1}(z) \quad \text{cd}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == K\left(\tfrac{1}{2}\right) - F\left(\sin^{-1}(z) \,\middle|\, \tfrac{1}{2}\right) \quad \text{cd}^{-1}(z \mid 1) == \tilde{\infty}$$
$$\text{cn}^{-1}(z \mid 0) == \cos^{-1}(z) \quad \text{cn}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == F\left(\cos^{-1}(z) \,\middle|\, \tfrac{1}{2}\right) \quad \text{cn}^{-1}(z \mid 1) == \text{sech}^{-1}(z)$$
$$\text{cs}^{-1}(z \mid 0) == \cot^{-1}(z) \quad \text{cs}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == -i\, F\left(i \sinh^{-1}\left(\tfrac{1}{z}\right) \,\middle|\, \tfrac{1}{2}\right) \quad \text{cs}^{-1}(z \mid 1) == \text{csch}^{-1}(z)$$
$$\text{dc}^{-1}(z \mid 0) == \sec^{-1}(z) \quad \text{dc}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == \sqrt{2}\, \left(K(2) - F\left(\sin^{-1}(z) \mid 2\right)\right) /; z > 1 \quad \text{dc}^{-1}(z \mid 1) == \tilde{\infty}$$
$$\text{dn}^{-1}(z \mid 0) == \tilde{\infty} \quad \text{dn}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == \frac{8\,(1+i)\,\pi^{3/2}}{\Gamma\left(-\frac{1}{4}\right)^2} - i\,\sqrt{2}\, F\left(\sin^{-1}(z) \mid 2\right) \quad \text{dn}^{-1}(z \mid 1) == \text{sech}^{-1}(z)$$
$$\text{ds}^{-1}(z \mid 0) == \csc^{-1}(z) \quad \text{ds}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == \sqrt{2}\, F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}\, z}\right) \,\middle|\, -1\right) /; z > 1 \quad \text{ds}^{-1}(z \mid 1) == \text{csch}^{-1}(z)$$
$$\text{nc}^{-1}(z \mid 0) == \sec^{-1}(z) \quad \text{nc}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == i\,\sqrt{2}\, \left(F\left(\sin^{-1}(z) \mid -1\right) - \frac{1}{4\sqrt{2\pi}}\, \Gamma\left(\tfrac{1}{4}\right)^2\right) /; z > 1 \quad \text{nc}^{-1}(z \mid 1) == \cosh^{-1}(z)$$
$$\text{nd}^{-1}(z \mid 0) == \tilde{\infty} \quad \text{nd}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == i\,\sqrt{2}\, F\left(\tfrac{\pi}{4} \,\middle|\, 2\right) - \sqrt{2}\, i\, F\left(\sin^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, 2\right) /; -1 < z < 1 \quad \text{nd}^{-1}(z \mid 1) == \cosh^{-1}(z)$$
$$\text{ns}^{-1}(z \mid 0) == \csc^{-1}(z) \quad \text{ns}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == \sqrt{2}\, F\left(\sin^{-1}(z) \mid 2\right) + \frac{i\,\pi^{3/2}}{2\,\Gamma\left(\frac{3}{4}\right)^2} \quad \text{ns}^{-1}(z \mid 1) == \coth^{-1}(z)$$
$$\text{sc}^{-1}(z \mid 0) == \tan^{-1}(z) \quad \text{sc}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == -i\,\sqrt{2}\, F\left(i \sinh^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, 2\right) \quad \text{sc}^{-1}(z \mid 1) == \sinh^{-1}(z)$$
$$\text{sd}^{-1}(z \mid 0) == \sin^{-1}(z) \quad \text{sd}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == -i\,\sqrt{2}\, F\left(i \sinh^{-1}\left(\frac{z}{\sqrt{2}}\right) \,\middle|\, -1\right) /; z > -1 \quad \text{sd}^{-1}(z \mid 1) == \sinh^{-1}(z)$$
$$\text{sn}^{-1}(z \mid 0) == \sin^{-1}(z) \quad \text{sn}^{-1}\left(z \,\middle|\, \tfrac{1}{2}\right) == F\left(\sin^{-1}(z) \,\middle|\, \tfrac{1}{2}\right) \quad \text{sn}^{-1}(z \mid 1) == \tanh^{-1}(z).$$

At the points $z == -1,\ -1/2,\ 0,\ 1/2,\ 1$, and i, the inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, $\text{sn}^{-1}(z \mid m)$ have the following representations through the elliptic integrals $F(w(z, m) \mid n)$ and $K(u(m))$:

$$\text{cd}^{-1}(-1 \mid m) == 2\,K(m) \quad \text{cd}^{-1}\left(-\tfrac{1}{2} \,\middle|\, m\right) == F\left(\tfrac{\pi}{6} \,\middle|\, m\right) + K(m) \quad \text{cd}^{-1}(0 \mid m) == K(m)$$
$$\text{cd}^{-1}\left(\tfrac{1}{2} \,\middle|\, m\right) == K(m) - F\left(\tfrac{\pi}{6} \,\middle|\, m\right) \quad \text{cd}^{-1}(1 \mid m) == 0 \quad \text{cd}^{-1}(i \mid m) == K(m) - F\left(\sin^{-1}(i) \mid m\right)$$

$\text{cn}^{-1}(-1 \mid m) = 2 K(m) \quad \text{cn}^{-1}\left(-\frac{1}{2} \middle| m\right) = F\left(\frac{2\pi}{3} \middle| m\right) \quad \text{cn}^{-1}(0 \mid m) = K(m) /; m \in \mathbb{R} \wedge m < 1$

$\text{cn}^{-1}\left(\frac{1}{2} \middle| m\right) = F\left(\frac{\pi}{3} \middle| m\right) \quad \text{cn}^{-1}(1 \mid m) = 0 \quad \text{cn}^{-1}(i \mid m) = \frac{1}{\sqrt{m-1}} \left(i \left(K\left(\frac{m}{m-1}\right) - F\left(\sin^{-1}(i) \middle| \frac{m}{m-1}\right)\right)\right)$

$\text{cs}^{-1}(-1 \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - i F\left(i \sinh^{-1}(1) \middle| \frac{1}{1-m}\right)\right) \quad \text{cs}^{-1}\left(-\frac{1}{2} \middle| m\right) = -i F\left(i \sinh^{-1}(2) \middle| 1-m\right) - \frac{2i}{\sqrt{1-m}} F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \middle| \frac{1}{1-m}\right) \quad \text{c}$

$\text{cs}^{-1}\left(\frac{1}{2} \middle| m\right) = -i F\left(i \sinh^{-1}(2) \middle| 1-m\right) \quad \text{cs}^{-1}(1 \mid m) = \frac{i}{\sqrt{1-m}} \left(F\left(i \sinh^{-1}(1) \middle| \frac{1}{1-m}\right) + K\left(\frac{m}{m-1}\right)\right) \quad \text{c}$

$\text{dc}^{-1}(-1 \mid m) = \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m > 1 \quad \text{dc}^{-1}\left(-\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + F\left(\frac{\pi}{6} \middle| \frac{1}{m}\right)\right) /; m > 1 \quad \text{dc}^{-1}(0 \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m >$

$\text{dc}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - F\left(\frac{\pi}{6} \middle| \frac{1}{m}\right)\right) /; m > 1 \quad \text{dc}^{-1}(1 \mid m) = 0 \quad \text{dc}^{-1}(i \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - F(\text{s}\right.$

$\text{dn}^{-1}(-1 \mid m) = \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1 \quad \text{dn}^{-1}\left(-\frac{1}{2} \middle| m\right) = -\frac{1}{\sqrt{m-1}} \left(F\left(\frac{\pi}{6} \middle| \frac{1}{1-m}\right) + K\left(\frac{1}{1-m}\right)\right) /; 0 < m < 1 \quad \text{dn}^{-1}(0 \mid m) =$

$\text{dn}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - F\left(\frac{\pi}{6} \middle| \frac{1}{1-m}\right)\right) /; m > 1 \quad \text{dn}^{-1}(1 \mid m) = 0 \quad \text{dn}^{-1}(i \mid m) =$

$\text{ds}^{-1}(-1 \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + i F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right)\right) /; m > 1 \quad \text{ds}^{-1}\left(-\frac{1}{2} \middle| m\right) = \frac{2i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{1}{2\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}(2\right.$

$\text{ds}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(2\sqrt{1-m}\right) \middle| \frac{m}{m-1}\right) /; m > 1 \quad \text{ds}^{-1}(1 \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - i F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right)\right) /; m > 1$

$\text{nc}^{-1}(-1 \mid m) = \frac{2i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) \quad \text{nc}^{-1}\left(-\frac{1}{2} \middle| m\right) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) + F\left(\frac{\pi}{6} \middle| \frac{m-1}{m}\right)\right) \quad \text{nc}^{-1}(0 \mid m) = \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right)$

$\text{nc}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - F\left(\frac{\pi}{6} \middle| \frac{m-1}{m}\right)\right) \quad \text{nc}^{-1}(1 \mid m) = 0 \quad \text{nc}^{-1}(i \mid m) = \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - F\left(\sin^{-1}(i) \right|\right.$

