# Introductions to Tanh

## Introduction to the hyperbolic functions

### General

The six well-known hyperbolic functions are the hyperbolic sine $\sinh(z)$, hyperbolic cosine $\cosh(z)$, hyperbolic tangent $\tanh(z)$, hyperbolic cotangent $\coth(z)$, hyperbolic cosecant $\operatorname{csch}(z)$, and hyperbolic secant $\operatorname{sech}(z)$. They are among the most used elementary functions. The hyperbolic functions share many common properties and they have many properties and formulas that are similar to those of the trigonometric functions.

### Definitions of the hyperbolic functions

All hyperbolic functions can be defined as simple rational functions of the exponential function of $z$:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{csch}(z) = \frac{2}{e^z - e^{-z}}$$

$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}.$$

The functions $\tanh(z)$, $\coth(z)$, $\operatorname{csch}(z)$, and $\operatorname{sech}(z)$ can also be defined through the functions $\sinh(z)$ and $\cosh(z)$ using the following formulas:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

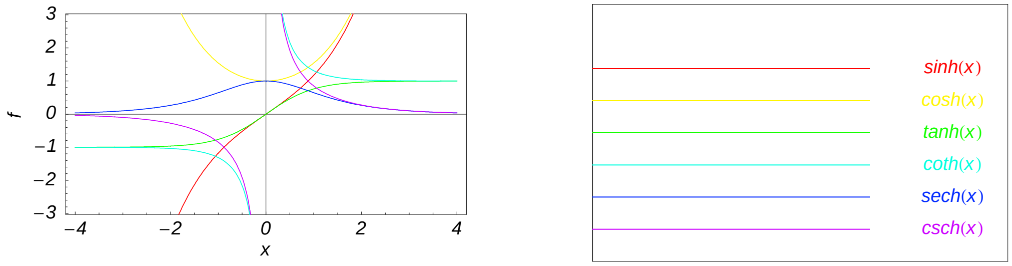
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)}.$$

### A quick look at the hyperbolic functions

Here is a quick look at the graphics of the six hyperbolic functions along the real axis.




## Connections within the group of hyperbolic functions and with other function groups

### Representations through more general functions

The hyperbolic functions are particular cases of more general functions. Among these more general functions, four classes of special functions are of special relevance: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, sinh($z$) and cosh($z$) have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, J_{1/2}(i z) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}}\, I_{1/2}(z) \quad \sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, Y_{-1/2}(i z) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) - \sqrt{z}\right.$$

$$\cosh(z) = \sqrt{\frac{\pi i z}{2}}\, J_{-1/2}(i z) \quad \cosh(z) = \sqrt{\frac{\pi z}{2}}\, I_{-1/2}(z) \quad \cosh(z) = -\sqrt{\frac{\pi i z}{2}}\, Y_{1/2}(i z) \quad \cosh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) + \sqrt{z}\right.$$

$$\sinh(z) = -i\,\mathrm{Se}(1, 0, i z) \quad \cosh(z) = \mathrm{Ce}(1, 0, i z)$$

$$\sinh(z) = z\, {}_0F_1\!\left(; \frac{3}{2}; \frac{z^2}{4}\right) \quad \cosh(z) = {}_0F_1\!\left(; \frac{1}{2}; \frac{z^2}{4}\right).$$

All hyperbolic functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{array}{ll}
\sinh(z) = -i\,\mathrm{sd}(i z \mid 0) = -i\,\mathrm{sn}(i z \mid 0) & \sinh(z) = \mathrm{sc}(z \mid 1) = \mathrm{sd}(z \mid 1) \\
\cosh(z) = \mathrm{cd}(i z \mid 0) = \mathrm{cn}(i z \mid 0) & \cosh(z) = \mathrm{nc}(z \mid 1) = \mathrm{nd}(z \mid 1) \\
\tanh(z) = -i\,\mathrm{sc}(i z \mid 0) & \tanh(z) = \mathrm{sn}(z \mid 1) \\
\coth(z) = i\,\mathrm{cs}(i z \mid 0) & \coth(z) = \mathrm{ns}(z \mid 1) \\
\mathrm{csch}(z) = i\,\mathrm{ds}(i z \mid 0) = i\,\mathrm{ns}(i z \mid 0) & \mathrm{csch}(z) = \mathrm{cs}(z \mid 1) = \mathrm{ds}(z \mid 1) \\
\mathrm{sech}(z) = \mathrm{dc}(i z \mid 0) = \mathrm{nc}(i z \mid 0) & \mathrm{sech}(z) = \mathrm{cn}(z \mid 1) = \mathrm{dn}(z \mid 1).
\end{array}$$

### Representations through related equivalent functions

Each of the six hyperbolic functions can be represented through the corresponding trigonometric function:

$$\begin{array}{ll}
\sinh(z) = -i\sin(i z) & \sinh(i z) = i\sin(z) \\
\cosh(z) = \cos(i z) & \cosh(i z) = \cos(z) \\
\tanh(z) = -i\tan(i z) & \tanh(i z) = i\tan(z) \\
\coth(z) = i\cot(i z) & \coth(i z) = -i\cot(z) \\
\mathrm{csch}(z) = i\csc(i z) & \mathrm{csch}(i z) = -i\csc(z) \\
\mathrm{sech}(z) = \sec(i z) & \mathrm{sech}(i z) = \sec(z).
\end{array}$$

### Relations to inverse functions

Each of the six hyperbolic functions is connected with a corresponding inverse hyperbolic function by two formulas. One direction can be expressed through a simple formula, but the other direction is much more complicated because of the multivalued nature of the inverse function:

$$\sinh(\sinh^{-1}(z)) == z \quad \sinh^{-1}(\sinh(z)) == z \;/;\; -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2} \bigvee \left(\mathrm{Im}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Re}(z) \le 0\right) \bigvee \left(\mathrm{Im}(z) == \frac{\pi}{2} \bigwedge \mathrm{Re}(z) \ge 0\right)$$

$$\cosh(\cosh^{-1}(z)) == z \quad \cosh^{-1}(\cosh(z)) == z \;/;\; \mathrm{Re}(z) > 0 \land -\pi < \mathrm{Im}(z) \le \pi \lor \mathrm{Re}(z) == 0 \land \mathrm{Im}(z) \ge 0$$

$$\tanh(\tanh^{-1}(z)) == z \quad \tanh^{-1}(\tanh(z)) == z \;/;\; -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2} \bigvee \left(\mathrm{Im}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Re}(z) > 0\right) \bigvee \left(\mathrm{Im}(z) == \frac{\pi}{2} \bigwedge \mathrm{Re}(z) < 0\right)$$

$$\coth(\coth^{-1}(z)) == z \quad \coth^{-1}(\coth(z)) == z \;/;\; -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2} \bigvee \left(\mathrm{Im}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Re}(z) > 0\right) \bigvee \left(\mathrm{Im}(z) == \frac{\pi}{2} \bigwedge \mathrm{Re}(z) \le 0\right)$$

$$\mathrm{csch}(\mathrm{csch}^{-1}(z)) == z \quad \mathrm{csch}^{-1}(\mathrm{csch}(z)) == z \;/;\; -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2} \bigvee \left(\mathrm{Im}(z) == -\frac{\pi}{2} \bigwedge \mathrm{Re}(z) \le 0\right) \bigvee \left(\mathrm{Im}(z) == \frac{\pi}{2} \bigwedge \mathrm{Re}(z) \ge 0\right)$$

$$\mathrm{sech}(\mathrm{sech}^{-1}(z)) == z \quad \mathrm{sech}^{-1}(\mathrm{sech}(z)) == z \;/;\; -\pi < \mathrm{Im}(z) \le \pi \land \mathrm{Re}(z) > 0 \lor \mathrm{Re}(z) == 0 \land \mathrm{Im}(z) \ge 0.$$

### Representations through other hyperbolic functions

Each of the six hyperbolic functions can be represented through any other function as a rational function of that function with a linear argument. For example, the hyperbolic sine can be representative as a group-defining function because the other five functions can be expressed as:

$$\cosh(z) == -i \sinh\left(\frac{\pi i}{2} - z\right) \qquad \cosh^2(z) == 1 + \sinh^2(z)$$

$$\tanh(z) == \frac{\sinh(z)}{\cosh(z)} == \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \tanh^2(z) == \frac{\sinh^2(z)}{1 + \sinh^2(z)}$$

$$\coth(z) == \frac{\cosh(z)}{\sinh(z)} == -\frac{i \sinh\left(\frac{\pi i}{2} - z\right)}{\sinh(z)} \qquad \coth^2(z) == \frac{1 + \sinh^2(z)}{\sinh^2(z)}$$

$$\mathrm{csch}(z) == \frac{1}{\sinh(z)} \qquad \mathrm{csch}^2(z) == \frac{1}{\sinh^2(z)}$$

$$\mathrm{sech}(z) == \frac{1}{\cosh(z)} == \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \mathrm{sech}^2(z) == \frac{1}{1 + \sinh^2(z)}.$$

All six hyperbolic functions can be transformed into any other function of the group of hyperbolic functions if the argument $z$ is replaced by $p \pi i / 2 + q z$ with $q^2 == 1 \bigwedge p \in \mathbb{Z}$:

$$\sinh(-z - 2\pi i) == -\sinh(z) \qquad \sinh(z - 2\pi i) == \sinh(z)$$

$$\sinh\left(-z - \frac{3\pi i}{2}\right) == i \cosh(z) \qquad \sinh\left(z - \frac{3\pi i}{2}\right) == i \cosh(z)$$

$$\sinh(-z - \pi i) == \sinh(z) \qquad \sinh(z - \pi i) == -\sinh(z)$$

$$\sinh\left(-z - \frac{\pi i}{2}\right) == -i \cosh(z) \qquad \sinh\left(z - \frac{\pi i}{2}\right) == -i \cosh(z)$$

$$\sinh\left(z + \frac{\pi i}{2}\right) == i \cosh(z) \qquad \sinh\left(\frac{\pi i}{2} - z\right) == i \cosh(z)$$

$$\sinh(z + \pi i) == -\sinh(z) \qquad \sinh(\pi i - z) == \sinh(z)$$

$$\sinh\left(z + \frac{3\pi i}{2}\right) == -i \cosh(z) \qquad \sinh\left(\frac{3\pi i}{2} - z\right) == -i \cosh(z)$$

$$\sinh(z + 2\pi i) == \sinh(z) \qquad \sinh(2\pi i - z) == -\sinh(z)$$

$\cosh(-z - 2\pi i) == \cosh(z) \qquad \cosh(z - 2\pi i) == \cosh(z)$
$\cosh\left(-z - \frac{3\pi i}{2}\right) == -i\sinh(z) \qquad \cosh\left(z - \frac{3\pi i}{2}\right) == i\sinh(z)$
$\cosh(-z - \pi i) == -\cosh(z) \qquad \cosh(z - \pi i) == -\cosh(z)$
$\cosh\left(-z - \frac{\pi i}{2}\right) == i\sinh(z) \qquad \cosh\left(z - \frac{\pi i}{2}\right) == -i\sinh(z)$
$\cosh\left(z + \frac{\pi i}{2}\right) == i\sinh(z) \qquad \cosh\left(\frac{\pi i}{2} - z\right) == -i\sinh(z)$
$\cosh(z + \pi i) == -\cosh(z) \qquad \cosh(\pi i - z) == -\cosh(z)$
$\cosh\left(z + \frac{3\pi i}{2}\right) == -i\sinh(z) \qquad \cosh\left(\frac{3\pi i}{2} - z\right) == i\sinh(z)$
$\cosh(z + 2\pi i) == \cosh(z) \qquad \cosh(2\pi i - z) == \cosh(z)$

