Introductions to StruveH

Introduction to the Struve functions

General

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ appeared as special solutions of the inhomogeneous Bessel second-order differential equations:

$$w''(z)\,z^2 + w'(z)\,z + \left(z^2 - \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} /;\ w(z) = \boldsymbol{H}_\nu(z) + c_1\, J_\nu(z) + c_2\, Y_\nu(z)$$

$$w''(z)\,z^2 + w'(z)\,z - \left(z^2 + \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} /;\ w(z) = \boldsymbol{L}_\nu(z) + c_1\, I_\nu(z) + c_2\, K_\nu(z),$$

where c_1 and c_2 are arbitrary constants and $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ are Bessel functions.

The last two differential equations are very similar and can be converted into each other by changing z to $i\,z$. Their solutions can be constructed in the form of a series with arbitrary coefficients:

$$w(z) = z^\nu \sum_{j=0}^{\infty} a_j\, z^j + z^{-\nu} \sum_{j=0}^{\infty} b_j\, z^j = z^\nu \left(\sum_{k=0}^{\infty} a_{2k}\, z^{2k} + \sum_{k=0}^{\infty} a_{2k+1}\, z^{2k+1}\right) + z^{-\nu} \left(\sum_{k=0}^{\infty} b_{2k}\, z^{2k} + \sum_{k=0}^{\infty} b_{2k+1}\, z^{2k+1}\right).$$

Substitution of this series into the first equation gives the following partial solution of the inhomogeneous equation:

$$w(z) = z^\nu \sum_{k=0}^{\infty} A_k\, z^{2k+1} /;\ A_0 = \frac{2^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{3}{2}\right)} \bigwedge A_1 = -\frac{2^{-1-\nu}}{3\sqrt{\pi}\,\Gamma\left(\nu + \frac{5}{2}\right)} \bigwedge A_k = a_{2k+1} = \frac{(-1)^k\, 2^{-\nu-2k-1}}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \nu + \frac{3}{2}\right)}.$$

This solution, which appeared in an article by H. Struve (1882), was later ascribed Struve's name and the special notation $\boldsymbol{H}_\nu(z)$.

A similar procedure carried out for the second inhomogeneous equation leads to the function $\boldsymbol{L}_\nu(z)$, which was introduced by J. W. Nicholson (1911).

Definitions of Struve functions

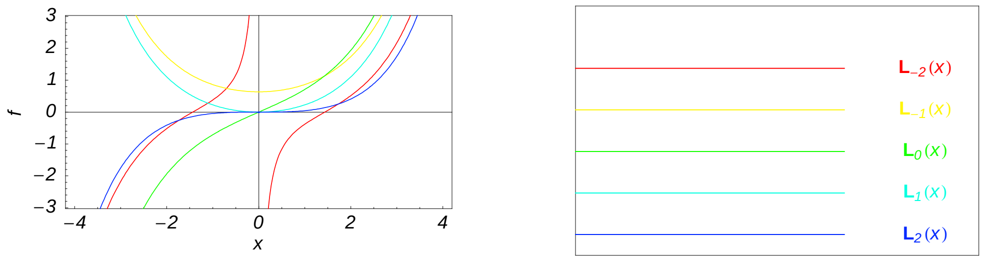
The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are defined as sums of the following infinite series:

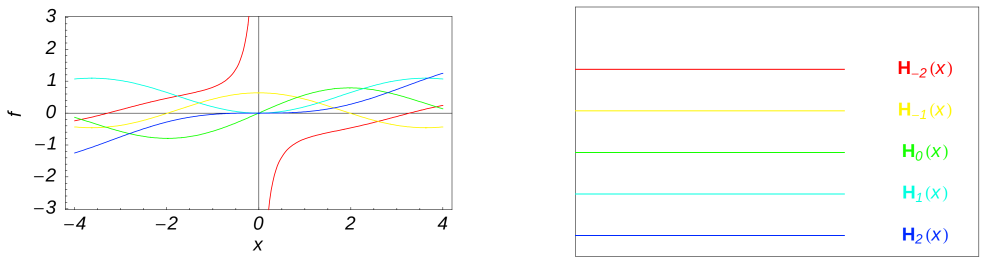
$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \nu + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}$$

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \nu + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}.$$

A quick look at the Struve functions

Here is a quick look at the graphics for the Struve functions along the real axis.