$\text{nd}^{-1}(-1 \mid m) = 2 i K(1-m) \quad \text{nd}^{-1}\left(-\frac{1}{2} \middle| m\right) = i \left(F\left(\frac{\pi}{6} \middle| 1-m\right) + K(1-m)\right) \quad \text{nd}^{-1}(0 \mid m) = i K(1-m)$

$\text{nd}^{-1}\left(\frac{1}{2} \middle| m\right) = i \left(K(1-m) - F\left(\frac{\pi}{6} \middle| 1-m\right)\right) \quad \text{nd}^{-1}(1 \mid m) = 0 \quad \text{nd}^{-1}(i \mid m) = i K(1-m) - i F\left(i \sinh^{-1}(1)\right.$

$\text{ns}^{-1}(-1 \mid m) = -K(m) \quad \text{ns}^{-1}\left(-\frac{1}{2} \middle| m\right) = K(m) - \frac{1}{\sqrt{m}} \left(F\left(\frac{\pi}{6} \middle| \frac{1}{m}\right) + K\left(\frac{1}{m}\right)\right) \quad \text{ns}^{-1}(0 \mid m) = K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)$

$\text{ns}^{-1}\left(\frac{1}{2} \middle| m\right) = \frac{1}{\sqrt{m}} \left(F\left(\frac{\pi}{6} \middle| \frac{1}{m}\right) - K\left(\frac{1}{m}\right)\right) + K(m) \quad \text{ns}^{-1}(1 \mid m) = K(m) \quad \text{ns}^{-1}(i \mid m) = K(m) - \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right)\right.$

$\text{sc}^{-1}(-1 \mid m) = i F\left(i \sinh^{-1}(1) \middle| 1-m\right) \quad \text{sc}^{-1}\left(-\frac{1}{2} \middle| m\right) = i F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \middle| 1-m\right) \quad \text{sc}^{-1}(0 \mid m) = 0$

$\text{sc}^{-1}\left(\frac{1}{2} \middle| m\right) = -i F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \middle| 1-m\right) \quad \text{sc}^{-1}(1 \mid m) = -i F\left(i \sinh^{-1}(1) \middle| 1-m\right) \quad \text{sc}^{-1}(i \mid m) = i K(1-m)$

$\text{sd}^{-1}(-1 \mid m) = \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\sqrt{m}\right) \middle| \frac{m-1}{m}\right) /; m > 0 \quad \text{sd}^{-1}\left(-\frac{1}{2} \middle| m\right) = \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \middle| \frac{m-1}{m}\right) /; m > 0 \quad \text{sd}^{-1}(0 \mid m) = 0$

$\text{sd}^{-1}\left(\frac{1}{2} \middle| m\right) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \middle| \frac{m-1}{m}\right) /; m > 0 \quad \text{sd}^{-1}(1 \mid m) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\sqrt{m}\right) \middle| \frac{m-1}{m}\right) /; m > 0 \quad \text{sd}^{-1}(i \mid m) = \frac{i}{\sqrt{m}} F$

$\text{sn}^{-1}(-1 \mid m) = -K(m) \quad \text{sn}^{-1}\left(-\frac{1}{2} \middle| m\right) = -F\left(\frac{\pi}{6} \middle| m\right) \quad \text{sn}^{-1}(0 \mid m) = 0$

$\text{sn}^{-1}\left(\frac{1}{2} \middle| m\right) = F\left(\frac{\pi}{6} \middle| m\right) \quad \text{sn}^{-1}(1 \mid m) = K(m) \quad \text{sn}^{-1}(i \mid m) = F\left(i \sinh^{-1}(1) \middle| m\right).$

At the points $m = \pm\infty$ or $z = \pm\infty$, the inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ have the following values:

$\text{cd}^{-1}(z \mid \infty) = 0 \quad \text{cd}^{-1}(z \mid -\infty) = 0$

$\text{cd}^{-1}(\infty \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) \quad \text{cd}^{-1}(-\infty \mid m) = 2 K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)$

$\text{cn}^{-1}(z \mid \infty) = 0 \quad \text{cn}^{-1}(z \mid -\infty) = 0$

$\text{cn}^{-1}(\infty \mid m) = -\frac{i}{\sqrt{m}} K\left(1 - \frac{1}{m}\right) \quad \text{cn}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right)$

$\text{cs}^{-1}(z \mid \infty) = 0 \quad \text{cs}^{-1}(z \mid -\infty) = 0$

$\text{cs}^{-1}(\infty \mid m) = 0 \quad \text{cs}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right)$

$\text{dc}^{-1}(z \mid \infty) = 0 \quad \text{dc}^{-1}(z \mid -\infty) = 0$

$\text{dc}^{-1}(\infty \mid m) = K(m) \quad \text{dc}^{-1}(-\infty \mid m) = -2 i K(1-m) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) - K(m) \text{ /; } m > 1$

$\text{dn}^{-1}(z \mid \infty) = 0 \quad \text{dn}^{-1}(z \mid -\infty) = 0$

$\text{dn}^{-1}(\infty \mid m) = i K(1-m) \quad \text{dn}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - i K(1-m) \text{ /; } m > 1$

$\text{ds}^{-1}(z \mid \infty) = 0 \quad \text{ds}^{-1}(z \mid -\infty) = 0$

$\text{ds}^{-1}(\infty \mid m) = 0 \quad \text{ds}^{-1}(-\infty \mid m) = \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) \text{ /; } m > 1$

$\text{nc}^{-1}(z \mid \infty) = 0 \quad \text{nc}^{-1}(z \mid -\infty) = 0$

$\text{nc}^{-1}(\infty \mid m) = -\frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) \quad \text{nc}^{-1}(-\infty \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{2 i}{\sqrt{m}} K\left(\frac{m-1}{m}\right)$

$\text{nd}^{-1}(z \mid \infty) = 0 \quad \text{nd}^{-1}(z \mid -\infty) = 0$

$\text{nd}^{-1}(\infty \mid m) = -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) \quad \text{nd}^{-1}(-\infty \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + 2 i K(1-m)$

$\text{ns}^{-1}(z \mid \infty) = 0 \quad \text{ns}^{-1}(z \mid -\infty) = 0$

$\text{ns}^{-1}(\infty \mid m) = 0 \quad \text{ns}^{-1}(-\infty \mid m) = 2 K(m) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right)$

$\text{sc}^{-1}(z \mid \infty) = 0 \quad \text{sc}^{-1}(z \mid -\infty) = 0$

$\text{sc}^{-1}(\infty \mid m) = K(m) \quad \text{sc}^{-1}(-\infty \mid m) = -K(m)$

$\text{sd}^{-1}(z \mid \infty) = 0 \quad \text{sd}^{-1}(z \mid -\infty) = 0$

$\text{sd}^{-1}(\infty \mid m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) \text{ /; } m > 1 \quad \text{sd}^{-1}(-\infty \mid m) = -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) \text{ /; } m > 1$

$\text{sn}^{-1}(z \mid \infty) = 0 \quad \text{sn}^{-1}(z \mid -\infty) = 0$

$\text{sn}^{-1}(\infty \mid m) = K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) \text{ /; } m > 1 \quad \text{sn}^{-1}(-\infty \mid m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - K(m) \text{ /; } m > 1.$

Analyticity

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ are analytical functions of z and m that are defined over $\mathbb{C}^2$.

Poles and essential singularities

The inverse Jacobi functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ do not have poles and essential singularities with respect to z and m.

Branch points and branch cuts

For fixed z, the point $m = \tilde{\infty}$ is the branch point for all twelve inverse Jacobi functions. Other branch points are the following: $m = \frac{1}{z^2}$ for $\mathrm{cd}^{-1}(z \mid m)$, $m = \frac{1}{1-z^2}$ for $\mathrm{cn}^{-1}(z \mid m)$, $m = 1 + z^2$ for $\mathrm{cs}^{-1}(z \mid m)$, $m = z^2$ for $\mathrm{dc}^{-1}(z \mid m)$, $m = 1 - z^2$ for $\mathrm{dn}^{-1}(z \mid m)$, $m = -z^2$ and $m == 1 - z^2$ for $\mathrm{ds}^{-1}(z \mid m)$, $m = \frac{z^2}{z^2-1}$ for $\mathrm{nc}^{-1}(z \mid m)$, $m = \frac{z^2-1}{z^2}$ for $\mathrm{nd}^{-1}(z \mid m)$, $m = z^2$ for $\mathrm{ns}^{-1}(z \mid m)$, $m = \frac{z^2+1}{z^2}$ for $\mathrm{sc}^{-1}(z \mid m)$, $m = -\frac{1}{z^2}$ and $m == \frac{z^2-1}{z^2}$ for $\mathrm{sd}^{-1}(z \mid m)$, and $m = \frac{1}{z^2}$ for $\mathrm{sn}^{-1}(z \mid m)$.