$\tanh(-z - \pi i) == -\tanh(z) \qquad \tanh(z - \pi i) == \tanh(z)$
$\tanh\left(-z - \frac{\pi i}{2}\right) == -\coth(z) \qquad \tanh\left(z - \frac{\pi i}{2}\right) == \coth(z)$
$\tanh\left(z + \frac{\pi i}{2}\right) == \coth(z) \qquad \tanh\left(\frac{\pi i}{2} - z\right) == -\coth(z)$
$\tanh(z + \pi i) == \tanh(z) \qquad \tanh(\pi i - z) == -\tanh(z)$

$\coth(-z - \pi i) == -\coth(z) \qquad \coth(z - \pi i) == \coth(z)$
$\coth\left(-z - \frac{\pi i}{2}\right) == -\tanh(z) \qquad \coth\left(z - \frac{\pi i}{2}\right) == \tanh(z)$
$\coth\left(z + \frac{\pi i}{2}\right) == \tanh(z) \qquad \coth\left(\frac{\pi i}{2} - z\right) == -\tanh(z)$
$\coth(z + \pi i) == \coth(z) \qquad \coth(\pi i - z) == -\coth(z)$

$\operatorname{csch}(-z - 2\pi i) == -\operatorname{csch}(z) \qquad \operatorname{csch}(z - 2\pi i) == \operatorname{csch}(z)$
$\operatorname{csch}\left(-z - \frac{3\pi i}{2}\right) == -i\operatorname{sech}(z) \qquad \operatorname{csch}\left(z - \frac{3\pi i}{2}\right) == -i\operatorname{sech}(z)$
$\operatorname{csch}(-z - \pi i) == \operatorname{csch}(z) \qquad \operatorname{csch}(z - \pi i) == -\operatorname{csch}(z)$
$\operatorname{csch}\left(-z - \frac{\pi i}{2}\right) == i\operatorname{sech}(z) \qquad \operatorname{csch}\left(z - \frac{\pi i}{2}\right) == i\operatorname{sech}(z)$
$\operatorname{csch}\left(z + \frac{\pi i}{2}\right) == -i\operatorname{sech}(z) \qquad \operatorname{csch}\left(\frac{\pi i}{2} - z\right) == -i\operatorname{sech}(z)$
$\operatorname{csch}(z + \pi i) == -\operatorname{csch}(z) \qquad \operatorname{csch}(\pi i - z) == \operatorname{csch}(z)$
$\operatorname{csch}\left(z + \frac{3\pi i}{2}\right) == i\operatorname{sech}(z) \qquad \operatorname{csch}\left(\frac{3\pi i}{2} - z\right) == i\operatorname{sech}(z)$
$\operatorname{csch}(z + 2\pi i) == \operatorname{csch}(z) \qquad \operatorname{csch}(2\pi - z i) == -\operatorname{csch}(z)$

$\operatorname{sech}(-z - 2\pi i) == \operatorname{sech}(z) \qquad \operatorname{sech}(z - 2\pi i) == \operatorname{sech}(z)$
$\operatorname{sech}\left(-z - \frac{3\pi i}{2}\right) == i\operatorname{csch}(z) \qquad \operatorname{sech}\left(z - \frac{3\pi i}{2}\right) == -i\operatorname{csch}(z)$
$\operatorname{sech}(-z - \pi i) == -\operatorname{sech}(z) \qquad \operatorname{sech}(z - \pi i) == -\operatorname{sech}(z)$
$\operatorname{sech}\left(-z - \frac{\pi i}{2}\right) == -i\operatorname{csch}(z) \qquad \operatorname{sech}\left(z - \frac{\pi i}{2}\right) == i\operatorname{csch}(z)$
$\operatorname{sech}\left(z + \frac{\pi i}{2}\right) == -i\operatorname{csch}(z) \qquad \operatorname{sech}\left(\frac{\pi i}{2} - z\right) == i\operatorname{csch}(z)$
$\operatorname{sech}(z + \pi i) == -\operatorname{sech}(z) \qquad \operatorname{sech}(\pi i - z) == -\operatorname{sech}(z)$
$\operatorname{sech}\left(z + \frac{3\pi i}{2}\right) == i\operatorname{csch}(z) \qquad \operatorname{sech}\left(\frac{3\pi i}{2} - z\right) == -i\operatorname{csch}(z)$
$\operatorname{sech}(z + 2\pi i) == \operatorname{sech}(z) \qquad \operatorname{sech}(2\pi i - z) == \operatorname{sech}(z).$

## The best-known properties and formulas for hyperbolic functions

### Real values for real arguments

For real values of argument $z$, the values of all the hyperbolic functions are real (or infinity).

In the points $z = 2\pi n i / m /; n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of the hyperbolic functions are algebraic. In several cases, they can even be rational numbers, 1, or $i$ (e.g. $\sinh(\pi i/2) = i$, $\operatorname{sech}(0) = 1$, or $\cosh(\pi i/3) = 1/2$). They can be expressed using only square roots if $n \in \mathbb{Z}$ and $m$ is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

### Simple values at zero

All hyperbolic functions has rather simple values for arguments $z == 0$ and $z == \pi i/2$:

$$\sinh(0) == 0 \quad \sinh\left(\frac{\pi i}{2}\right) == i$$
$$\cosh(0) == 1 \quad \cosh\left(\frac{\pi i}{2}\right) == 0$$
$$\tanh(0) == 0 \quad \tanh\left(\frac{\pi i}{2}\right) == \tilde{\infty}$$
$$\coth(0) == \tilde{\infty} \quad \coth\left(\frac{\pi i}{2}\right) == 0$$
$$\operatorname{csch}(0) == \tilde{\infty} \quad \operatorname{csch}\left(\frac{\pi i}{2}\right) == -i$$
$$\operatorname{sech}(0) == 1 \quad \operatorname{sech}\left(\frac{\pi i}{2}\right) == \tilde{\infty}.$$

### Analyticity

All hyperbolic functions are defined for all complex values of $z$, and they are analytical functions of $z$ over the whole complex $z$-plane and do not have branch cuts or branch points. The two functions $\sinh(z)$ and $\cosh(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other hyperbolic functions are meromorphic functions with simple poles at points $z == \pi k i /; k \in \mathbb{Z}$ (for $\operatorname{csch}(z)$ and $\coth(z)$) and at points $z == \pi i/2 + \pi k i /; k \in \mathbb{Z}$ (for $\operatorname{sech}(z)$ and $\tanh(z)$).

### Periodicity

All hyperbolic functions are periodic functions with a real period ($2\pi i$ or $\pi i$):

$$\sinh(z) == \sinh(z + 2\pi i) \quad \sinh(z + 2\pi i k) == \sinh(z) /; k \in \mathbb{Z}$$
$$\cosh(z) == \cosh(z + 2\pi i) \quad \cosh(z + 2\pi i k) == \cosh(z) /; k \in \mathbb{Z}$$
$$\tanh(z) == \tanh(z + \pi i) \quad \tanh(z + \pi i k) == \tanh(z) /; k \in \mathbb{Z}$$
$$\coth(z) == \coth(z + \pi i) \quad \coth(z + \pi i k) == \coth(z) /; k \in \mathbb{Z}$$
$$\operatorname{csch}(z) == \operatorname{csch}(z + 2\pi i) \quad \operatorname{csch}(z + 2\pi i k) == \operatorname{csch}(z) /; k \in \mathbb{Z}$$
$$\operatorname{sech}(z) == \operatorname{sech}(z + 2\pi i) \quad \operatorname{sech}(z + 2\pi i k) == \operatorname{sech}(z) /; k \in \mathbb{Z}.$$

### Parity and symmetry

All hyperbolic functions have parity (either odd or even) and mirror symmetry:

$$\sinh(-z) == -\sinh(z) \quad \sinh(\bar{z}) == \overline{\sinh(z)}$$
$$\cosh(-z) == \cosh(z) \quad \cosh(\bar{z}) == \overline{\cosh(z)}$$
$$\tanh(-z) == -\tanh(z) \quad \tanh(\bar{z}) == \overline{\tanh(z)}$$
$$\coth(-z) == -\coth(z) \quad \coth(\bar{z}) == \overline{\coth(z)}$$
$$\operatorname{csch}(-z) == -\operatorname{csch}(z) \quad \operatorname{csch}(\bar{z}) == \overline{\operatorname{csch}(z)}$$
$$\operatorname{sech}(-z) == \operatorname{sech}(z) \quad \operatorname{sech}(\bar{z}) == \overline{\operatorname{sech}(z)}.$$

### Simple representations of derivatives

The derivatives of all hyperbolic functions have simple representations that can be expressed through other hyperbolic functions:

$$\frac{\partial \sinh(z)}{\partial z} == \cosh(z) \qquad \frac{\partial \cosh(z)}{\partial z} == \sinh(z) \qquad \frac{\partial \tanh(z)}{\partial z} == \operatorname{sech}^2(z)$$
$$\frac{\partial \coth(z)}{\partial z} == -\operatorname{csch}^2(z) \qquad \frac{\partial \operatorname{csch}(z)}{\partial z} == -\coth(z)\operatorname{csch}(z) \qquad \frac{\partial \operatorname{sech}(z)}{\partial z} == -\operatorname{sech}(z)\tanh(z).$$

**Simple differential equations**

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh($z$) and cosh($z$). The other hyperbolic functions satisfy first-order nonlinear differential equations:

$$w''(z) - w(z) == 0 /; w(z) == \cosh(z) \land w(0) == 1 \land w'(0) == 0$$
$$w''(z) - w(z) == 0 /; w(z) == \sinh(z) \land w(0) == 0 \land w'(0) == 1$$
$$w''(z) - w(z) == 0 /; w(z) == c_1 \cosh(z) + c_2 \sinh(z).$$

All six hyperbolic functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 + (w(z))^2} == 0 /; w(z) == \sinh(z) \bigwedge w(0) == 0 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{-1 + (w(z))^2} == 0 /; w(z) == \cosh(z) \bigwedge w(0) == 1 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) + w(z)^2 - 1 == 0 /; w(z) == \tanh(z) \land w(0) == 0$$
$$w'(z) + w(z)^2 - 1 == 0 /; w(z) == \coth(z) \bigwedge w\left(\frac{\pi i}{2}\right) == 0$$
$$w'(z)^2 - w(z)^4 - w(z)^2 == 0 /; w(z) == \operatorname{csch}(z)$$
$$w'(z)^2 + w(z)^4 - w(z)^2 == 0 /; w(z) == \operatorname{sech}(z).$$

## Applications of hyperbolic functions

Trigonometric functions are intimately related to triangle geometry. Functions like sine and cosine are often introduced as edge lengths of right-angled triangles. Hyperbolic functions occur in the theory of triangles in hyperbolic spaces.