Connections within the group of Struve functions and with other function groups

Representations through more general functions

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are particular cases of the more general hypergeometric and Meijer G functions.

For example, they can be represented through regularized hypergeometric functions ${}_1\tilde{F}_2$:

$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} {}_1\tilde{F}_2\left(1; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{z^2}{4}\right)$$

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} {}_1\tilde{F}_2\left(1; \frac{3}{2}, \nu + \frac{3}{2}; \frac{z^2}{4}\right).$$

In the cases when $\nu + \frac{3}{2} = 0, -1, -2, \ldots$, the previous formulas degenerate into the following:

$$\boldsymbol{H}_\nu(z) = (-1)^{-\nu-\frac{1}{2}} \left(\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; -\frac{z^2}{4}\right) /; -\nu - \frac{3}{2} \in \mathbb{N}$$

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; \frac{z^2}{4}\right) /; -\nu - \frac{3}{2} \in \mathbb{N}.$$

For general values of parameter ν, the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ cannot be represented through classical hypergeometric functions without restrictions on parameter ν:

$$\boldsymbol{H}_\nu(z) = \frac{z^{\nu+1}}{2^\nu \sqrt{\pi}\, \Gamma\left(\nu + \frac{3}{2}\right)} {}_1F_2\left(1; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{z^2}{4}\right) /; -\nu - \frac{3}{2} \notin \mathbb{N}$$

$$\boldsymbol{L}_\nu(z) = \frac{z^{\nu+1}}{2^\nu \sqrt{\pi}\, \Gamma\left(\nu + \frac{3}{2}\right)} \, {}_1F_2\left(1; \frac{3}{2}, \nu + \frac{3}{2}; \frac{z^2}{4}\right) /; -\nu - \frac{3}{2} \notin \mathbb{N}.$$

Similar conclusion can be drawn from the following representations of the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ through generalized and classical Meijer G functions:

$$\boldsymbol{H}_\nu(z) = G_{1,3}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) G_{2,4}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

$$\boldsymbol{H}_\nu(z) = z^{\nu+1} \left(z^2\right)^{-\frac{\nu+1}{2}} G_{1,3}^{1,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

$$\boldsymbol{L}_\nu(z) = -\pi \csc\left(\frac{\pi \nu}{2}\right) z^{\nu-1} \left(z^2\right)^{\frac{1-\nu}{2}} G_{2,4}^{1,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1}{2} \\ \frac{\nu+1}{2}, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right).$$

The first two formulas are simpler than the last two classical representations that include factors like $z^{\nu+1} \left(z^2\right)^{-\frac{\nu+1}{2}}$.

Transformation inside the group (Interconnections)

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are connected to each other by the formulas:

$$\boldsymbol{H}_\nu(z) = -i\,(i\,z)^{-\nu} z^\nu \boldsymbol{L}_\nu(i\,z) \quad \boldsymbol{H}_\nu(i\,z) = i\,(i\,z)^\nu z^{-\nu} \boldsymbol{L}_\nu(z)$$

$$\boldsymbol{L}_\nu(z) = -i\,(i\,z)^{-\nu} z^\nu \boldsymbol{H}_\nu(i\,z) \quad \boldsymbol{L}_\nu(i\,z) = i\,(i\,z)^\nu z^{-\nu} \boldsymbol{H}_\nu(z)\,.$$

The best-known properties and formulas for Struve functions

Real values for real arguments

For real values of parameter ν and positive argument z, the values of the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are real.