For fixed m, the point $m = \tilde{\infty}$ is the branch point for all twelve inverse Jacobi functions. There are four or five other branch points that include the following: $z = \pm 1, z == \pm \frac{1}{\sqrt{m}}$ for $\mathrm{cd}^{-1}(z \mid m)$, $z = \pm i, z == \pm \sqrt{m-1}$ for $\mathrm{cn}^{-1}(z \mid m)$, $z = \pm i, z == \pm \sqrt{m-1}$ for $\mathrm{cs}^{-1}(z \mid m)$, $z == 0, z = \pm 1, z == \pm \sqrt{m}$ for $\mathrm{dc}^{-1}(z \mid m)$, $z == 0$, $z = \pm 1, z == \pm \sqrt{1-m}$ for $\mathrm{dn}^{-1}(z \mid m)$, $z == 0$, $z = \pm \sqrt{-m}, z == \pm \sqrt{1-m}$ for $\mathrm{ds}^{-1}(z \mid m)$, $z = \pm 1, z == \pm \sqrt{\frac{m}{m-1}}$ for $\mathrm{nc}^{-1}(z \mid m)$, $z = \pm 1, z == \pm \frac{1}{\sqrt{1-m}}$ for $\mathrm{nd}^{-1}(z \mid m)$, $z = \pm 1, z == \pm \sqrt{m}$ for $\mathrm{ns}^{-1}(z \mid m)$, $z = \pm i, z == \pm \frac{1}{\sqrt{m-1}}$ for $\mathrm{sc}^{-1}(z \mid m)$, $z = \pm \frac{1}{\sqrt{-m}}, z == \pm \frac{1}{\sqrt{1-m}}$ for $\mathrm{sd}^{-1}(z \mid m)$, and $z = \pm 1, z == \pm \frac{1}{\sqrt{m}}$ for $\mathrm{sn}^{-1}(z \mid m)$.

Parity and symmetry

The inverse Jacobi functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ have mirror symmetry:

$$\mathrm{cd}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{cd}^{-1}(z \mid m)} \quad \mathrm{cn}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{cn}^{-1}(z \mid m)} \quad \mathrm{cs}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{cs}^{-1}(z \mid m)}$$
$$\mathrm{dc}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{dc}^{-1}(z \mid m)} \quad \mathrm{dn}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{dn}^{-1}(z \mid m)} \quad \mathrm{ds}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{ds}^{-1}(z \mid m)}$$
$$\mathrm{nc}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{nc}^{-1}(z \mid m)} \quad \mathrm{nd}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{nd}^{-1}(z \mid m)} \quad \mathrm{ns}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{ns}^{-1}(z \mid m)}$$
$$\mathrm{sc}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{sc}^{-1}(z \mid m)} \quad \mathrm{sd}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{sd}^{-1}(z \mid m)} \quad \mathrm{sn}^{-1}(\bar{z} \mid \bar{m}) == \overline{\mathrm{sn}^{-1}(z \mid m)}.$$

Nine inverse Jacobi functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$ have the following quasi-reflection symmetry with respect to z:

$$\mathrm{cd}^{-1}(-z \mid m) == 2\,K(m) - \mathrm{cd}^{-1}(z \mid m) \quad \mathrm{cn}^{-1}(-z \mid m) == \frac{2}{\sqrt{1-m}}\,F\left(\sin^{-1}(z) \,\middle|\, \frac{m}{m-1}\right) + \mathrm{cn}^{-1}(z \mid m)$$

$$\mathrm{cs}^{-1}(-z \mid m) == \mathrm{cs}^{-1}(z \mid m) - \frac{2\,i}{\sqrt{1-m}}\,F\left(i \sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) \quad \mathrm{dc}^{-1}(-z \mid m) == \frac{2}{\sqrt{m}}\,F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right) + \mathrm{dc}^{-1}(z \mid m)$$

$$\mathrm{dn}^{-1}(-z \mid m) == \mathrm{dn}^{-1}(z \mid m) - \frac{2}{\sqrt{m-1}}\,F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) \quad \mathrm{ds}^{-1}(-z \mid m) == \frac{2\,i}{\sqrt{m}}\,F\left(\sin^{-1}\left(\frac{z}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) + \mathrm{ds}^{-1}(z \mid m)$$

$$\mathrm{nc}^{-1}(-z \mid m) == \frac{2\,i}{\sqrt{m}}\,F\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) + \mathrm{nc}^{-1}(z \mid m) \;/;\; m < 1 \quad \mathrm{nd}^{-1}(-z \mid m) == 2\,i\,F\left(\sin^{-1}(z) \,\middle|\, 1-m\right) + \mathrm{nd}^{-1}(z \mid m)$$

$$\text{ns}^{-1}(-z \mid m) = \text{ns}^{-1}(z \mid m) - \frac{2}{\sqrt{m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right).$$

The other three inverse Jacobi functions $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ are odd functions with respect to z:

$$\text{sc}^{-1}(-z \mid m) = -\text{sc}^{-1}(z \mid m) \quad \text{sd}^{-1}(-z \mid m) = -\text{sd}^{-1}(z \mid m) \quad \text{sn}^{-1}(-z \mid m) = -\text{sn}^{-1}(z \mid m).$$

Series representations

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ have the following series expansions at the point $z = 0$:

$$\text{cd}^{-1}(z \mid m) = K(m) - z - \frac{m+1}{6} z^3 - \frac{3+2m+3m^2}{40} z^5 - \ldots \;/;\; (z \to 0)$$

$$\text{cn}^{-1}(z \mid m) = K(m) - \frac{z}{\sqrt{1-m}} \left(1 + \frac{2m-1}{6(m-1)} z^2 + \frac{3-8m+8m^2}{40(m-1)^2} z^4 + \ldots\right) /;\; (z \to 0)$$

$$\text{cs}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m) - \frac{z}{\sqrt{m-1}} \left(1 - \frac{m-2}{6(m-1)} z^2 + \frac{8-8m+3m^2}{40(m-1)^2} z^4 - \ldots\right)\right) /;\; (z \to 0)$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - z - \frac{(1+m)z^3}{6m} - \frac{\left(3+2m+3m^2\right) z^5}{40m^2} \ldots\right) /;\; (z \to 0)$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\left(\frac{1}{1-m}\right) - z - \frac{(2-m)z^3}{6(1-m)} - \frac{\left(8-8m+3m^2\right) z^5}{40(-1+m)^2} - \ldots\right) /;\; (z \to 0)$$

$$\text{ds}^{-1}(z \mid m) = \sqrt{\frac{1}{m}}\, K\left(\frac{1}{m}\right) - \frac{z}{\sqrt{m-1}\,\sqrt{m}} + \frac{(2m-1)z^3}{6(m-1)^{3/2} m^{3/2}} - \ldots \;/;\; (z \to 0)$$

$$\text{nc}^{-1}(z \mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{m}\,\sqrt{z^2-1}} \left(z + \frac{2m-1}{6m} z^3 + \frac{8m^2-8m+3}{40m^2} z^5 + \ldots\right) /;\; (z \to 0)$$

$$\text{nd}^{-1}(z \mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \left(z + \frac{2-m}{6} z^3 + \frac{8-8m+3m^2}{40} z^5 + \ldots\right) /;\; (z \to 0)$$

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{-m}} \left(\left(-\frac{1}{m}\right)^{-1/2} K(m) + i\,K\left(\frac{1}{m}\right)\right) + \frac{1}{\sqrt{m}} \left(z + \frac{1+m}{6m} z^3 + \frac{3+2m+3m^2}{40m^2} z^5 + \ldots\right) /;\; (z \to 0)$$

$$\text{sc}^{-1}(z \mid m) = z + \frac{m-2}{6} z^3 + \frac{3m^2-8m+8}{40} z^5 + \ldots \;/;\; (z \to 0)$$

$$\text{sd}^{-1}(z \mid m) = z + \frac{1-2m}{6} z^3 + \frac{3-8m+8m^2}{40} z^5 - \ldots \;/;\; (z \to 0)$$

$$\text{sn}^{-1}(z \mid m) = z + \frac{1+m}{6} z^3 + \frac{3+2m+3m^2}{40} z^5 + \ldots /; (z \to 0).$$

The previous expansions are the particular cases of the following series representations of the twelve inverse Jacobi functions near the point $z = 0$:

$$\text{cd}^{-1}(z \mid m) = K(m) - \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{m}\right) z^{2k+1}$$

$$\text{cn}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m}{m-1}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m-1}{m}\right) z^{2k+1}$$

$$\text{cs}^{-1}(z \mid m) = i \left(\frac{1}{\sqrt{1-m}} K\!\left(\frac{1}{1-m}\right) - K(1-m) - \sum_{k=0}^{\infty} \frac{(m-1)^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1} \right)$$

$$\text{dc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} K\!\left(\frac{1}{m}\right) - \frac{1}{\sqrt{m}} \sum_{k=0}^{\infty} \frac{m^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; m\right) z^{2k+1}$$

$$\text{dn}^{-1}(z \mid m) = \frac{1}{\sqrt{m-1}} \left(K\!\left(\frac{1}{1-m}\right) - \sum_{k=0}^{\infty} \frac{(1-m)^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1} \right)$$

$$\text{ds}^{-1}(z \mid m) = \sqrt{\frac{1}{m}} \, K\!\left(\frac{1}{m}\right) - \frac{1}{\sqrt{m-1}\,\sqrt{m}} \sum_{k=0}^{\infty} \frac{(-1)^k m^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\text{nc}^{-1}(z \mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}\,\sqrt{m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m-1}{m}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\text{nd}^{-1}(z \mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{1-m}\right) z^{2k+1}$$