Lobachevsky (1829) and J. Bolyai (1832) independently recognized that Euclid's fifth postulate—saying that for a given line and a point not on the line, there is exactly one line parallel to the first—might be changed and still be a consistent geometry. In the hyperbolic geometry it is allowable for more than one line to be parallel to the first (meaning that the parallel lines will never meet the first, however far they are extended). Translated into triangles, this means that the sum of the three angles is always less than $\pi$.

A particularly nice representation of the hyperbolic geometry can be realized in the unit disk of complex numbers (the Poincaré disk model). In this model, points are complex numbers in the unit disk, and the lines are either arcs of circles that meet the boundary of the unit circle orthogonal or diameters of the unit circle.

The distance $d$ between two points (meaning complex numbers) $A$ and $B$ in the Poincaré disk is:

$$d(A, B) = 2\tanh^{-1}\left(\left|\frac{A - B}{1 - \overline{B}A}\right|\right).$$

The attractive feature of the Poincaré disk model is that the hyperbolic angles agree with the Euclidean angles. Formally, the angle $\alpha$ at a point $A$ of two hyperbolic lines $\overline{A\,B}$ and $\overline{A\,C}$ is described by the formula:

$$\cos(\alpha) = \frac{\frac{-A+B}{1-\overline{A}\,B}\,\overline{\frac{-A+C}{1-\overline{A}\,C}}}{\left|\frac{-A+B}{1-\overline{A}\,B}\right|\left|\frac{-A+C}{1-\overline{A}\,C}\right|}.$$

In the following, the values of the three angles of an hyperbolic triangle at the vertices $A$, $B$, and $C$ are denoted through $\alpha$, $\beta$, and $\gamma$. The hyperbolic length of the three edges opposite to the angles are denoted $a$, $b$, and $c$.

The cosine rule and the second cosine rule for hyperbolic triangles are:

$$\sinh(b)\sinh(c)\cos(\alpha) == \cosh(b)\cosh(c) - \cosh(a)$$
$$\sinh(a)\sinh(c)\cos(\beta) == \cosh(a)\cosh(c) - \cosh(b)$$
$$\sinh(a)\sinh(b)\cos(\gamma) == \cosh(a)\cosh(b) - \cosh(c)$$

$$\sin(\beta)\sin(\gamma)\cosh(a) == \cos(\beta)\cos(\gamma) + \cos(\alpha)$$
$$\sin(\alpha)\sin(\gamma)\cosh(b) == \cos(\alpha)\cos(\alpha) + \cos(\beta)$$
$$\sin(\alpha)\sin(\beta)\cosh(c) == \cos(\alpha)\cos(\beta) + \cos(\gamma).$$

The sine rule for hyperbolic triangles is:

$$\frac{\sin(\alpha)}{\sinh(a)} == \frac{\sin(\beta)}{\sinh(b)} == \frac{\sin(\gamma)}{\sinh(c)}.$$

For a right-angle triangle, the hyperbolic version of the Pythagorean theorem follows from the preceding formulas (the right angle is taken at vertex $A$):

$$\cosh(a) == \cosh(b)\cosh(c).$$

Using the series expansion $\cosh(x) \approx 1 + x^2 / 2$ at small scales the hyperbolic geometry is approximated by the familar Euclidean geometry. The cosine formulas and the sine formulas for hyperbolic triangles with a right angle at vertex $A$ become:

$$\cos(\beta) = \frac{\tanh(c)}{\tanh(a)},\quad \sin(\beta) = \frac{\sinh(b)}{\sinh(a)}$$
$$\cos(\gamma) = \frac{\tanh(b)}{\tanh(a)},\quad \sin(\gamma) = \frac{\text{sinhh}(c)}{\tanh(a)}.$$

The inscribed circle has the radius:

$$\rho = \sqrt{\tanh^{-1}\left(\frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2\,(1+\cos(\alpha))\,(1+\cos(\beta))\,(1+\cos(\gamma))}\right)}.$$

The circumscribed circle has the radius:

$$\rho = \tanh^{-1}\left(\frac{4\sinh\left(\frac{a}{2}\right)\sinh\left(\frac{b}{2}\right)\sinh\left(\frac{c}{2}\right)}{\sin(\gamma)\sinh(a)\sinh(b)}\right).$$

### Other applications

As rational functions of the exponential function, the hyperbolic functions appear virtually everywhere in quantitative sciences. It is impossible to list their numerous applications in teaching, science, engineering, and art.

# Introduction to the Hyperbolic Tangent Function

## Defining the hyperbolic tangent function

The hyperbolic tangent function is an old mathematical function. It was first used in the work by L'Abbe Sauri (1774).

This function is easily defined as the ratio between the hyperbolic sine and the cosine functions (or expanded, as the ratio of the half-difference and half-sum of two exponential functions in the points $z$ and $-z$):

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

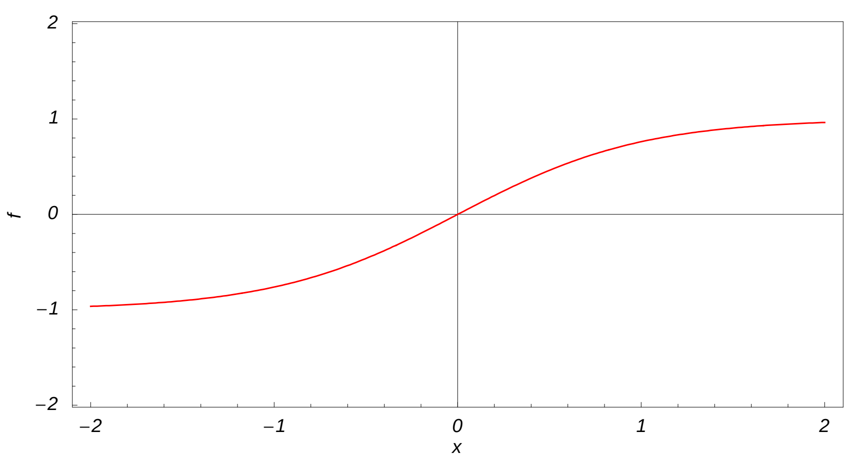
After comparison with the famous Euler formulas for the sine and cosine functions, $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$, it is easy to derive the following representation of the hyperbolic tangent through the circular tangent function:

$$\tanh(z) = -i \tan(i z).$$

This formula allows the derivation of all the properties and formulas for the hyperbolic tangent from the corresponding properties and formulas for the circular tangent.

## A quick look at the hyperbolic tangent function

Here is a graphic of the hyperbolic tangent function $f(x) = \tanh(x)$ for real values of its argument $x$.




## Representation through more general functions

The hyperbolic tangent function $\tanh(z)$ can be represented using more general mathematical functions. As the ratio of the hyperbolic sine and cosine functions that are particular cases of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the hyperbolic tangent function can also be represented as ratios of those special functions. But these representations are not very useful. It is more useful to write the hyperbolic tangent function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the hyperbolic tangent function when their second parameter is equal to 0 or 1:

$$\tanh(z) = \mathrm{sn}(z \mid 1) = -\mathrm{ns}\left(\frac{\pi i}{2} - z \,\middle|\, 1\right) = -i\,\mathrm{sc}(i z \mid 0) = -i\,\mathrm{cs}\left(\frac{\pi}{2} - i z \,\middle|\, 0\right).$$

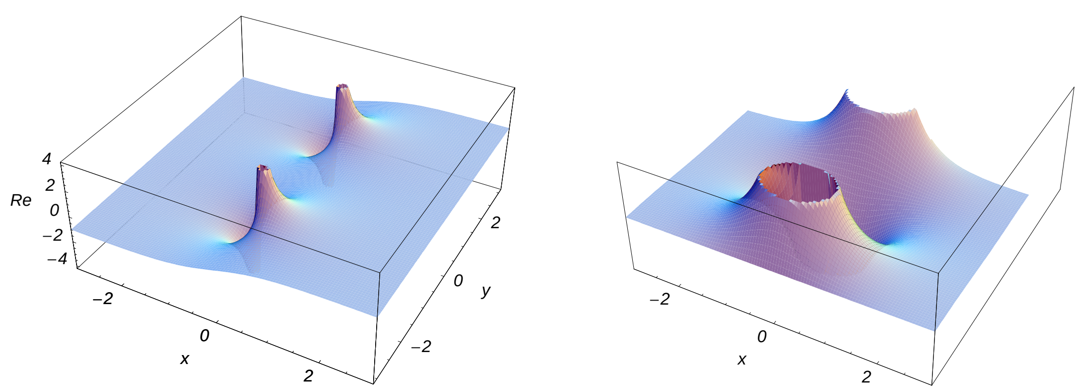
## Definition of the hyperbolic tangent function for a complex argument

In the complex $z$-plane, the function $\tanh(z)$ is defined by the same formula that is used for real values:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

In the points $z = \pi i/2 + \pi k i /; k \in \mathbb{Z}$, where $\cosh(z)$ has zeros, the denominator of the last formula equals zero and $\tanh(z)$ has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the hyperbolic tangent function over the complex plane.