Simple values at zero

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ have rather simple values for the argument $z = 0$:

$$\boldsymbol{H}_0(0) = 0$$

$$\boldsymbol{L}_0(0) = 0$$

$$\boldsymbol{H}_\nu(0) = 0 /; \operatorname{Re}(\nu) > -1$$

$$\boldsymbol{L}_\nu(0) = 0 /; \operatorname{Re}(\nu) > -1.$$

Specific values for specialized parameter

In the cases when parameter ν is equal to $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, …, the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ can be expressed through the sine and cosine (or hyperbolic sine and cosine) multiplied by rational and sqrt functions, for example:

$$\boldsymbol{H}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \quad \boldsymbol{H}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} (1 - \cos(z))$$

$$\boldsymbol{L}_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh(z) \quad \boldsymbol{L}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} (\cosh(z) - 1).$$

The previous formulas are the particular cases of the following general formulas:

$$\boldsymbol{H}_\nu(z) = \frac{1}{\left(\nu - \frac{1}{2}\right)! \sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu-1} \sum_{k=0}^{\nu-\frac{1}{2}} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(-\frac{z^2}{4}\right)^{-k} +$$
$$\frac{\sqrt{\frac{2}{\pi}} (-1)^{\nu+\frac{1}{2}}}{\sqrt{z}} \left(\sin\left(\frac{1}{2}\pi\left(\nu + \frac{1}{2}\right) + z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2\nu-1) \right\rfloor} \frac{(-1)^k \left(2k + \nu - \frac{1}{2}\right)!}{(2k)! \left(-2k + \nu - \frac{1}{2}\right)! (2z)^{2k}} + \right.$$
$$\left. \cos\left(\frac{1}{2}\pi\left(\nu + \frac{1}{2}\right) + z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2\nu-3) \right\rfloor} \frac{(-1)^k \left(2k + \nu + \frac{1}{2}\right)! (2z)^{-2k-1}}{(2k+1)! \left(-2k + \nu - \frac{3}{2}\right)!} \right) /; \nu - \frac{1}{2} \in \mathbb{Z}$$

$$\boldsymbol{L}_\nu(z) = -\frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi} \left(\nu - \frac{1}{2}\right)!} \sum_{k=0}^{\nu-\frac{1}{2}} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(\frac{z^2}{4}\right)^{-k} +$$
$$-\frac{1}{\sqrt{z}} e^{\frac{1}{2}\pi i \left(\nu+\frac{1}{2}\right)} \sqrt{\frac{2}{\pi}} \left(\sinh\left(\frac{1}{2} i \pi \left(\nu + \frac{1}{2}\right) - z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2|\nu|-1) \right\rfloor} \frac{\left(2k + |\nu| - \frac{1}{2}\right)!}{(2k)! \left(|\nu| - 2k - \frac{1}{2}\right)! (2z)^{2k}} + \right.$$
$$\left. \cosh\left(\frac{1}{2} i \pi \left(\nu + \frac{1}{2}\right) - z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2|\nu|-3) \right\rfloor} \frac{\left(2k + |\nu| + \frac{1}{2}\right)! (2z)^{-2k-1}}{(2k+1)! \left(|\nu| - 2k - \frac{3}{2}\right)!} \right) /; \nu - \frac{1}{2} \in \mathbb{Z}.$$

Analyticity

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are defined for all complex values of their parameter ν and variable z. They are analytical functions of ν and z over the whole complex ν- and z-planes excluding the branch cuts. For fixed integer ν, the functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are entire functions of z. For fixed z, the functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ are entire functions of ν.

Poles and essential singularities

For fixed ν, the functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ have an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point (except cases for integer ν).

With respect to ν, the Struve functions have only one essential singular point at $\nu = \tilde{\infty}$.

Branch points and branch cuts.

For fixed noninteger ν, the functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ have two branch points: $z == 0$ and $z == \tilde{\infty}$.

If functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ have branch cuts, they are single-valued functions on the z-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \mathbf{H}_\nu(x + i\,\epsilon) == \mathbf{H}_\nu(x) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} \mathbf{L}_\nu(x + i\,\epsilon) == \mathbf{L}_\nu(x) \;/;\; x < 0.$$

From below, functions have discontinuities that are described by the formulas:

$$\lim_{\epsilon \to +0} \mathbf{H}_\nu(x - i\,\epsilon) == -e^{-i\,\pi\,\nu}\,\mathbf{H}_\nu(-x) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} \mathbf{L}_\nu(x - i\,\epsilon) == -e^{-i\,\pi\,\nu}\,\mathbf{L}_\nu(-x) \;/;\; x < 0.$$

Periodicity

The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ do not have periodicity.

Parity and symmetry

The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ have mirror symmetry (except on the branch cut interval $(-\infty, 0)$):

$$\mathbf{H}_{\bar{\nu}}(\bar{z}) == \overline{\mathbf{H}_\nu(z)} \;/;\; z \notin (-\infty, 0)$$

$$\mathbf{L}_{\bar{\nu}}(\bar{z}) == \overline{\mathbf{L}_\nu(z)} \;/;\; z \notin (-\infty, 0).$$

The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ have generalized parity (either odd or even) with respect to variable z:

$$\mathbf{H}_\nu(-z) == -(-z)^\nu\, z^{-\nu}\, \mathbf{H}_\nu(z)$$

$$\mathbf{L}_\nu(-z) == -(-z)^\nu\, z^{-\nu}\, \mathbf{L}_\nu(z).$$

Series representations

The Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ have the following series expansions through series that converge on the whole z-plane:

$$\mathbf{H}_\nu(z) \propto \frac{2^{-\nu}\, z^{\nu+1}}{\sqrt{\pi}\;\Gamma\!\left(\nu + \frac{3}{2}\right)} \left(1 - \frac{z^2}{3\,(2\,\nu + 3)} + \frac{z^4}{15\,(2\,\nu + 3)\,(2\,\nu + 5)} - \ldots\right) /;\; (z \to 0)$$

$$\mathbf{H}_\nu(z) == \frac{2}{\sqrt{\pi}\;\Gamma\!\left(\nu + \frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2\,k}}{4^k \left(\frac{3}{2}\right)_k \left(\nu + \frac{3}{2}\right)_k}$$

$$\mathbf{L}_\nu(z) \propto \frac{2^{-\nu}\, z^{\nu+1}}{\sqrt{\pi}\;\Gamma\!\left(\nu + \frac{3}{2}\right)} \left(1 + \frac{z^2}{3\,(2\,\nu + 3)} + \frac{z^4}{15\,(2\,\nu + 3)\,(2\,\nu + 5)} + \ldots\right) /;\; (z \to 0)$$

$$\boldsymbol{L}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}.$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through the generalized hypergeometric function ${}_2F_2$, for example:

$$\boldsymbol{H}_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_n(z, \nu) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{n} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} = \boldsymbol{H}_\nu(z) + \frac{(-1)^n}{\Gamma\left(n+\frac{5}{2}\right)\Gamma\left(n+\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu+3} {}_1F_2\left(1; n+\frac{5}{2}, n+\nu+\frac{5}{2}; -\frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ can be described by the following formulas (only the main terms of asymptotic expansion are given):

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\frac{4\nu^2-1}{8\sqrt{z^2}} \cos\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right) + \sin\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\boldsymbol{L}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(-z^2\right)^{-\frac{2\nu+3}{4}} \left(\frac{4\nu^2-1}{8\sqrt{-z^2}} \cos\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right) + \sin\left(\sqrt{-z^2} - \frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty).$$

The previous formulas are valid in any directions approaching point z to infinity ($|z| \to \infty$). In particular cases when $|\operatorname{Arg}(z)| < \pi$ or $|\operatorname{Arg}(z)| < \frac{\pi}{2}$, these formulas can be simplified to the following relations:

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{z}} \left(\frac{4\nu^2-1}{8z} \cos\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right) + \sin\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1+O\left(\frac{1}{z^2}\right)\right) /; |\operatorname{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$\boldsymbol{L}_\nu(z) \propto \frac{e^z}{\sqrt{2\pi z}} \left(1+O\left(\frac{1}{z}\right)\right) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1+O\left(\frac{1}{z^2}\right)\right) /; |\operatorname{Arg}(z)| < \frac{\pi}{2} \bigwedge (|z| \to \infty).$$

Integral representations

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ have simple integral representations through the sine (or hyperbolic sine) and power functions:

$$\boldsymbol{H}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \sin(t\, z)\, dt \;/;\; \mathrm{Re}(\nu) > -\frac{1}{2}$$

$$\boldsymbol{L}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \sinh(t\, z)\, dt \;/;\; \mathrm{Re}(\nu) > -\frac{1}{2}.$$

Transformations

Arguments of the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ with square root arguments can sometimes be simplified:

$$\boldsymbol{H}_\nu\left(\sqrt{z^2}\right) = z^{-\nu-1} \left(z^2\right)^{\frac{\nu+1}{2}} \boldsymbol{H}_\nu(z)$$

$$\boldsymbol{L}_\nu\left(\sqrt{z^2}\right) = z^{-\nu-1} \left(z^2\right)^{\frac{\nu+1}{2}} \boldsymbol{L}_\nu(z).$$

Identities

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ satisfy the following recurrence identities:

$$\boldsymbol{H}_\nu(z) = \frac{2(\nu+1)}{z} \boldsymbol{H}_{\nu+1}(z) - \boldsymbol{H}_{\nu+2}(z) + \frac{2^{-\nu-1} z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{5}{2}\right)}$$

$$\boldsymbol{H}_\nu(z) = \frac{2(\nu-1)}{z} \boldsymbol{H}_{\nu-1}(z) - \boldsymbol{H}_{\nu-2}(z) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)}$$

$$\boldsymbol{L}_\nu(z) = \frac{2(\nu+1)}{z} \boldsymbol{L}_{\nu+1}(z) + \boldsymbol{L}_{\nu+2}(z) + \frac{2^{-\nu-1} z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{5}{2}\right)}$$

$$\boldsymbol{L}_\nu(z) = -\frac{2(\nu-1)}{z} \boldsymbol{L}_{\nu-1}(z) + \boldsymbol{L}_{\nu-2}(z) - \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)}.$$

The previous identities can be generalized to the following recurrence identities with a jump of length n:

$$\boldsymbol{H}_\nu(z) = C_n(\nu, z)\, \boldsymbol{H}_{\nu+n}(z) - C_{n-1}(\nu, z)\, \boldsymbol{H}_{\nu+n+1}(z) + \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(j+\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{j+\nu+1} C_j(\nu, z) \;/;$$

$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$\boldsymbol{H}_\nu(z) = C_n(\nu, z)\, \boldsymbol{H}_{\nu-n}(z) - C_{n-1}(\nu, z)\, \boldsymbol{H}_{\nu-n-1}(z) + \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(\nu+\frac{1}{2}-j\right)} \left(\frac{z}{2}\right)^{\nu-j-1} C_j(\nu, z) \;/;$$

$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = \frac{2(\nu-n)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$\boldsymbol{L}_\nu(z) = \mathcal{C}_n(\nu, z)\, \boldsymbol{L}_{\nu+n}(z) + \mathcal{C}_{n-1}(\nu, z)\, \boldsymbol{L}_{\nu+n+1}(z) + \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(j+\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{j+\nu+1} \mathcal{C}_j(\nu, z) \;/;$$

$$\mathcal{C}_0(\nu, z) = 1 \bigwedge \mathcal{C}_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge \mathcal{C}_n(\nu, z) = \frac{2(n+\nu)}{z} \mathcal{C}_{n-1}(\nu, z) + \mathcal{C}_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