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{-m}} \left(\left(-\frac{1}{m}\right)^{-1/2} K(m) + i\,K\!\left(\frac{1}{m}\right) \right) + \sum_{k=0}^{\infty} \frac{m^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; m\right) z^{2k+1}$$

$$\text{sc}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(m-1)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{1-m}\right) z^{2k+1}$$

$$\text{sd}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\text{sn}^{-1}(z\mid m)=\sum_{k=0}^{\infty}\frac{m^k\left(\frac{1}{2}\right)_k}{(2k+1)\,k!}\,{}_2F_1\!\left(\frac{1}{2},-k;\frac{1}{2}-k;\frac{1}{m}\right)z^{2k+1}.$$

The inverse Jacobi functions $\text{cd}^{-1}(z\mid m)$, $\text{cn}^{-1}(z\mid m)$, $\text{cs}^{-1}(z\mid m)$, $\text{dc}^{-1}(z\mid m)$, $\text{dn}^{-1}(z\mid m)$, $\text{ds}^{-1}(z\mid m)$, $\text{nc}^{-1}(z\mid m)$, $\text{nd}^{-1}(z\mid m)$, $\text{ns}^{-1}(z\mid m)$, $\text{sc}^{-1}(z\mid m)$, $\text{sd}^{-1}(z\mid m)$, and $\text{sn}^{-1}(z\mid m)$ have the following series expansions at the point $m=0$:

$$\text{cd}^{-1}(z\mid m)=\cos^{-1}(z)+\frac{1}{4}\left(\sqrt{1-z^2}\,z+\cos^{-1}(z)\right)m+\frac{3}{64}\left(z\sqrt{1-z^2}\,(2z^2+3)+\cos^{-1}(z)\right)m^2+\ldots\;/;\;(m\to 0)$$

$$\text{cn}^{-1}(z\mid m)=\cos^{-1}(z)+\frac{1}{4}\left(\cos^{-1}(z)-z\sqrt{1-z^2}\right)m+\frac{3}{64}\left((2z^2-5)\sqrt{1-z^2}\,z+3\cos^{-1}(z)\right)m^2+\ldots\;/;\;(m\to 0)$$

$$\text{cs}^{-1}(z\mid m)=\cot^{-1}(z)+\frac{(z^2+1)\cot^{-1}(z)-z}{4(z^2+1)}m+\frac{3\left(-3z^3-5z+3(z^2+1)^2\cot^{-1}(z)\right)}{64(z^2+1)^2}m^2+\ldots\;/;\;(m\to 0)$$

$$\text{dc}^{-1}(z\mid m)=\sec^{-1}(z)+\frac{1}{4z}\left(z\sec^{-1}(z)+\sqrt{1-\frac{1}{z^2}}\right)m+\frac{3}{64z^3}\left(3\sec^{-1}(z)\,z^3+(3z^2+2)\sqrt{1-\frac{1}{z^2}}\right)m^2+\ldots\;/;\;(m\to 0)$$

$$\text{dn}^{-1}(z\mid m)=\frac{\sqrt{1-m}}{\sqrt{m-1}}\left(-\frac{1}{2}\left(1+\frac{m}{4}+\frac{9m^2}{64}+\ldots\right)\log(-m)+\log(4)+\frac{\log(4)-1}{4}m+\frac{3(6\log(4)-7)}{128}m^2+\ldots\right)-$$
$$\frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}}\left(\frac{\tanh^{-1}(z)}{z}+\frac{1}{4}\left(\frac{\tanh^{-1}(z)}{z}+\frac{1}{1-z^2}\right)m+\frac{3\left(-3z^3+5z+3(z^2-1)^2\tanh^{-1}(z)\right)}{64z(z^2-1)^2}m^2+\ldots\right)/;\;(m\to 0)$$

$$\text{ds}^{-1}(z\mid m)=$$
$$\csc^{-1}(z)+\frac{1}{4}\left(\csc^{-1}(z)+\frac{z^2+1}{z(1-z^2)}\sqrt{1-\frac{1}{z^2}}\right)m+\frac{1}{64}\left(9\csc^{-1}(z)-\frac{(9z^6-12z^4-11z^2+6)}{z^3(z^2-1)^2}\sqrt{1-\frac{1}{z^2}}\right)m^2+\ldots\;/;\;(m\to 0)$$

$$\text{nc}^{-1}(z\mid m)=\sec^{-1}(z)+\frac{1}{4z}\left(z\sec^{-1}(z)-\sqrt{1-\frac{1}{z^2}}\right)m+\frac{3}{64z^3}\left(3\sec^{-1}(z)\,z^3+(2-5z^2)\sqrt{1-\frac{1}{z^2}}\right)m^2+\ldots\;/;\;(m\to 0)$$

$$\text{nd}^{-1}(z\mid m)=\left(-\frac{i}{2}\left(1+\frac{m}{4}+\frac{9m^2}{64}+\ldots\right)\log(m)+i\log(4)+\frac{i}{4}(\log(4)-1)\,m+\frac{3i}{128}(6\log(4)-7)\,m^2+\ldots\right)+$$
$$\frac{\sqrt{1-z^2}}{\sqrt{z^2-1}}\left(\tanh^{-1}(z)+\frac{1}{4}\left(\frac{z}{z^2-1}+\tanh^{-1}(z)\right)m+\frac{3}{64}\left(\frac{(5z^2-3)\,z}{(z^2-1)^2}+3\tanh^{-1}(z)\right)m^2+\ldots\right)/;\;(m\to 0)$$

$$\text{ns}^{-1}(z\mid m)=\csc^{-1}(z)-\frac{1}{4z}\left(\sqrt{1-\frac{1}{z^2}}-z\csc^{-1}(z)\right)m-\frac{3}{64z^3}\left(\sqrt{1-\frac{1}{z^2}}\,(3z^2+2)-3z^3\csc^{-1}(z)\right)m^2-\ldots\;/;\;(m\to 0)$$

$$\text{sc}^{-1}(z\mid m) = \tan^{-1}(z) + \frac{(z^2+1)\tan^{-1}(z) - z}{4(z^2+1)}\, m + \frac{3\left(-5z^3 - 3z + 3(z^2+1)^2\tan^{-1}(z)\right)}{64(z^2+1)^2}\, m^2 + \ldots \;/;\; (m \to 0)$$

$$\text{sd}^{-1}(z\mid m) = \sin^{-1}(z) + \frac{z\sqrt{1-z^2}\,(z^2+1) + (z^2-1)\sin^{-1}(z)}{4(z^2-1)}\, m - \frac{z\sqrt{1-z^2}\,(6z^6 - 11z^4 - 12z^2 + 9) - 9(z^2-1)^2\sin^{-1}(z)}{64(z^2-1)^2}\, m^2 + \ldots \;/;\; (m \to 0)$$

$$\text{sn}^{-1}(z\mid m) = \sin^{-1}(z) - \frac{1}{4}\left(z\sqrt{1-z^2} - \sin^{-1}(z)\right) m - \frac{3}{64}\left(z(2z^2+3)\sqrt{1-z^2} - 3\sin^{-1}(z)\right) m^2 + \ldots \;/;\; (m \to 0).$$

The previous expansions are the particular cases of the following series representations of the twelve inverse Jacobi functions near the point $m = 0$:

$$\text{cd}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\left(\frac{\pi\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{2\,k!^2} - \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)\,k!}\,{}_2F_1\!\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; z^2\right)\right) m^k$$

$$\text{cn}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{\sqrt{\pi}}{2\,k!}\,\Gamma\!\left(k+\frac{1}{2}\right) - z\,{}_2F_1\!\left(\frac{1}{2}, \frac{1}{2}-k; \frac{3}{2}; z^2\right)\right) m^k$$

$$\text{cs}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k z^{-2k-1}}{(2k+1)\,k!}\,{}_2F_1\!\left(k+\frac{1}{2}, k+1; k+\frac{3}{2}; -\frac{1}{z^2}\right) m^k$$

$$\text{dc}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{\sqrt{\pi}\,\Gamma\!\left(k+\frac{3}{2}\right)}{(2k+1)\,k!} - \frac{z^{-2k-1}}{2k+1}\,{}_2F_1\!\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; \frac{1}{z^2}\right)\right) m^k$$

$$\text{dn}^{-1}(z\mid m) = \frac{1}{\sqrt{m-1}}\,K\!\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\,{}_2F_1\!\left(\frac{1}{2}, k+1; \frac{3}{2}; z^2\right) m^k$$

$$\text{ds}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{z^{-2j-2k-1}(-1)^{j+k}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k}{(2j+2k+1)\,j!\,k!}\,{}_2F_1\!\left(j+\frac{1}{2}, j+k+\frac{1}{2}; j+k+\frac{3}{2}; \frac{1}{z^2}\right) m^{j+k}$$

$$\text{nc}^{-1}(z\mid m) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{\sqrt{\pi}}{2\,k!}\,\Gamma\!\left(k+\frac{1}{2}\right) - \frac{1}{z}\,{}_2F_1\!\left(\frac{1}{2}, \frac{1}{2}-k; \frac{3}{2}; \frac{1}{z^2}\right)\right) m^k$$