## The best-known properties and formulas for the hyperbolic tangent function

### Values in points

The values of the hyperbolic tangent for special values of its argument can be easily derived from corresponding values of the circular tangent in the special points of the circle:

$\tanh(0) = 0 \quad \tanh\left(\frac{\pi i}{6}\right) = \frac{i}{\sqrt{3}} \quad \tanh\left(\frac{\pi i}{4}\right) = i \quad \tanh\left(\frac{\pi i}{3}\right) = \sqrt{3}\, i$

$\tanh\left(\frac{\pi i}{2}\right) = \tilde{\infty} \quad \tanh\left(\frac{2\pi i}{3}\right) = -\sqrt{3}\, i \quad \tanh\left(\frac{3\pi i}{4}\right) = -i \quad \tanh\left(\frac{5\pi i}{6}\right) = -\frac{i}{\sqrt{3}}$

$\tanh(\pi i) = 0$

$\tanh(\pi i m) = 0 /; m \in \mathbb{Z} \quad \tanh\left(\pi i \left(\frac{1}{2} + m\right)\right) = \tilde{\infty} /; m \in \mathbb{Z}.$

The values at infinity can be expressed by the following formulas:

$\tanh(\infty) = 1 \quad \tanh(-\infty) = -1.$

## General characteristics

For real values of argument $z$, the values of $\tanh(z)$ are real.

In the points $z = \pi n i / m /; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\tanh(z)$ are algebraic. In several cases, they can be $-i$, 0, or $i$:

$\tanh\left(-\frac{\pi i}{4}\right) = -i \quad \tanh(0) = 0 \quad \tanh\left(\frac{\pi i}{4}\right) = i.$

The values of $\tanh\left(\frac{n \pi i}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and $m$ is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

The function $\tanh(z)$ is an analytical function of $z$ that is defined over the whole complex $z$-plane and does not have branch cuts and branch points. It has an infinite set of singular points:

(a) $z = \pi i / 2 + \pi i k /; k \in \mathbb{Z}$ are the simple poles with residues 1.
(b) $z = \tilde{\infty}$ is an essential singular point.

It is a periodic function with period $\pi i$:

$\tanh(z + \pi i) = \tanh(z)$

$\tanh(z) = \tanh(z + \pi i k) /; k \in \mathbb{Z}.$

The function $\tanh(z)$ is an odd function with mirror symmetry:

$\tanh(-z) = -\tanh(z) \quad \tanh(\bar{z}) = \overline{\tanh(z)}.$

## Differentiation

The first derivative of $\tanh(z)$ has simple representations using either the $\cosh(z)$ function or the $\operatorname{sech}(z)$ function:

$$\frac{\partial \tanh(z)}{\partial z} = \frac{1}{\cosh^2(z)} = \operatorname{sech}^2(z).$$

The $n^{\text{th}}$ derivative of $\tanh(z)$ has much more complicated representations than the symbolic $n^{\text{th}}$ derivatives for $\sinh(z)$ and $\cosh(z)$:

$$\frac{\partial^n \tanh(z)}{\partial z^n} = \tanh(z)\,\delta_n + \operatorname{sech}^2(z)\,\delta_{n-1} - i^n n \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^k \cosh^{-2k-2}(z)\, 2^{n-2k} (k-j)^{n-1}}{k+1} \binom{n-1}{k} \binom{2k}{j} \sinh\left(\frac{i n \pi}{2} + 2(j-k) z\right) /; n \in \mathbb{N},$$

where $\delta_n$ is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 /; n \neq 0$.

### Ordinary differential equation

The function tanh($z$) satisfies the following first-order nonlinear differential equation:

$$w'(z) + w(z)^2 - 1 = 0 \;/;\; w(z) = \tanh(z) \bigwedge w(0) = 0.$$

### Series representation

The function tanh($z$) has a simple series expansion at the origin that converges for all finite values $z$ with $|z| < \frac{\pi}{2}$:

$$\tanh(z) = z - \frac{z^3}{3} + \frac{2 z^5}{15} - \ldots = \sum_{k=1}^{\infty} \frac{2^{2k} \left(2^{2k} - 1\right) B_{2k} z^{2k-1}}{(2k)!},$$

where $B_{2k}$ are the Bernoulli numbers.

### Integral representation

The function tanh($z$) has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\tanh(z) = -\frac{2 i}{\pi} \int_0^{\infty} \frac{t^{\frac{2 i z}{\pi}} - 1}{t^2 - 1} \, dt \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < 0.$$

### Continued fraction representations

The function tanh($z$) has the following simple continued fraction representations:

$$\tanh(z) = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{7 + \cfrac{z^2}{9 + \cfrac{z^2}{11 + \ldots}}}}}} \;/;\; \frac{i z}{\pi} - \frac{1}{2} \notin \mathbb{Z}$$

$$\tanh(z) = \cfrac{1}{\frac{1}{z} + \cfrac{1}{\frac{3}{z} + \cfrac{1}{\frac{5}{z} + \cfrac{1}{\frac{7}{z} + \cfrac{1}{\frac{9}{z} + \cfrac{1}{\frac{11}{z} + \ldots}}}}}} \;/;\; \frac{i z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

### Indefinite integration

Indefinite integrals of expressions involving the hyperbolic tangent function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \tanh(z)\, dz = \log(\cosh(z))$$

$$\int \sqrt{\tanh(z)}\, dz = -\tan^{-1}\left(\tanh^{\frac{1}{2}}(z)\right) - \frac{1}{2}\log\left(\tanh^{\frac{1}{2}}(z) - 1\right) + \frac{1}{2}\log\left(\tanh^{\frac{1}{2}}(z) + 1\right)$$

$$\int \tanh^{\nu}(c\, z)\, dz = \frac{\tanh^{\nu+1}(c\, z)}{\nu\, c + c}\, {}_2F_1\left(\frac{\nu+1}{2}, 1; \frac{\nu+3}{2}; \tanh^2(c\, z)\right).$$

### Definite integration

Definite integrals that contains the hyperbolic tangent function are sometimes simple. For example, the famous Catalan constant $C$ can be defined through the following integral:

$$\int_0^{\infty} t\, e^{-t} \tanh(t)\, dt = 2\, C - 1.$$

Some special functions can be used to evaluate more complicated definite integrals. For example, the hypergeometric function is needed to express the following integral:

$$\int_0^1 \tanh(t)^{\nu}\, dt = \frac{1}{\nu+1}\, {}_2F_1\left(\frac{\nu+1}{2}, 1; \frac{\nu+1}{2} + 1; \tanh^2(1)\right) \tanh^{\nu+1}(1) \;/;\; \mathrm{Re}(\nu) > -1.$$

### Finite summation

The following finite sum that contains the hyperbolic tangent function can be expressed using hyperbolic cotangent functions:

$$\sum_{k=0}^{n} \frac{1}{2^k} \tanh\left(\frac{a}{2^k}\right) = 2\coth(2\, a) - \frac{1}{2^n}\coth\left(\frac{a}{2^n}\right).$$

### Addition formulas

The hyperbolic tangent of a sum can be represented by the rule: "the hyperbolic tangent of a sum is equal to the sum of the hyperbolic tangents divided by one plus the product of the hyperbolic tangents". A similar rule is valid for the hyperbolic tangent of the difference:

$$\tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{\tanh(a)\tanh(b) + 1}$$

$$\tanh(a-b) = \frac{\tanh(a) - \tanh(b)}{1 - \tanh(a)\tanh(b)}.$$

### Multiple arguments

In the case of multiple arguments $2\, z$, $3\, z$, …, the function $\tanh(n\, z)$ can be represented as the ratio of the finite sums that includes powers of hyperbolic tangents:

$$\tanh(2z) = \frac{2\tanh(z)}{\tanh^2(z)+1}$$

$$\tanh(3z) = \frac{\tanh^3(z)+3\tanh(z)}{3\tanh^2(z)+1}$$

$$\tanh(nz) = \frac{1}{\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2k}\tanh^{2k}(z)}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}\tanh^{2k+1}(z)\ /;\ n\in\mathbb{N}^+.$$

### Half-angle formulas

The hyperbolic tangent of a half-angle can be represented using two hyperbolic functions by the following simple formulas:

$$\tanh\left(\frac{z}{2}\right) = \coth(z) - \operatorname{csch}(z)$$

$$\tanh\left(\frac{z}{2}\right) = \frac{\sinh(z)}{\cosh(z)+1}.$$

The hyperbolic sine function in the last formula can be replaced by the hyperbolic cosine function. But it leads to a more complicated representation that is valid in a horizontal strip:

$$\tanh\left(\frac{z}{2}\right) = -\frac{\sqrt{-z^2}}{z}\sqrt{\frac{1-\cosh(z)}{1+\cosh(z)}}\ /;\ |\operatorname{Im}(z)|<\pi\ \lor \operatorname{Im}(z) = -\pi\land\operatorname{Re}(z)>0\lor\operatorname{Im}(z) = \pi\land\operatorname{Re}(z)<0.$$

The last restrictions can be removed by slightly modifying the formula (now the identity is valid for all complex $z$):

$$\tanh\left(\frac{z}{2}\right) = \frac{\sqrt{z^2}}{z}\sqrt{\frac{\cosh(z)-1}{\cosh(z)+1}}\ .$$

### Sums of two direct functions

The sum of two hyperbolic tangent functions can be described by the rule: "the sum of hyperbolic tangents is equal to the hyperbolic sine of the sum multiplied by the hyperbolic secants". A similar rule is valid for the difference of two hyperbolic tangents:

$$\tanh(a)+\tanh(b) = \operatorname{sech}(a)\operatorname{sech}(b)\sinh(a+b)$$
$$\tanh(a)-\tanh(b) = \operatorname{sech}(a)\operatorname{sech}(b)\sinh(a-b).$$

### Products involving the direct function

The product of two hyperbolic tangent functions and the product of the hyperbolic tangent and cotangent have the following representations:

$$\tanh(a)\tanh(b) = \frac{\cosh(a+b)-\cosh(a-b)}{\cosh(a-b)+\cosh(a+b)}$$

$$\tanh(a)\coth(b) = \frac{\sinh(a-b)+\sinh(a+b)}{\sinh(a+b)-\sinh(a-b)}.$$

### Inequalities

The most famous inequality for the hyperbolic tangent function is the following:

$$\tanh(x) < x \,/;\, x > 0 \land x \in \mathbb{R}.$$

### Relations with its inverse function

There are simple relations between the function $\tanh(z)$ and its inverse function $\tanh^{-1}(z)$:

$$\tanh(\tanh^{-1}(z)) = z$$

$$\tanh^{-1}(\tanh(z)) = z \,/;\, -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) < 0\right).$$