$$\boldsymbol{L}_\nu(z) = \mathcal{C}_n(\nu, z)\, \boldsymbol{L}_{\nu-n}(z) + \mathcal{C}_{n-1}(\nu, z)\, \boldsymbol{L}_{\nu-n-1}(z) - \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\left(\nu+\frac{1}{2}-j\right)} \left(\frac{z}{2}\right)^{\nu-j-1} \mathcal{C}_j(\nu, z) \;/;$$

$$\mathcal{C}_0(\nu, z) = 1 \bigwedge \mathcal{C}_1(\nu, z) = -\frac{2(\nu-1)}{z} \bigwedge \mathcal{C}_n(\nu, z) = -\frac{2(\nu-n)}{z} \mathcal{C}_{n-1}(\nu, z) + \mathcal{C}_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+.$$

Simple representations of derivatives

The derivatives of the Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ have simple representations that can also be expressed through Struve functions with different indices:

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \frac{1}{2} \left(\frac{2^{-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{H}_{\nu-1}(z) - \boldsymbol{H}_{\nu+1}(z) \right)$$

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \boldsymbol{H}_{\nu-1}(z) - \frac{\nu}{z} \boldsymbol{H}_\nu(z)$$

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \frac{2^{-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} - \boldsymbol{H}_{\nu+1}(z) + \frac{\nu}{z} \boldsymbol{H}_\nu(z)$$

$$\frac{\partial \boldsymbol{L}_\nu(z)}{\partial z} = \frac{1}{2} \left(\frac{2^{-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{L}_{\nu-1}(z) + \boldsymbol{L}_{\nu+1}(z) \right)$$

$$\frac{\partial \boldsymbol{L}_\nu(z)}{\partial z} = \boldsymbol{L}_{\nu-1}(z) - \frac{\nu}{z} \boldsymbol{L}_\nu(z)$$

$$\frac{\partial \boldsymbol{L}_\nu(z)}{\partial z} = \frac{2^{-\nu} z^\nu}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{L}_{\nu+1}(z) + \frac{\nu}{z} \boldsymbol{L}_\nu(z).$$

The symbolic n^{th}-order derivatives have the following representations:

$$\frac{\partial^n \boldsymbol{H}_\nu(z)}{\partial z^n} = 2^{n-2\nu-2} \sqrt{\pi}\; z^{\nu-n+1}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-n}{2}+1, \frac{\nu-n+3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right) /; n \in \mathbb{N}$$

$$\frac{\partial^n \boldsymbol{L}_\nu(z)}{\partial z^n} = 2^{n-2\nu-2} \sqrt{\pi}\; z^{\nu-n+1}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-n}{2}+1, \frac{\nu-n+3}{2}, \nu+\frac{3}{2}; \frac{z^2}{4}\right) /; n \in \mathbb{N}.$$

Differential equations

The Struve functions $\boldsymbol{H}_\nu(z)$ and $\boldsymbol{L}_\nu(z)$ appeared as special solutions of the special inhomogeneous Bessel second-order linear differential equations:

$$w''(z)\,z^2 + w'(z)\,z + \left(z^2 - \nu^2\right) w(z) == \frac{4}{\sqrt{\pi}\;\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \;/;\; w(z) == \boldsymbol{H}_\nu(z) + c_1\, J_\nu(z) + c_2\, Y_\nu(z)$$

$$w''(z)\,z^2 + w'(z)\,z - \left(z^2 + \nu^2\right) w(z) == \frac{4}{\sqrt{\pi}\;\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \;/;\; w(z) == \boldsymbol{L}_\nu(z) + c_1\, I_\nu(z) + c_2\, K_\nu(z),$$

where c_1 and c_2 are arbitrary constants and $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ are Bessel functions.
The previous equations are very similar and can be converted into each other by changing z to $i\,z$.

Applications of Struve functions

Applications of Struve functions include electrodynamics, potential theory, and optics.

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