$$\text{nd}^{-1}(z\mid m) = i\,K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)\,k!}\,{}_2F_1\!\left(k+\frac{1}{2}, k+1; k+\frac{3}{2}; z^2\right) m^k$$

$$\text{ns}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{z^{-2k-1} \left(\frac{1}{2}\right)_k}{k!\,(2k+1)}\, {}_2F_1\!\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; \frac{1}{z^2}\right) m^k$$

$$\text{sc}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{z^{2k+1} \left(\frac{1}{2}\right)_k}{k!\,(2k+1)}\, {}_2F_1\!\left(k+\frac{1}{2}, k+1; k+\frac{3}{2}; -z^2\right) m^k$$

$$\text{sd}^{-1}(z \mid m) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-1)^j \left(\frac{1}{2}\right)_{j-k} \left(\frac{1}{2}\right)_k}{(2j+1)\,(j-k)!\,k!}\, z^{2j+1}\, {}_2F_1\!\left(j+\frac{1}{2}, k+\frac{1}{2}; j+\frac{3}{2}; z^2\right) m^j$$

$$\text{sn}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)\,k!}\, {}_2F_1\!\left(\frac{1}{2}, k+\frac{1}{2}; k+\frac{3}{2}; z^2\right) m^k.$$

Integral representations

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ have the following integral representations, which can be used for their definitions:

$$\text{cd}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt \;/;\; -1 < z < 1 \land m < 1$$

$$\text{cn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{m\,t^2-m+1}}\,dt \;/;\; -1 < z < 1 \bigwedge m\left(z^2-1\right) > -1$$

$$\text{cs}^{-1}(z \mid m) = \int_z^{\infty} \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 - m > -1$$

$$\text{dc}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 - m > 0 \bigwedge m < 1$$

$$\text{dn}^{-1}(z \mid m) = \int_z^1 \frac{1}{\sqrt{1-t^2}\,\sqrt{t^2+m-1}}\,dt \;/;\; -1 < z < 1 \bigwedge z^2 + m > 1$$

$$\text{ds}^{-1}(z \mid m) = \int_z^{\infty} \frac{1}{\sqrt{t^2+m}\,\sqrt{t^2+m-1}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 + m > 1$$

$$\text{nc}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{(1-m)\,t^2+m}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\,z^2 + m > 0$$

$$\text{nd}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2-1}\,\sqrt{1-(1-m)\,t^2}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\,z^2 < 1 \bigwedge m > 0$$

$$\text{ns}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\,dt\ /;\ z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 > m$$

$$\text{sc}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{t^2+1}\,\sqrt{(1-m)\,t^2+1}}\,dt\ /;\ z \in \mathbb{R} \bigwedge (1-m)\,z^2 > -1$$

$$\text{sd}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{m\,t^2+1}\,\sqrt{1-(1-m)\,t^2}}\,dt\ /;\ z \in \mathbb{R} \bigwedge m\,z^2 > -1 \bigwedge (1-m)\,z^2 < 1$$

$$\text{sn}^{-1}(z \mid m) = \int_0^z \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt\ /;\ -1 < z < 1 \bigwedge m\,z^2 < 1.$$

Transformations

Some inverse Jacobi functions satisfy additional formulas, for example:

$$\text{cn}^{-1}(z_1 \mid m) + \text{cn}^{-1}(z_2 \mid m) = \text{cn}^{-1}\left(\frac{z_1\,z_2 - \sqrt{\left(1-z_1^2\right)\left(m\,z_1^2+(1-m)\right)\left(1-z_2^2\right)\left(m\,z_2^2+(1-m)\right)}}{1-m\left(1-z_1^2\right)\left(1-z_2^2\right)} \,\middle|\, m\right)$$

$$\text{dn}^{-1}(z_1 \mid m) + \text{dn}^{-1}(z_2 \mid m) = \text{dn}^{-1}\left(\frac{m\,z_1\,z_2 + \sqrt{\left(1-z_1^2\right)\left(z_1^2+m-1\right)\left(1-z_2^2\right)\left(z_2^2+m-1\right)}}{m-\left(1-z_1^2\right)\left(1-z_2^2\right)} \,\middle|\, m\right)$$

$$\text{sn}^{-1}(z_1 \mid m) + \text{sn}^{-1}(z_2 \mid m) = \text{sn}^{-1}\left(\frac{\sqrt{\left(1-z_2^2\right)\left(1-m\,z_2^2\right)}\,z_1 + \sqrt{\left(1-z_1^2\right)\left(1-m\,z_1^2\right)}\,z_2}{1-m\,z_1^2\,z_2^2} \,\middle|\, m\right) /;$$

$$z_1 \in \mathbb{R} \bigwedge z_2 \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge -\frac{1}{m} < z_1 < \frac{1}{m} \bigwedge -\frac{1}{m} < z_2 < \frac{1}{m}.$$

Identities

The inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ satisfy nonlinear functional equations:

$$\left(\left(z_2^2-1\right)m\,z_1^2 - m\,z_2^2+1\right)\text{cd}(w(z_1)+w(z_2) \mid m)^2 + 2\,(m-1)\,z_1\,z_2\,\text{cd}(w(z_1)+w(z_2) \mid m) + z_2^2 + z_1^2\left(1-m\,z_2^2\right) = 1\ /;\ w(z) = \text{cd}^{-1}(z \mid m)$$

$$\left(\left(z_2^2-1\right)m\,z_1^2 - m\,z_2^2+m-1\right)\text{cn}(w(z_1)+w(z_2) \mid m)^2 + 2\,z_1\,z_2\,\text{cn}(w(z_1)+w(z_2) \mid m) + \left(-m\,z_2^2+m-1\right)z_1^2 + (m-1)\left(z_2^2-1\right) = 0\ /;\ w(z) = \text{cn}^{-1}(z \mid m)$$

$$\left(z_1^2-z_2^2\right)^2\text{cs}(w(z_1)+w(z_2) \mid m)^4 + 2\left(-z_2^2\,z_1^4 + \left(-z_2^4+2\,(m-2)\,z_2^2+m-1\right)z_1^2 + (m-1)\,z_2^2\right)\text{cs}(w(z_1)+w(z_2) \mid m)^2 + \left(z_1^2\,z_2^2+m-1\right)^2 = 0\ /;\ w(z) = \text{cs}^{-1}(z \mid m)$$

$$\left(\left(z_2^2-1\right)z_1^2 - z_2^2+m\right)\text{dc}(w(z_1)+w(z_2) \mid m)^2 - 2\,(m-1)\,z_1\,z_2\,\text{dc}(w(z_1)+w(z_2) \mid m) + \left(z_2^2-1\right)m + z_1^2\left(m-z_2^2\right) = 0\ /;\ w(z) = \text{dc}^{-1}(z \mid m)$$

$$\left(z_2^2+m+\left(z_2^2-1\right)\left(-z_1^2\right)-1\right)\text{dn}(w(z_1)+w(z_2) \mid m)^2 - 2\,m\,z_1\,z_2\,\text{dn}(w(z_1)+w(z_2) \mid m) + \left(z_2^2+m-1\right)z_1^2 + (m-1)\left(z_2^2-1\right) = 0\ /;\ w(z) = \text{dn}^{-1}(z \mid m)$$

$$\left(z_1^2-z_2^2\right)^2 \operatorname{ds}(w(z_1)+w(z_2)\mid m)^4-2\left(z_2^2 z_1^4+\left(z_2^4+(4m-2)z_2^2+(m-1)m\right)z_1^2+(m-1)m z_2^2\right)\operatorname{ds}(w(z_1)+w(z_2)\mid m)^2+\left((m-1)m-z_1^2 z_2^2\right)^2=0\;/;\;w(z)=\operatorname{ds}^{-1}(z\mid m)$$

$$\left((m-1)\left(z_2^2-1\right)z_1^2-(m-1)z_2^2+m\right)\operatorname{nc}(w(z_1)+w(z_2)\mid m)^2-2z_1 z_2\operatorname{nc}(w(z_1)+w(z_2)\mid m)+\left(z_2^2-1\right)m+z_1^2\left(m-(m-1)z_2^2\right)=0\;/;\;w(z)=\operatorname{nc}^{-1}(z\mid m)$$

$$\left((m-1)\left(z_2^2-1\right)z_1^2-(m-1)z_2^2-1\right)\operatorname{nd}(w(z_1)+w(z_2)\mid m)^2+2m z_1 z_2\operatorname{nd}(w(z_1)+w(z_2)\mid m)-z_2^2-z_1^2\left((m-1)z_2^2+1\right)+1=0\;/;\;w(z)=\operatorname{nd}^{-1}(z\mid m)$$

$$\left(z_1^2-z_2^2\right)^2\operatorname{ns}(w(z_1)+w(z_2)\mid m)^4+\left(-2z_2^2 z_1^4+\left(-2z_2^4+4(m+1)z_2^2-2m\right)z_1^2-2m z_2^2\right)\operatorname{ns}(w(z_1)+w(z_2)\mid m)^2+\left(m-z_1^2 z_2^2\right)^2=0\;/;\;w(z)=\operatorname{ns}^{-1}(z\mid m)$$