The second formula is valid at least in the horizontal strip $-\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2}$. Outside of this strip a much more complicated relation (that contains the unit step, real part, and the floor functions) holds:

$$\tanh^{-1}(\tanh(z)) = z + i\pi \left\lfloor \frac{1}{2} - \frac{\operatorname{Im}(z)}{\pi} \right\rfloor - \frac{\pi i}{2}\left(1 + (-1)^{\left\lfloor \frac{\operatorname{Im}(z)}{\pi} + \frac{1}{2} \right\rfloor + \left\lfloor -\frac{\operatorname{Im}(z)}{\pi} - \frac{1}{2} \right\rfloor}\right)\theta(\operatorname{Re}(z)) \,/;\, \frac{i z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

### Representations through other hyperbolic functions

The hyperbolic tangent and cotangent functions are connected by a very simple formula that contains the linear function in the argument:

$$\tanh(z) = \coth\left(z - \frac{\pi i}{2}\right).$$

The hyperbolic tangent function can also be represented through other hyperbolic functions by the following formulas:

$$\tanh(z) = \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \tanh(z) = \frac{i \cosh\left(\frac{\pi i}{2} - z\right)}{\cosh(z)}$$

$$\tanh(z) = \frac{i \operatorname{csch}\left(\frac{\pi i}{2} - z\right)}{\operatorname{csch}(z)} \quad \tanh(z) = \frac{i \operatorname{sech}(z)}{\operatorname{sech}\left(\frac{\pi i}{2} - z\right)}.$$

### Representations through trigonometric functions

The hyperbolic tangent function has representations that use the trigonometric functions:

$$\tanh(z) = -\frac{i \sin(i z)}{\sin\left(\frac{\pi}{2} - i z\right)} \quad \tanh(z) = -\frac{i \cos\left(\frac{\pi}{2} - i z\right)}{\cos(i z)} \quad \tanh(z) = -i \tan(i z) \quad \tanh(i z) = i \tan(z)$$

$$\tanh(z) = i \cot\left(i z - \frac{\pi}{2}\right) \quad \tanh(z) = -\frac{i \csc\left(\frac{\pi}{2} - i z\right)}{\csc(i z)} \quad \tanh(z) = -\frac{i \sec(i z)}{\sec\left(\frac{\pi}{2} - i z\right)}.$$

### Applications

The hyperbolic tangent function is used throughout mathematics, the exact sciences, and engineering.

# Introduction to the Hyperbolic Functions in *Mathematica*

## Overview

The following shows how the six hyperbolic functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic functions or return them are shown. These involve numeric and symbolic calculations and plots.

## Notations

### *Mathematica* forms of notations

All six hyperbolic functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `hypFunctions` of the six hyperbolic functions in `StandardForm`.

```
hypFunctions = {Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

Here is a list `hypFunctions` of the six trigonometric functions in `TraditionalForm`.

```
hypFunctions // TraditionalForm
```

$\{\sinh(z), \cosh(z), \tanh(z), \coth(z), \operatorname{sech}(z), \cosh(z)\}$

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the hyperbolic functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
hypFunctions /. {z → 2 π z} // CForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

```
hypFunctions /. {z → 2 π z} // TeXForm
```

```
\{ \sinh (2\,\pi \,z),\cosh (2\,\pi \,z),\tanh (2\,\pi \,z),\coth (2\,\pi \,z),
  \Mfunction{Sech}(2\,\pi \,z),\cosh (2\,\pi \,z)\}
```

```
hypFunctions /. {z → 2 π z} // FortranForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns an exact result. For instance, for the argument $\pi i/6$, the `Sinh` function evaluates to $i/2$.

```
Sinh[π ⅈ/6]
```

$\frac{i}{2}$

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → π ⅈ/6
```

$\left\{\frac{i}{2}, \frac{\sqrt{3}}{2}, \frac{i}{\sqrt{3}}, -i\sqrt{3}, -2i, \frac{2}{\sqrt{3}}\right\}$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cosh[3.]
```

```
10.0677
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 2.
```

```
{3.62686, 3.7622, 0.964028, 1.03731, 0.275721, 0.265802}
```

The next inputs calculate 100-digit approximations of the six hyperbolic functions at $z = 1$.

```
N[Tanh[1], 40]
```

```
0.7615941559557648881194582826047935904128
```

```
Coth[1] // N[#, 50] &
```

```
1.3130352854993313036361612469308478329120139412405
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 100]
```

```
{1.175201193643801456882381850595600815155717981334095870229565413013307567304323895
  607117452089623392,
 1.543080634815243778477905620757061682601529112365863704737402214710769063049223698
  964264726435543036,
 0.761594155955764888119458282604793590412768597257936551596810500121953244576638483
  4589475216736767144,
 1.313035285499331303636161246930847832912013941240452655543152967567084270461874382
  674679241480856303,
 0.850918128239321545133842763287175284181724660910339616990421151729003364321465103
  8997301773288938124,
 0.648054273663885399574977353226150323108489312071942023037865337318717595646712830
  2808547853078928924}
```

Within a second, it is possible to calculate thousands of digits for the hyperbolic functions. The next input calculates 10000 digits for sinh(1), cosh(1), tanh(1), coth(1), sech(1), and csch(1) and analyzes the frequency of the occurrence of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 10 000]]
```

```
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014}, {4, 986}, {5, 1001},
  {6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}, {{0, 1015}, {1, 960}, {2, 997},
  {3, 1037}, {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}},
 {{0, 971}, {1, 1023}, {2, 1016}, {3, 970}, {4, 949}, {5, 1052}, {6, 981},
  {7, 1056}, {8, 1010}, {9, 972}}, {{0, 975}, {1, 986}, {2, 1023},
  {3, 1004}, {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}},
 {{0, 979}, {1, 1030}, {2, 987}, {3, 992}, {4, 1016}, {5, 1030}, {6, 1021},
  {7, 969}, {8, 974}, {9, 1002}}, {{0, 1009}, {1, 971}, {2, 1018},
  {3, 994}, {4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here are 50-digit approximations to the six hyperbolic functions at the complex argument $z = 3 + 5\,i$.

```
N[Csch[3 + 5 ⅈ], 100]
```

```
0.028058516423080075996315984260274369705154012388728593163173673096445331808273091↘
  1484269546408531396 +
 0.095323634674178402851915930706256451645442166878775479803879772793331583262276221↘
   38939784445056701747 ⅈ
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{2.8416922956063519438168753953062364359281841632360 -
  9.6541254768548391365515436340301659921919691213853 ⅈ,
 2.8558150042273872913639018630946098374643609536732 -
  9.6063834484325811198111562160434163877218590394033 ⅈ,
 1.0041647106948152119205166259313184311852454735738 -
  0.0027082358362240721322640353684331035927960259125751 ⅈ,
 0.99584531857585412976042001587164841711026557204102 +
  0.0026857984057585256446537711012814749378977439361108 ⅈ,
 0.028058516423080075996315984260274369705154012388729 +
  0.095323634674178402851915930706256451645442166878775 ⅈ,
 0.028433530909971667358833684958646399417265586614624 +
  0.095644640955286344684316595933099452259073530811833 ⅈ}
```

*Mathematica* always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
```

```
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}
```

```
% // InputForm
```

```
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,
 3.626860407847018767668213982801262016441`20}
```

```
Precision[%%]
```

```
16
```

### Simplification of the argument

*Mathematica* uses symmetries and periodicities of all the hyperbolic functions to simplify expressions. Here are some examples.

```
Sinh[-z]
```

```
-Sinh[z]
```

```
Sinh[z + π ⅈ]
```

```
-Sinh[z]
```

```
Sinh[z + 2 π ⅈ]
```

```
Sinh[z]
```

```
Sinh[z + 34 π ⅈ]
```

```
Sinh[z]
```

```
{Sinh[-z], Cosh[-z], Tanh[-z], Coth[-z], Csch[-z], Sech[-z]}
```

```
{-Sinh[z], Cosh[z], -Tanh[z], -Coth[z], -Csch[z], Sech[z]}
```

```
{Sinh[z + π ⅈ], Cosh[z + π ⅈ], Tanh[z + π ⅈ], Coth[z + π ⅈ], Csch[z + π ⅈ], Sech[z + π ⅈ]}
```

```
{-Sinh[z], -Cosh[z], Tanh[z], Coth[z], -Csch[z], -Sech[z]}
```

```
{Sinh[z + 2 π ⅈ], Cosh[z + 2 π ⅈ], Tanh[z + 2 π ⅈ],
 Coth[z + 2 π ⅈ], Csch[z + 2 π ⅈ], Sech[z + 2 π ⅈ]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
{Sinh[z + 342 π ⅈ], Cosh[z + 342 π ⅈ], Tanh[z + 342 π ⅈ],
 Coth[z + 342 π ⅈ], Csch[z + 342 π ⅈ], Sech[z + 342 π ⅈ]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

*Mathematica* automatically simplifies the composition of the direct and the inverse hyperbolic functions into the argument.

```
{Sinh[ArcSinh[z]], Cosh[ArcCosh[z]], Tanh[ArcTanh[z]],
 Coth[ArcCoth[z]], Csch[ArcCsch[z]], Sech[ArcSech[z]]}
```

```
{z, z, z, z, z, z}
```

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

```
{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]],
 Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}
```

$$\left\{z,\ \sqrt{\frac{-1+z}{1+z}}\ (1+z),\ \frac{z}{\sqrt{1-z^2}},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}\ z},\ \frac{1}{z},\ \frac{\sqrt{\frac{1-z}{1+z}}\ (1+z)}{z}\right\}$$

**{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]], Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}**

$$\left\{\sqrt{1+z^2},\ z,\ \frac{1}{\sqrt{1-z^2}},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}},\ \sqrt{1+\frac{1}{z^2}},\ \frac{1}{z}\right\}$$

**{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]], Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tanh[ArcSech[z]]}**

$$\left\{\frac{z}{\sqrt{1+z^2}},\ \frac{\sqrt{\frac{-1+z}{1+z}}\ (1+z)}{z},\ z,\ \frac{1}{z},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\ z},\ \sqrt{\frac{1-z}{1+z}}\ (1+z)\right\}$$

**{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]], Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}**

$$\left\{\frac{\sqrt{1+z^2}}{z},\ \frac{z}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{1}{z},\ z,\ \sqrt{1+\frac{1}{z^2}}\ z,\ \frac{1}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

**{Csch[ArcSinh[z]], Csch[ArcCosh[z]], Csch[ArcTanh[z]], Csch[ArcCoth[z]], Csch[ArcCsch[z]], Csch[ArcSech[z]]}**

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{\sqrt{1-z^2}}{z},\ \sqrt{1-\frac{1}{z^2}}\ z,\ z,\ \frac{z}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

**{Sech[ArcSinh[z]], Sech[ArcCosh[z]], Sech[ArcTanh[z]], Sech[ArcCoth[z]], Sech[ArcCsch[z]], Sech[ArcSech[z]]}**

$$\left\{\frac{1}{\sqrt{1+z^2}},\ \frac{1}{z},\ \sqrt{1-z^2},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, e $\pi k/2+i z$ or $\pi k/2-i z$ with integer $k$, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\mathbf{Tanh}\left[\frac{\pi\, i}{2}-z\right]$$

```
-Coth[z]
```

**Csch[ⅈ z]**

```
-ⅈ Csc[z]
```

**$\{\text{Sinh}[\frac{\pi i}{2} - z], \text{Cosh}[\frac{\pi i}{2} - z], \text{Tanh}[\frac{\pi i}{2} - z], \text{Coth}[\frac{\pi i}{2} - z], \text{Csch}[\frac{\pi i}{2} - z], \text{Sech}[\frac{\pi i}{2} - z]\}$**

```
{ⅈ Cosh[z], -ⅈ Sinh[z], -Coth[z], -Tanh[z], -ⅈ Sech[z], ⅈ Csch[z]}
```

**{Sinh[ⅈ z], Cosh[ⅈ z], Tanh[ⅈ z], Coth[ⅈ z], Csch[ⅈ z], Sech[ⅈ z]}**

```
{ⅈ Sin[z], Cos[z], ⅈ Tan[z], -ⅈ Cot[z], -ⅈ Csc[z], Sec[z]}
```

### Simplification of simple expressions containing hyperbolic functions

Sometimes simple arithmetic operations containing hyperbolic functions can automatically produce other hyperbolic functions.

**1 / Sech[z]**

```
Cosh[z]
```

**{1 / Sinh[z], 1 / Cosh[z], 1 / Tanh[z], 1 / Coth[z], 1 / Csch[z], 1 / Sech[z],
Sinh[z] / Cosh[z], Cosh[z] / Sinh[z], Sinh[z] / Sinh[π ⅈ / 2 - z], Cosh[z] / Sinh[z] ^2}**

```
{Csch[z], Sech[z], Coth[z], Tanh[z], Sinh[z],
 Cosh[z], Tanh[z], Coth[z], -ⅈ Tanh[z], Coth[z] Csch[z]}
```

### Hyperbolic functions as special cases of more general functions

All hyperbolic functions can be treated as particular cases of some more advanced special functions. For example, sinh($z$) and cosh($z$) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

**$\text{BesselI}[\frac{1}{2}, z]$**

$$\frac{\sqrt{\frac{2}{\pi}}\ \text{Sinh}[z]}{\sqrt{z}}$$

**MathieuC[1, 0, ⅈ z]**

```
Cosh[z]
```

**JacobiSN[z, 1]**

```
Tanh[z]
```

```
{BesselI[1/2, z], MathieuS[1, 0, ⅈ z], JacobiSD[ⅈ z, 0],
 HypergeometricPFQ[{}, {3/2}, z^2/4], MeijerG[{{}, {}}, {{0}, {-1/2}}, -z^2/4]}
```

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Sinh}[z]}{\sqrt{z}},\ i\,\mathrm{Sinh}[z],\ i\,\mathrm{Sinh}[z],\ \frac{\mathrm{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}},\ \frac{2\,\mathrm{Sinh}[z]}{\sqrt{\pi}\,z}\right\}$$

```
{BesselI[-1/2, z], MathieuC[1, 0, ⅈ z], JacobiCD[ⅈ z, 0],
 Hypergeometric0F1[1/2, z^2/4], MeijerG[{{}, {}}, {{0}, {1/2}}, -z^2/4]}
```

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Cosh}[z]}{\sqrt{z}},\ \mathrm{Cosh}[z],\ \mathrm{Cosh}[z],\ \mathrm{Cosh}\left[\sqrt{z^2}\right],\ \frac{\mathrm{Cosh}[z]}{\sqrt{\pi}}\right\}$$

```
{JacobiSC[ⅈ z, 0], JacobiNS[z, 1], JacobiNS[ⅈ z, 0], JacobiDC[ⅈ z, 0]}
```

```
{ⅈ Tanh[z], Coth[z], -ⅈ Csch[z], Sech[z]}
```

## Equivalence transformations carried out by specialized *Mathematica* functions

### General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one chosen direction and the result can be overly complicated. For example, the expression $i\cosh(z)/2$ is generally preferable to the more complicated $\sinh(\pi\, i/2 - z)\cosh(\pi\, i/3)$. *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like $\sinh(2\,z)\cosh(\pi\, i/16)$ should not be automatically expanded into the more complicated expression $\sinh(z)\cosh(z)\left(2+\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce these types of expansions. Some of them are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sinh[x - y]]
```

```
Cosh[y] Sinh[x] - Cosh[x] Sinh[y]
```

```
Cosh[4 z] // TrigExpand
```

$$\mathrm{Cosh}[z]^4 + 6\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]^2 + \mathrm{Sinh}[z]^4$$

```
TrigExpand[{{Sinh[x + y], Sinh[3 z]},
            {Cosh[x + y], Cosh[3 z]},
            {Tanh[x + y], Tanh[3 z]},
            {Coth[x + y], Coth[3 z]},
            {Csch[x + y], Csch[3 z]},
            {Sech[x + y], Sech[3 z]}}]
```

$$\Big\{\{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y],\ 3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3\},$$
$$\{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y],\ \mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2\},$$
$$\Big\{\frac{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}+\frac{\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]},$$
$$\frac{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}+\frac{\mathrm{Sinh}[z]^3}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}\Big\},$$
$$\Big\{\frac{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}+\frac{\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]},$$
$$\frac{\mathrm{Cosh}[z]^3}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}+\frac{3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]},\ \frac{1}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}\Big\},$$
$$\Big\{\frac{1}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]},\ \frac{1}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}\Big\}\Big\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sinh[x + y], Sinh[3 z]}, {Cosh[x + y], Cosh[3 z]}, {Tanh[x + y], Tanh[3 z]},
    {Coth[x + y], Coth[3 z]}, {Csch[x + y], Csch[3 z]}, {Sech[x + y], Sech[3 z]}}]]
```

$$\mathrm{Sinh}[x+y]==\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]$$
$$\mathrm{Sinh}[3\,z]==3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3$$
$$\mathrm{Cosh}[x+y]==\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]$$
$$\mathrm{Cosh}[3\,z]==\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2$$
$$\mathrm{Tanh}[x+y]==\frac{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}+\frac{\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}$$
$$\mathrm{Tanh}[3\,z]==\frac{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}+\frac{\mathrm{Sinh}[z]^3}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}$$
$$\mathrm{Coth}[x+y]==\frac{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}+\frac{\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}$$
$$\mathrm{Coth}[3\,z]==\frac{\mathrm{Cosh}[z]^3}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}+\frac{3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}$$
$$\mathrm{Csch}[x+y]==\frac{1}{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]+\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}$$
$$\mathrm{Csch}[3\,z]==\frac{1}{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]+\mathrm{Sinh}[z]^3}$$
$$\mathrm{Sech}[x+y]==\frac{1}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y]+\mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}$$
$$\mathrm{Sech}[3\,z]==\frac{1}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}$$

## TrigFactor

The command `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sinh[x] + ⅈ Cosh[y]]
```

$$\left(i\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]+\text{Sinh}\left[\frac{x}{2}-\frac{y}{2}\right]\right)\left(\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right]-i\,\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right]\right)$$

```
Tanh[x] - Coth[y] // TrigFactor
```

$$-\text{Cosh}[x-y]\,\text{Csch}[y]\,\text{Sech}[x]$$

```
TrigFactor[{Sinh[x] + Sinh[y],
            Cosh[x] + Cosh[y],
            Tanh[x] + Tanh[y],
            Coth[x] + Coth[y],
            Csch[x] + Csch[y],
            Sech[x] + Sech[y]}]
```

$$\left\{2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right],\ 2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right],\ \text{Sech}[x]\,\text{Sech}[y]\,\text{Sinh}[x+y],\right.$$
$$\text{Csch}[x]\,\text{Csch}[y]\,\text{Sinh}[x+y],\ \frac{1}{2}\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Csch}\left[\frac{x}{2}\right]\text{Csch}\left[\frac{y}{2}\right]\text{Sech}\left[\frac{x}{2}\right]\text{Sech}\left[\frac{y}{2}\right]\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right],$$
$$\left.\frac{2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right]}{\left(\text{Cosh}\left[\frac{x}{2}\right]-i\,\text{Sinh}\left[\frac{x}{2}\right]\right)\left(\text{Cosh}\left[\frac{x}{2}\right]+i\,\text{Sinh}\left[\frac{x}{2}\right]\right)\left(\text{Cosh}\left[\frac{y}{2}\right]-i\,\text{Sinh}\left[\frac{y}{2}\right]\right)\left(\text{Cosh}\left[\frac{y}{2}\right]+i\,\text{Sinh}\left[\frac{y}{2}\right]\right)}\right\}$$

### TrigReduce

The command `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sinh[z]^3]
```

$$\frac{1}{4}(-3\,\text{Sinh}[z]+\text{Sinh}[3\,z])$$

```
Sinh[x] Cosh[y] // TrigReduce
```

$$\frac{1}{2}(\text{Sinh}[x-y]+\text{Sinh}[x+y])$$

```
TrigReduce[{Sinh[z]^2, Cosh[z]^2, Tanh[z]^2, Coth[z]^2, Csch[z]^2, Sech[z]^2}]
```

$$\left\{\frac{1}{2}(-1+\text{Cosh}[2\,z]),\ \frac{1}{2}(1+\text{Cosh}[2\,z]),\right.$$
$$\left.\frac{-1+\text{Cosh}[2\,z]}{1+\text{Cosh}[2\,z]},\ \frac{1+\text{Cosh}[2\,z]}{-1+\text{Cosh}[2\,z]},\ \frac{2}{-1+\text{Cosh}[2\,z]},\ \frac{2}{1+\text{Cosh}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sinh[x + y], Sinh[3 z], Sinh[x] Sinh[y]},
                        {Cosh[x + y], Cosh[3 z], Cosh[x] Cosh[y]},
                        {Tanh[x + y], Tanh[3 z], Tanh[x] Tanh[y]},
                        {Coth[x + y], Coth[3 z], Coth[x] Coth[y]},
                        {Csch[x + y], Csch[3 z], Csch[x] Csch[y]},
                        {Sech[x + y], Sech[3 z], Sech[x] Sech[y]}}]]
```

$$\left\{\left\{\text{Sinh}[x+y],\ \text{Sinh}[3\,z],\ \frac{1}{2}\,(-\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},\right.$$
$$\left\{\text{Cosh}[x+y],\ \text{Cosh}[3\,z],\ \frac{1}{2}\,(\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},$$
$$\left\{\text{Tanh}[x+y],\ \text{Tanh}[3\,z],\ \frac{-\text{Cosh}[x-y]+\text{Cosh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Coth}[x+y],\ \text{Coth}[3\,z],\ \frac{-\text{Cosh}[x-y]-\text{Cosh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Csch}[x+y],\ \text{Csch}[3\,z],\ -\frac{2}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left.\left\{\text{Sech}[x+y],\ \text{Sech}[3\,z],\ \frac{2}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sinh[x] + Sinh[y], Cosh[x] + Cosh[y],
   Tanh[x] + Tanh[y], Coth[x] + Coth[y], Csch[x] + Csch[y], Sech[x] + Sech[y]}]]
```

$$\left\{\text{Sinh}[x]+\text{Sinh}[y],\ \text{Cosh}[x]+\text{Cosh}[y],\ \frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]},\right.$$
$$\left.-\frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ -\frac{2\,(\text{Sinh}[x]+\text{Sinh}[y])}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ \frac{2\,(\text{Cosh}[x]+\text{Cosh}[y])}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}$$