$$\left((m-1)z_1^2 z_2^2+1\right)^2\operatorname{sc}(w(z_1)+w(z_2)\mid m)^4+2\left(z_1^2\left((m-1)z_2^4+\left((m-1)z_1^2+2(m-2)\right)z_2^2-1\right)-z_2^2\right)\operatorname{sc}(w(z_1)+w(z_2)\mid m)^2+\left(z_1^2-z_2^2\right)^2=0\;/;\;w(z)=\operatorname{sc}^{-1}(z\mid m)$$

$$\left((m-1)m z_1^2 z_2^2-1\right)^2\operatorname{sd}(w(z_1)+w(z_2)\mid m)^4-2\left(\left(\left((m-1)\left(z_1^2+z_2^2\right)m+4m-2\right)z_2^2+1\right)z_1^2+z_2^2\right)\operatorname{sd}(w(z_1)+w(z_2)\mid m)^2+\left(z_1^2-z_2^2\right)^2=0\;/;\;w(z)=\operatorname{sd}^{-1}(z\mid m)$$

$$\left(m z_1^2 z_2^2-1\right)^2\operatorname{sn}(w(z_1)+w(z_2)\mid m)^4-2\left(\left(\left(m\left(z_1^2+z_2^2\right)-2(m+1)\right)z_2^2+1\right)z_1^2+z_2^2\right)\operatorname{sn}(w(z_1)+w(z_2)\mid m)^2+\left(z_1^2-z_2^2\right)^2=0\;/;\;w(z)=\operatorname{sn}^{-1}(z\mid m).$$

Representations of derivatives

The derivatives of the inverse Jacobi functions $\operatorname{cd}^{-1}(z\mid m)$, $\operatorname{cn}^{-1}(z\mid m)$, $\operatorname{cs}^{-1}(z\mid m)$, $\operatorname{dc}^{-1}(z\mid m)$, $\operatorname{dn}^{-1}(z\mid m)$, $\operatorname{ds}^{-1}(z\mid m)$, $\operatorname{nc}^{-1}(z\mid m)$, $\operatorname{nd}^{-1}(z\mid m)$, $\operatorname{ns}^{-1}(z\mid m)$, $\operatorname{sc}^{-1}(z\mid m)$, $\operatorname{sd}^{-1}(z\mid m)$, and $\operatorname{sn}^{-1}(z\mid m)$ with respect to variable z can be expressed through direct and inverse Jacobi functions:

$$\frac{\partial\operatorname{cd}^{-1}(z\mid m)}{\partial z}=\frac{1}{(m-1)\operatorname{nd}\left(\operatorname{cd}^{-1}(z\mid m)\mid m\right)\operatorname{sd}\left(\operatorname{cd}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial\operatorname{cn}^{-1}(z\mid m)}{\partial z}=-\frac{1}{\operatorname{dn}\left(\operatorname{cn}^{-1}(z\mid m)\mid m\right)\operatorname{sn}\left(\operatorname{cn}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial\operatorname{cs}^{-1}(z\mid m)}{\partial z}=-\frac{1}{\operatorname{ds}\left(\operatorname{cs}^{-1}(z\mid m)\mid m\right)\operatorname{ns}\left(\operatorname{cs}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial\operatorname{dc}^{-1}(z\mid m)}{\partial z}=\frac{1}{(1-m)\operatorname{nc}\left(\operatorname{dc}^{-1}(z\mid m)\mid m\right)\operatorname{sc}\left(\operatorname{dc}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial\operatorname{dn}^{-1}(z\mid m)}{\partial z}=-\frac{1}{m\operatorname{cn}\left(\operatorname{dn}^{-1}(z\mid m)\mid m\right)\operatorname{sn}\left(\operatorname{dn}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial\operatorname{ds}^{-1}(z\mid m)}{\partial z}=-\frac{1}{\operatorname{cs}\left(\operatorname{ds}^{-1}(z\mid m)\mid m\right)\operatorname{ns}\left(\operatorname{ds}^{-1}(z\mid m)\mid m\right)}$$

$$\frac{\partial \text{nc}^{-1}(z \mid m)}{\partial z} = \frac{1}{\text{dc}\left(\text{nc}^{-1}(z \mid m) \mid m\right) \text{sc}\left(\text{nc}^{-1}(z \mid m) \mid m\right)}$$

$$\frac{\partial \text{nd}^{-1}(z \mid m)}{\partial z} = \frac{1}{m\, \text{cd}\left(\text{nd}^{-1}(z \mid m) \mid m\right) \text{sd}\left(\text{nd}^{-1}(z \mid m) \mid m\right)}$$

$$\frac{\partial \text{ns}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\text{cs}\left(\text{ns}^{-1}(z \mid m) \mid m\right) \text{ds}\left(\text{ns}^{-1}(z \mid m) \mid m\right)}$$

$$\frac{\partial \text{sc}^{-1}(z \mid m)}{\partial z} = \frac{1}{\text{dc}\left(\text{sc}^{-1}(z \mid m) \mid m\right) \text{nc}\left(\text{sc}^{-1}(z \mid m) \mid m\right)}$$

$$\frac{\partial \text{sd}^{-1}(z \mid m)}{\partial z} = \frac{1}{\text{cd}\left(\text{sd}^{-1}(z \mid m) \mid m\right) \text{nd}\left(\text{sd}^{-1}(z \mid m) \mid m\right)}$$

$$\frac{\partial \text{sn}^{-1}(z \mid m)}{\partial z} = \frac{1}{\text{cn}\left(\text{sn}^{-1}(z \mid m) \mid m\right) \text{dn}\left(\text{sn}^{-1}(z \mid m) \mid m\right)}.$$

The previous formulas can be generalized to the following symbolic derivatives of the n^{th} order with respect to variable z:

$$\frac{\partial^n \text{cd}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{cd}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} \left(1-z^2\right)^{-k-\frac{1}{2}} \left(1-m z^2\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \text{cn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{cn}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} \left(1-z^2\right)^{-k-\frac{1}{2}} \left(m z^2 - m + 1\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \text{cs}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{cs}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(-1)^j (1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2+1\right)^{-k-\frac{1}{2}} \left(z^2 - m + 1\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \text{dc}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{dc}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(-1)^{j-1} (1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2-1\right)^{-k-\frac{1}{2}} \left(z^2 - m\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \text{dn}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{dn}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(1-z^2\right)^{-k-\frac{1}{2}} \left(z^2 + m - 1\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \text{ds}^{-1}(z \mid m)}{\partial z^n} = \delta_n \text{ds}^{-1}(z \mid m) + \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2 + m - 1\right)^{-k-\frac{1}{2}} \left(z^2 + m\right)^{k-j-\frac{1}{2}} /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial z^n} == \delta_n \operatorname{nc}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (m-1)^{j-k} \left(z^2-1\right)^{-k-\frac{1}{2}} \left((1-m)z^2+m\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{nd}^{-1}(z \mid m)}{\partial z^n} == \delta_n \operatorname{nd}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-m)^{j-k} \left(z^2-1\right)^{-k-\frac{1}{2}} \left(1-(1-m)z^2\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{ns}^{-1}(z \mid m)}{\partial z^n} == \operatorname{ns}^{-1}(z \mid m)\,\delta_n + \sum_{j=0}^{n-1} \frac{(-1)^{j-1}(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2-1\right)^{-k-\frac{1}{2}} \left(z^2-m\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{sc}^{-1}(z \mid m)}{\partial z^n} == \delta_n \operatorname{sc}^{-1}(z \mid m) - \sum_{j=0}^{n-1} \frac{(-1)^{j-1}(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-m)^{j-k} \left(z^2+1\right)^{-k-\frac{1}{2}} \left((1-m)z^2+1\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{sd}^{-1}(z \mid m)}{\partial z^n} == \delta_n \operatorname{sd}^{-1}(z \mid m) + \sum_{j=0}^{n-1} \frac{(-1)^j m^j (1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(\frac{m-1}{m}\right)^{j-k} \left(m z^2+1\right)^{-k-\frac{1}{2}} \left(1-(1-m)z^2\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{sn}^{-1}(z \mid m)}{\partial z^n} == \operatorname{sn}^{-1}(z \mid m)\,\delta_n + \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} \left(1-z^2\right)^{-k-\frac{1}{2}} \left(1-m z^2\right)^{k-j-\frac{1}{2}} /;\, n \in \mathbb{N}.$$

The derivatives of the inverse Jacobi functions $\operatorname{cd}^{-1}(z \mid m)$, $\operatorname{cn}^{-1}(z \mid m)$, $\operatorname{cs}^{-1}(z \mid m)$, $\operatorname{dc}^{-1}(z \mid m)$, $\operatorname{dn}^{-1}(z \mid m)$, $\operatorname{ds}^{-1}(z \mid m)$, $\operatorname{nc}^{-1}(z \mid m)$, $\operatorname{nd}^{-1}(z \mid m)$, $\operatorname{ns}^{-1}(z \mid m)$, $\operatorname{sc}^{-1}(z \mid m)$, $\operatorname{sd}^{-1}(z \mid m)$, and $\operatorname{sn}^{-1}(z \mid m)$ with respect to variable m have more complicated representations that include direct and inverse Jacobi functions and the elliptic integral $E(\operatorname{am}(z \mid m) \mid m)$:

$$\frac{\partial \operatorname{cd}^{-1}(z \mid m)}{\partial m} == \frac{E\left(\operatorname{am}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\operatorname{cd}^{-1}(z \mid m)}{2(1-m)m}$$

$$\frac{\partial \operatorname{cn}^{-1}(z \mid m)}{\partial m} == \frac{E\left(\operatorname{am}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\operatorname{cn}^{-1}(z \mid m) - m\operatorname{cd}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)\operatorname{sn}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)}{2(1-m)m}$$

$$\frac{\partial \operatorname{cs}^{-1}(z \mid m)}{\partial m} == \frac{-E\left(\operatorname{am}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{cs}^{-1}(z \mid m) + m\operatorname{cd}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right)\operatorname{sn}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right)}{2(m-1)m}$$

$$\frac{\partial \operatorname{dc}^{-1}(z \mid m)}{\partial m} == \frac{E\left(\operatorname{am}\left(\operatorname{dc}^{-1}(z \mid m) \mid m\right) \mid m\right) - (1-m)\operatorname{dc}^{-1}(z \mid m)}{2(1-m)m}$$

$$\frac{\partial \operatorname{dn}^{-1}(z \mid m)}{\partial m} = \frac{E\left(\operatorname{am}\left(\operatorname{dn}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)\operatorname{dn}^{-1}(z \mid m) - z\operatorname{sc}\left(\operatorname{dn}^{-1}(z \mid m) \mid m\right)}{2(1-m)m}$$

$$\frac{\partial \operatorname{ds}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(-E\left(\operatorname{am}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{ds}^{-1}(z \mid m) + m\operatorname{dn}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)\operatorname{sc}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)\right)$$

$$\frac{\partial \operatorname{nc}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(-E\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{nc}^{-1}(z \mid m) + m\operatorname{cd}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)\operatorname{sn}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)\right)$$

$$\frac{\partial \operatorname{nd}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(\frac{\operatorname{sc}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right)}{z} - E\left(\operatorname{am}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{nd}^{-1}(z \mid m)\right)$$

$$\frac{\partial \operatorname{ns}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(\frac{m\operatorname{cd}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right)}{z} - E\left(\operatorname{am}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{ns}^{-1}(z \mid m)\right)$$

$$\frac{\partial \operatorname{sc}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(-E\left(\operatorname{am}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{sc}^{-1}(z \mid m) + m\operatorname{cd}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)\operatorname{sn}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right)\right)$$

$$\frac{\partial \operatorname{sd}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(-E\left(\operatorname{am}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\operatorname{sd}^{-1}(z \mid m) + m\operatorname{dn}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)\operatorname{sc}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)\right)$$

$$\frac{\partial \operatorname{sn}^{-1}(z \mid m)}{\partial m} = \frac{E\left(\sin^{-1}(z) \mid m\right) - (1-m)F\left(\sin^{-1}(z) \mid m\right) - m z\operatorname{cd}\left(F\left(\sin^{-1}(z) \mid m\right) \mid m\right)}{2(1-m)m}.$$

The previous formulas can be generalized to the following symbolic derivatives of the n^{th} order with respect to variable z:

$$\frac{\partial^n \operatorname{cd}^{-1}(z \mid m)}{\partial m^n} = \frac{\pi m^{-n}}{2}\,{}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; m\right) - \frac{(-1)^n\sqrt{\pi}\, z^{2n+1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, m z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cn}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n\sqrt{\pi}\,\left(1-z^2\right)^{n+\frac{1}{2}}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; 1-z^2, m\left(1-z^2\right)\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cs}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n\sqrt{\pi}\, z^{-2n-1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; -\frac{1}{z^2}, -\frac{1-m}{z^2}\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dc}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^{n-1}\sqrt{\pi}\, z^{-2n-1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) + \frac{\pi m^{-n}}{2}\,{}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; m\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dn}^{-1}(z \mid m)}{\partial m^n} = \frac{\sqrt{\pi}\,(m-1)^{-n-\frac{1}{2}}}{2\,\Gamma\left(\frac{1}{2}-n\right)}\left(\pi\,{}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; 1; \frac{1}{1-m}\right) - 2z\,F_1\left(\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{1-m}\right)\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{ds}^{-1}(z \mid m)}{\partial m^n} = \frac{z^{-2n-1}}{2n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}-k\right)_k \left(k-n+\frac{1}{2}\right)_{n-k} F_1\left(n+\frac{1}{2}; -k+n+\frac{1}{2}, k+\frac{1}{2}; n+\frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right) /;$$
$$|z| > 1 \land |m| > 1 \land n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{nc}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} \left(1-\frac{1}{z^2}\right)^{n+\frac{1}{2}} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; 1-\frac{1}{z^2}, m\left(1-\frac{1}{z^2}\right)\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{nd}^{-1}(z \mid m)}{\partial m^n} = \frac{\pi i}{2} (m-1)^{-n} \, {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; 1-m\right) + \frac{i \sqrt{\pi}\, z^{2n+1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, (1-m) z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, z^{-2n-1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) /; |z| > 1 \land n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{sc}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, z^{2n+1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; -z^2, (m-1) z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{sd}^{-1}(z \mid m)}{\partial m^n} = \frac{z^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}-k\right)_k \left(k-n+\frac{1}{2}\right)_{n-k} F_1\left(n+\frac{1}{2}; \frac{1}{2}-k+n, k+\frac{1}{2}; n+\frac{3}{2}; (1-m) z^2, -m z^2\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{sn}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, z^{2n+1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, m z^2\right) /; n \in \mathbb{N}.$$

Integration

The indefinite integrals of the twelve inverse Jacobi functions $\mathrm{cd}^{-1}(z \mid m)$, $\mathrm{cn}^{-1}(z \mid m)$, $\mathrm{cs}^{-1}(z \mid m)$, $\mathrm{dc}^{-1}(z \mid m)$, $\mathrm{dn}^{-1}(z \mid m)$, $\mathrm{ds}^{-1}(z \mid m)$, $\mathrm{nc}^{-1}(z \mid m)$, $\mathrm{nd}^{-1}(z \mid m)$, $\mathrm{ns}^{-1}(z \mid m)$, $\mathrm{sc}^{-1}(z \mid m)$, $\mathrm{sd}^{-1}(z \mid m)$, and $\mathrm{sn}^{-1}(z \mid m)$ with respect to variable z can be expressed through direct and inverse Jacobi and elementary functions by the following formulas:

$$\int \mathrm{cd}^{-1}(z \mid m)\, dz = z\, \mathrm{cd}^{-1}(z \mid m) + \frac{\log\left(\sqrt{m}\, \mathrm{sd}\left(\mathrm{cd}^{-1}(z \mid m) \mid m\right) - \mathrm{nd}\left(\mathrm{cd}^{-1}(z \mid m) \mid m\right)\right)}{\sqrt{m}}$$

$$\int \mathrm{cn}^{-1}(z \mid m)\, dz = z\, \mathrm{cn}^{-1}(z \mid m) + \frac{i}{\sqrt{m}} \log\left(\frac{i\, \mathrm{dn}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)}{\sqrt{m}} - \mathrm{sn}\left(\mathrm{cn}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \mathrm{cs}^{-1}(z \mid m)\, dz = z\, \mathrm{cs}^{-1}(z \mid m) + \log\left(\mathrm{ds}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) + \mathrm{ns}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \mathrm{dc}^{-1}(z \mid m)\, dz = \mathrm{dc}^{-1}(z \mid m)\, z - \log\left(\mathrm{nc}\left(\mathrm{dc}^{-1}(z \mid m) \mid m\right) + \mathrm{sc}\left(\mathrm{dc}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \mathrm{dn}^{-1}(z \mid m)\, dz = \mathrm{dn}^{-1}(z \mid m)\, z - i \log\left(i\, \mathrm{cn}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right) + \mathrm{sn}\left(\mathrm{dn}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \mathrm{ds}^{-1}(z \mid m)\, dz = z\, \mathrm{ds}^{-1}(z \mid m) + \log\left(\mathrm{cs}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right) + \mathrm{ns}\left(\mathrm{ds}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \text{nc}^{-1}(z \mid m)\, dz = \text{nc}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\text{dc}\left(\text{nc}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}} + \text{sc}\left(\text{nc}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \text{nd}^{-1}(z \mid m)\, dz = \text{nd}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{m-1}} \log\left(\frac{\text{cd}\left(\text{nd}^{-1}(z \mid m) \mid m\right)}{\sqrt{m-1}} + \text{sd}\left(\text{nd}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \text{ns}^{-1}(z \mid m)\, dz = z\, \text{ns}^{-1}(z \mid m) + \log\left(\text{cs}\left(\text{ns}^{-1}(z \mid m) \mid m\right) + \text{ds}\left(\text{ns}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \text{sc}^{-1}(z \mid m)\, dz = \text{sc}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\text{dc}\left(\text{sc}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}} + \text{nc}\left(\text{sc}^{-1}(z \mid m) \mid m\right)\right)$$

$$\int \text{sd}^{-1}(z \mid m)\, dz = \text{sd}^{-1}(z \mid m)\, z - \frac{1}{\sqrt{m-1}\,\sqrt{m}} \log\left(\frac{\text{cd}\left(\text{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{m-1}} + \frac{\text{nd}\left(\text{sd}^{-1}(z \mid m) \mid m\right)}{\sqrt{m}}\right)$$

$$\int \text{sn}^{-1}(z \mid m)\, dz = \text{sn}^{-1}(z \mid m)\, z - \frac{\log\left(\text{dn}\left(\text{sn}^{-1}(z \mid m) \mid m\right) - \sqrt{m}\, \text{cn}\left(\text{sn}^{-1}(z \mid m) \mid m\right)\right)}{\sqrt{m}}.$$