## TrigToExp

The command `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponentials or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sinh[2 z]]
```

$$-\frac{1}{2}\,e^{-2\,z}+\frac{e^{2\,z}}{2}$$

```
Sinh[z] Tanh[2 z] // TrigToExp
```

$$\frac{(-e^{-z}+e^{z})\,\left(-e^{-2\,z}+e^{2\,z}\right)}{2\,\left(e^{-2\,z}+e^{2\,z}\right)}$$

```
TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]
```

$$\left\{-\frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{-e^{-z}+e^{z}}{e^{-z}+e^{z}},\ \frac{e^{-z}+e^{z}}{-e^{-z}+e^{z}},\ \frac{2}{-e^{-z}+e^{z}},\ \frac{2}{e^{-z}+e^{z}}\right\}$$

## ExpToTrig

The command ExpToTrig converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately opposite to TrigToExp. Here are some examples.

**ExpToTrig**$\left[e^{x\beta}\right]$

Cosh[x β] + Sinh[x β]

$\dfrac{e^{x\alpha}-e^{x\beta}}{e^{x\gamma}+e^{x\delta}}$ **// ExpToTrig**

$$\frac{\text{Cosh}[x\,\alpha]-\text{Cosh}[x\,\beta]+\text{Sinh}[x\,\alpha]-\text{Sinh}[x\,\beta]}{\text{Cosh}[x\,\gamma]+\text{Cosh}[x\,\delta]+\text{Sinh}[x\,\gamma]+\text{Sinh}[x\,\delta]}$$

**ExpToTrig[TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]]**

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

**ExpToTrig**$\left[\left\{\alpha\, e^{-x\beta}+\alpha\, e^{x\beta},\ \alpha\, e^{-x\beta}+\gamma\, e^{i x\beta}\right\}\right]$

{2 α Cosh[x β], γ Cos[x β] + α Cosh[x β] + ⅈ γ Sin[x β] − α Sinh[x β]}

### ComplexExpand

The function ComplexExpand expands expressions assuming that all the occurring variables are real. The value option TargetFunctions is a list of functions from the set {Re, Im, Abs, Arg, Conjugate, Sign}. ComplexExpand tries to give results in terms of the specified functions. Here are some examples.

**ComplexExpand[Sinh[x + ⅈ y] Cosh[x − ⅈ y]]**

$$\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x]+ i\left(\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2\right)$$

**Csch[x + ⅈ y] Sech[x − ⅈ y] // ComplexExpand**

$$-\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}-\frac{4\,\text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}+ i\left(\frac{4\,\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y]}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}-\frac{4\,\text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])\,(\text{Cos}[2\,y]+\text{Cosh}[2\,x])}\right)$$

**li1 = {Sinh[x + ⅈ y], Cosh[x + ⅈ y], Tanh[x + ⅈ y], Coth[x + ⅈ y], Csch[x + ⅈ y], Sech[x + ⅈ y]}**

{Sinh[x + ⅈ y], Cosh[x + ⅈ y], Tanh[x + ⅈ y], Coth[x + ⅈ y], Csch[x + ⅈ y], Sech[x + ⅈ y]}

**ComplexExpand[li1]**

$$\Big\{i\,\text{Cosh}[x]\,\text{Sin}[y]+\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x]+i\,\text{Sin}[y]\,\text{Sinh}[x],$$
$$\frac{i\,\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}+\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ \frac{i\,\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},$$
$$\frac{2\,i\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}-\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}-\frac{2\,i\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\Big\{\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x],\ \frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},$$
$$-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\Big\{\text{Cosh}[x]\,\text{Sin}[y],\ \text{Sin}[y]\,\text{Sinh}[x],\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},$$
$$\frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\Big\}$$

```
ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\Big\{\sqrt{\text{Cosh}[x]^2\,\text{Sin}[y]^2+\text{Cos}[y]^2\,\text{Sinh}[x]^2},\ \sqrt{\text{Cos}[y]^2\,\text{Cosh}[x]^2+\text{Sin}[y]^2\,\text{Sinh}[x]^2},$$
$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}},$$
$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$
$$\sqrt{\frac{4\,\text{Cosh}[x]^2\,\text{Sin}[y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{4\,\text{Cos}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$
$$\sqrt{\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{4\,\text{Sin}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}}\Big\}$$

```
% // Simplify[#, {x, y} ∈ Reals] &
```

$$\Big\{\frac{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Sin}[2\,y]^2+\text{Sinh}[2\,x]^2}}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},$$
$$\sqrt{-\frac{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}\Big\}$$

```
ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]
```

$$\left\{\text{ArcTan}[\text{Cos}[y]\,\text{Sinh}[x],\,\text{Cosh}[x]\,\text{Sin}[y]],\,\text{ArcTan}[\text{Cos}[y]\,\text{Cosh}[x],\,\text{Sin}[y]\,\text{Sinh}[x]],\right.$$
$$\text{ArcTan}\left[\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\,\frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right],$$
$$\text{ArcTan}\left[-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\,\frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}\right],$$
$$\text{ArcTan}\left[-\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\,\frac{2\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}\right],$$
$$\left.\text{ArcTan}\left[\frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\,-\frac{2\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right]\right\}$$

**% // Simplify[#, {x, y} ∈ Reals] &**

```
{ArcTan[Cos[y] Sinh[x], Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]],
 ArcTan[Sinh[2 x], Sin[2 y]], ArcTan[Cosh[x] Sinh[x], -Cos[y] Sin[y]],
 ArcTan[Cos[y] Sinh[x], -Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], -Sin[y] Sinh[x]]}
```

**ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify**

$$\left\{-i\,\text{Cosh}[x]\,\text{Sin}[y]+\text{Cos}[y]\,\text{Sinh}[x],\,\text{Cos}[y]\,\text{Cosh}[x]-i\,\text{Sin}[y]\,\text{Sinh}[x],\right.$$
$$\frac{-i\,\text{Sin}[2\,y]+\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\,-\frac{i\,\text{Sin}[2\,y]+\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},$$
$$\left.\frac{1}{-i\,\text{Cosh}[x]\,\text{Sin}[y]+\text{Cos}[y]\,\text{Sinh}[x]},\,\frac{1}{\text{Cos}[y]\,\text{Cosh}[x]-i\,\text{Sin}[y]\,\text{Sinh}[x]}\right\}$$

### Simplify

The command `Simplify` performs a sequence of algebraic transformations on an expression, and returns the simplest form it finds. Here are two examples.

**Simplify[Sinh[2 z] / Sinh[z]]**

2 Cosh[z]

**Sinh[2 z] / Cosh[z] // Simplify**

2 Sinh[z]

Here is a large collection of hyperbolic identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ (Cosh[z]^2 - Sinh[z]^2 == 1 ∧
  Sinh[z]^2 == (Cosh[2 z] - 1)/2 ∧ Cosh[z]^2 == (1 + Cosh[2 z])/2 ∧
  Tanh[z]^2 == (Cosh[2 z] - 1)/(Cosh[2 z] + 1) ∧ Coth[z]^2 == (Cosh[2 z] + 1)/(Cosh[2 z] - 1) ∧
  Sinh[2 z] == 2 Sinh[z] Cosh[z] ∧ Cosh[2 z] == Cosh[z]^2 + Sinh[z]^2 == 2 Cosh[z]^2 - 1 ∧
  Sinh[a + b] == Sinh[a] Cosh[b] + Cosh[a] Sinh[b] ∧
    Sinh[a - b] == Sinh[a] Cosh[b] - Cosh[a] Sinh[b] ∧
  Cosh[a + b] == Cosh[a] Cosh[b] + Sinh[a] Sinh[b] ∧
    Cosh[a - b] == Cosh[a] Cosh[b] - Sinh[a] Sinh[b] ∧
  Sinh[a] + Sinh[b] == 2 Sinh[(a + b)/2] Cosh[(a - b)/2] ∧
    Sinh[a] - Sinh[b] == 2 Cosh[(a + b)/2] Sinh[(a - b)/2] ∧
  Cosh[a] + Cosh[b] == 2 Cosh[(a + b)/2] Cosh[(a - b)/2] ∧
    Cosh[a] - Cosh[b] == -2 Sinh[(a + b)/2] Sinh[(b - a)/2] ∧
  Tanh[a] + Tanh[b] == Sinh[a + b]/(Cosh[a] Cosh[b]) ∧ Tanh[a] - Tanh[b] == Sinh[a - b]/(Cosh[a] Cosh[b]) ∧
  A Sinh[z] + B Cosh[z] == A Sqrt[1 - B^2/A^2] Sinh[z + ArcTanh[B/A]] ∧
  Sinh[a] Sinh[b] == (Cosh[a + b] - Cosh[a - b])/2 ∧ Cosh[a] Cosh[b] ==
    (Cosh[a - b] + Cosh[a + b])/2 ∧ Sinh[a] Cosh[b] == (Sinh[a + b] + Sinh[a - b])/2 ∧
  Sinh[z/2]^2 == (Cosh[z] - 1)/2 ∧ Cosh[z/2]^2 == (1 + Cosh[z])/2 ∧
  Tanh[z/2] == (Cosh[z] - 1)/Sinh[z] == Sinh[z]/(1 + Cosh[z]) ∧ Coth[z/2] == Sinh[z]/(Cosh[z] - 1) == (1 + Cosh[z])/Sinh[z])
```

```
True
```

The command `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\sinh(x) > 0$ for all real positive $x$, and uses the periodicity of hyperbolic functions for the symbolic integer coefficient $k$ of $k \pi i$.