The indefinite integrals of the twelve inverse Jacobi functions $\text{cd}^{-1}(z \mid m)$, $\text{cn}^{-1}(z \mid m)$, $\text{cs}^{-1}(z \mid m)$, $\text{dc}^{-1}(z \mid m)$, $\text{dn}^{-1}(z \mid m)$, $\text{ds}^{-1}(z \mid m)$, $\text{nc}^{-1}(z \mid m)$, $\text{nd}^{-1}(z \mid m)$, $\text{ns}^{-1}(z \mid m)$, $\text{sc}^{-1}(z \mid m)$, $\text{sd}^{-1}(z \mid m)$, and $\text{sn}^{-1}(z \mid m)$ with respect to variable m can be expressed through direct and inverse Jacobi and elementary functions by the following formulas:

$$\int \text{cd}^{-1}(z \mid m)\, dm = 2\left(E(m) - \frac{1}{z}\left(z\, E\left(\sin^{-1}(z) \mid m\right) + (m-1)\, z\, F\left(\sin^{-1}(z) \mid m\right) + \sqrt{1-z^2}\, \sqrt{1-m z^2}\right) + (m-1)\, K(m)\right) /;$$
$$-1 < z < 1 \land m < 1$$

$$\int \text{cn}^{-1}(z \mid m)\, dm = 2\left(\frac{z\sqrt{m\left(z^2-1\right)+1} - z}{\sqrt{1-z^2}} + i\sqrt{m}\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) - E\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) - \right.\right.$$
$$\left.\left. F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right) + F\left(i \sinh^{-1}\left(\frac{\sqrt{m}\, z}{\sqrt{1-m}}\right) \,\middle|\, \frac{m-1}{m}\right)\right)\right) /; z < 1 \land 0 < m < 1$$

$$\int \text{dc}^{-1}(z \mid m)\, dm = 2\sqrt{m}\left(E\left(\frac{1}{m}\right) - E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right)\right) /; -1 < z < 1 \land m > 1 \lor z > 1 \land m < 1$$

$$\int \text{dn}^{-1}(z \mid m)\, dm = 2\sqrt{m-1}\left(E\left(\frac{1}{1-m}\right) - E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)\right) /; z < 1 \land m > 1$$

$$\int \text{ds}^{-1}(z \mid m)\, dm = 2\left(\frac{\sqrt{z^2+m-1}\,\sqrt{z^2+m}}{z} - z\left(\log\left(\frac{\sqrt{z^2+m-1} + \sqrt{z^2+m}}{2z}\right) + 1\right) + \right.$$
$$\left. \sqrt{m-1}\; i\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right) \,\middle|\, \frac{m}{m-1}\right) - F\left(i \sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right) \,\middle|\, \frac{m}{m-1}\right)\right)\right) /; z > 0 \land m > 0$$

$$\int \operatorname{nc}^{-1}(z \mid m)\, dm = -2\left(\frac{z - z\sqrt{m - m z^2 + z^2}}{\sqrt{z^2-1}} + \sqrt{1-m}\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) - E\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}\, z}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) - F\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) + F\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}\, z}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right)\right)\right) /; z > 1 \land m > 0$$

$$\int \operatorname{nd}^{-1}(z \mid m)\, dm = \frac{2}{z}\left(-i\sqrt{1-z^2}\,\sqrt{(m-1) z^2 + 1} - \sqrt{m-1}\, z\, E\left(i \sinh^{-1}\left(\sqrt{m-1}\right) \middle| \frac{1}{1-m}\right) + \sqrt{m-1}\, z\, E\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \middle| \frac{1}{1-m}\right)\right) /; z > 1 \land m > 1$$

$$\int \operatorname{ns}^{-1}(z \mid m)\, dm = 2\left(-z - \sqrt{m}\, E\left(\frac{1}{m}\right) + E(m) + \sqrt{m}\, E\left(\sin^{-1}(z) \middle| \frac{1}{m}\right) + (m-1) K(m)\right) /; z > 1 \land m < 1$$

$$\int \operatorname{sc}^{-1}(z \mid m)\, dm = \frac{2}{z}\left(i\sqrt{1-m}\, z\, E\left(i \sinh^{-1}\left(\sqrt{1-m}\, z\right) \middle| \frac{1}{1-m}\right) + \sqrt{z^2+1}\,\sqrt{1 - m z^2 + z^2} - 1\right) /; z \in \mathbb{R} \bigwedge (1-m) z^2 > -1$$

$$\int \operatorname{sd}^{-1}(z \mid m)\, dm = 2\sqrt{m-1}\, i\left(E\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \middle| \frac{m}{m-1}\right) - F\left(i \sinh^{-1}\left(\sqrt{m-1}\, z\right) \middle| \frac{m}{m-1}\right)\right) + \frac{1}{z\sqrt{(m-1) z^2 + 1}}\left(2 (m-1)\sqrt{m z^2 + 1}\, z^2 - \sqrt{(m-1) z^2 + 1}\, \log\left(\frac{1}{4}\left((2m-1) z^2 + 2\sqrt{(m-1) z^2 + 1}\,\sqrt{m z^2 + 1} + 2\right)\right) - 2\sqrt{(m-1) z^2 + 1} + 2\sqrt{m z^2 + 1}\right) /; z > 0 \land m > 0$$

$$\int \operatorname{sn}^{-1}(z \mid m)\, dm = 2\left(\frac{\sqrt{1-z^2}\,\sqrt{1 - m z^2} - 1}{z} + E\left(\sin^{-1}(z) \mid m\right) + (m-1) F\left(\sin^{-1}(z) \mid m\right)\right) /; -1 < z < 1 \land m < 1.$$

Differential equations

The twelve inverse Jacobi functions $\operatorname{cd}^{-1}(z \mid m)$, $\operatorname{cn}^{-1}(z \mid m)$, $\operatorname{cs}^{-1}(z \mid m)$, $\operatorname{dc}^{-1}(z \mid m)$, $\operatorname{dn}^{-1}(z \mid m)$, $\operatorname{ds}^{-1}(z \mid m)$, $\operatorname{nc}^{-1}(z \mid m)$, $\operatorname{nd}^{-1}(z \mid m)$, $\operatorname{ns}^{-1}(z \mid m)$, $\operatorname{sc}^{-1}(z \mid m)$, $\operatorname{sd}^{-1}(z \mid m)$, and $\operatorname{sn}^{-1}(z \mid m)$ are the special solutions of the following second-order ordinary nonlinear differential equations:

$$w''(z) + \left(2 m z^2 - m - 1\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{cd}^{-1}(z \mid m)$$

$$w''(z) - \left(2 m z^2 - 2m + 1\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{cn}^{-1}(z \mid m)$$

$$w''(z) + \left(2 z^2 - m + 2\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{cs}^{-1}(z \mid m)$$

$$w''(z) + \left(2 z^2 - m - 1\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{dc}^{-1}(z \mid m)$$

$$w''(z) - \left(2 z^2 + m - 2\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{dn}^{-1}(z \mid m)$$

$$w''(z) + \left(2 z^2 + 2m - 1\right) z\, w'(z)^3 = 0 /; w(z) = \operatorname{ds}^{-1}(z \mid m)$$

$$w''(z) + \left(2\,(1-m)\,z^2 + 2\,m - 1\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{nc}^{-1}(z \mid m)$$

$$w''(z) - \left(2\,(1-m)\,z^2 + m - 2\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{nd}^{-1}(z \mid m)$$

$$w''(z) + \left(2\,z^2 - m - 1\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{ns}^{-1}(z \mid m)$$

$$w''(z) + \left(2\,(1-m)\,z^2 - m + 2\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{sc}^{-1}(z \mid m)$$

$$w''(z) - \left(2\,(1-m)\,m\,z^2 - 2\,m + 1\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{sd}^{-1}(z \mid m)$$

$$w''(z) + \left(2\,m\,z^2 - m - 1\right) z\,w'(z)^3 == 0\ /;\ w(z) == \mathrm{sn}^{-1}(z \mid m).$$

Applications of the inverse Jacobi functions

Fields of application of the inverse Jacobi functions include most of the application areas of the direct functions. In many applications, the need for the inversion of the elliptic function fortunately does not arise. In cases where inversion is needed, the invese Jacobi elliptic functions are very useful tools for calculations.

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