```
Simplify[Abs[Sinh[x]] > 0, x > 0]
```

```
True
```

```
Abs[Sinh[x]] > 0 // Simplify[#, x > 0] &
```

```
True
```

```
Simplify[{Sinh[z + 2 k π ⅈ], Cosh[z + 2 k π ⅈ], Tanh[z + k π ⅈ],
  Coth[z + k π ⅈ], Csch[z + 2 k π ⅈ], Sech[z + 2 k π ⅈ]}, k ∈ Integers]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
Simplify[{Sinh[z + k π ⅈ] / Sinh[z], Cosh[z + k π ⅈ] / Cosh[z], Tanh[z + k π ⅈ] / Tanh[z],
  Coth[z + k π ⅈ] / Coth[z], Csch[z + k π ⅈ] / Csch[z], Sech[z + k π ⅈ] / Sech[z]}, k ∈ Integers]
```

$$\left\{(-1)^k,\ (-1)^k,\ 1,\ 1,\ (-1)^k,\ (-1)^k\right\}$$

*Mathematica* also knows that the composition of inverse and direct hyperbolic functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSinh[Sinh[z]], ArcTanh[Tanh[z]],
  ArcCoth[Coth[z]], ArcCsch[Csch[z]]}, -π / 2 < Im[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCosh[Cosh[z]], ArcSech[Sech[z]]}, -π < Im[z] < π ∧ Re[z] > 0]
```

```
{z, z}
```

### FunctionExpand (and Together)

While the hyperbolic functions auto-evaluate for simple fractions of $\pi\, i$, for more complicated cases they stay as hyperbolic functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\mathrm{Cosh}\left[\frac{\pi\, i}{32}\right]$$

$$\mathrm{Cos}\left[\frac{\pi}{32}\right]$$

$$\mathrm{FunctionExpand}\left[\mathrm{Cosh}\left[\frac{\pi\, i}{32}\right]\right]$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\mathrm{Coth}\left[\frac{\pi\, i}{24}\right]\ //\ \mathrm{FunctionExpand}$$

$$-\frac{i\left(\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}\right)}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{\text{Sinh}\left[\frac{\pi i}{16}\right],\ \text{Cosh}\left[\frac{\pi i}{16}\right],\ \text{Tanh}\left[\frac{\pi i}{16}\right],\ \text{Coth}\left[\frac{\pi i}{16}\right],\ \text{Csch}\left[\frac{\pi i}{16}\right],\ \text{Sech}\left[\frac{\pi i}{16}\right]\right\}$$

$$\left\{i\,\text{Sin}\left[\frac{\pi}{16}\right],\ \text{Cos}\left[\frac{\pi}{16}\right],\ i\,\text{Tan}\left[\frac{\pi}{16}\right],\ -i\,\text{Cot}\left[\frac{\pi}{16}\right],\ -i\,\text{Csc}\left[\frac{\pi}{16}\right],\ \text{Sec}\left[\frac{\pi}{16}\right]\right\}$$

**FunctionExpand[%]**

$$\left\{\frac{1}{2}\,i\,\sqrt{2-\sqrt{2+\sqrt{2}}},\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ i\,\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\right.$$
$$\left.-i\,\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ -\frac{2\,i}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

$$\left\{\text{Sinh}\left[\frac{\pi i}{60}\right],\ \text{Cosh}\left[\frac{\pi i}{60}\right],\ \text{Tanh}\left[\frac{\pi i}{60}\right],\ \text{Coth}\left[\frac{\pi i}{60}\right],\ \text{Csch}\left[\frac{\pi i}{60}\right],\ \text{Sech}\left[\frac{\pi i}{60}\right]\right\}$$

$$\left\{i\,\text{Sin}\left[\frac{\pi}{60}\right],\ \text{Cos}\left[\frac{\pi}{60}\right],\ i\,\text{Tan}\left[\frac{\pi}{60}\right],\ -i\,\text{Cot}\left[\frac{\pi}{60}\right],\ -i\,\text{Csc}\left[\frac{\pi}{60}\right],\ \text{Sec}\left[\frac{\pi}{60}\right]\right\}$$

**Together**[**FunctionExpand**[**%**]]

$$\left\{\frac{1}{16}\,i\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),\right.$$
$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$
$$-\frac{i\left(1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$
$$\frac{i\left(1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$
$$-\frac{16\,i}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$
$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\text{Sinh}\left[\frac{\pi i}{9}\right], \text{Cosh}\left[\frac{\pi i}{9}\right], \text{Tanh}\left[\frac{\pi i}{9}\right], \text{Coth}\left[\frac{\pi i}{9}\right], \text{Csch}\left[\frac{\pi i}{9}\right], \text{Sech}\left[\frac{\pi i}{9}\right]\right\}$$

$$\left\{i \,\text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], i \,\text{Tan}\left[\frac{\pi}{9}\right], -i \,\text{Cot}\left[\frac{\pi}{9}\right], -i \,\text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\left\{\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+ i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),\right.$$

$$\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}- i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$-(8i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+ 2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\left.-(8i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}- i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$$\{\text{Root}[3+36\,\#1^2+96\,\#1^4+64\,\#1^6\,\&, 4], \text{Root}[-1-6\,\#1+8\,\#1^3\,\&, 3],$$
$$\text{Root}[3+27\,\#1^2+33\,\#1^4+\#1^6\,\&, 4], \text{Root}[1+33\,\#1^2+27\,\#1^4+3\,\#1^6\,\&, 3],$$
$$\text{Root}[64+96\,\#1^2+36\,\#1^4+3\,\#1^6\,\&, 5], \text{Root}[-8+6\,\#1^2+\#1^3\,\&, 3]\}$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler forms. Here are some examples.

$$\text{FunctionExpand}\left[\text{Coth}\left[\sqrt{-z^2}\right]\right]$$

$$-\frac{\sqrt{-z}\,\mathrm{Cot}[z]}{\sqrt{z}}$$

**Tanh**$\left[\sqrt{i z^2}\right]$ **// FunctionExpand**

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4} z}\,\sqrt{(-1)^{3/4} z}\,\mathrm{Tanh}\left[(-1)^{1/4} z\right]}{z}$$

$\left\{\mathrm{Sinh}\left[\sqrt{z^2}\right], \mathrm{Cosh}\left[\sqrt{z^2}\right], \mathrm{Tanh}\left[\sqrt{z^2}\right], \mathrm{Coth}\left[\sqrt{z^2}\right], \mathrm{Csch}\left[\sqrt{z^2}\right], \mathrm{Sech}\left[\sqrt{z^2}\right]\right\}$ **// FunctionExpand**

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{-i z}\,\sqrt{i z}\,\mathrm{Tanh}[z]}{z}, \frac{\sqrt{-i z}\,\sqrt{i z}\,\mathrm{Coth}[z]}{z}, \frac{\sqrt{-i z}\,\sqrt{i z}\,\mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

**Simplify[%]**

$$\left\{\frac{\sqrt{z^2}\,\mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{z^2}\,\mathrm{Tanh}[z]}{z}, \frac{\sqrt{z^2}\,\mathrm{Coth}[z]}{z}, \frac{\sqrt{z^2}\,\mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Here are some similar examples.

**Sinh[2 ArcTanh[z]] // FunctionExpand**

$$\frac{2 z}{1 - z^2}$$

**Cosh**$\left[\frac{\mathrm{ArcCoth}[z]}{2}\right]$ **// FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-i z}\,\sqrt{i z}}{\sqrt{(-1+z)\,(1+z)}}}}{\sqrt{2}}$$

**{Sinh[2 ArcSinh[z]], Cosh[2 ArcCosh[z]], Tanh[2 ArcTanh[z]], Coth[2 ArcCoth[z]], Csch[2 ArcCsch[z]], Sech[2 ArcSech[z]]} // FunctionExpand**

$$\left\{2 z \sqrt{i\,(-i + z)}\,\sqrt{-i\,(i + z)}, z^2 + (-1 + z)\,(1 + z), -\frac{2\,(-1 + z)\,z\,(1 + z)}{\left(1 - z^2\right)\left(1 + z^2\right)}, \frac{1}{2}\left(1 - \frac{1}{z^2}\right) z \left(\frac{1}{(-1 + z)\,(1 + z)} + \frac{z^2}{(-1 + z)\,(1 + z)}\right), \frac{\sqrt{-z}\,z^{3/2}}{2\sqrt{-1 - z^2}}, \frac{z^2}{2 - z^2}\right\}$$

$$\left\{\text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Tanh}\left[\frac{\text{ArcTanh}[z]}{2}\right], \text{Coth}\left[\frac{\text{ArcCoth}[z]}{2}\right], \text{Csch}\left[\frac{\text{ArcCsch}[z]}{2}\right], \text{Sech}\left[\frac{\text{ArcSech}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{-1+\sqrt{i(-i+z)}\sqrt{-i(i+z)}}}{\sqrt{2}\sqrt{-iz}\sqrt{iz}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z}\sqrt{1+z}}, z\left(1+\frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-iz}\sqrt{iz}}\right), \frac{\sqrt{2}\sqrt{-\frac{i}{z}}\sqrt{\frac{i}{z}}\,z}{\sqrt{-1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\sqrt{z}}}}, \frac{\sqrt{2}\sqrt{-z}}{\sqrt{-1-z}}\right\}$$

**Simplify[%]**

$$\left\{\frac{z\sqrt{-1+\sqrt{1+z^2}}}{\sqrt{2}\sqrt{z^2}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z^2}}, z+\frac{\sqrt{z^2}\sqrt{-1+z^2}}{z}, \frac{\sqrt{2}\sqrt{\frac{1}{z^2}}\,z}{\sqrt{-1+\sqrt{1+\frac{1}{z^2}}}}, \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

### FullSimplify

The function FullSimplify tries a wider range of transformations than the function Simplify and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cosh}\left[\frac{1}{2}\text{Log}[1-iz]-\frac{1}{2}\text{Log}[1+iz]\right] \text{ // Simplify}$$

$$\text{Cosh}\left[\frac{1}{2}(\text{Log}[1-iz]-\text{Log}[1+iz])\right]$$

**% // FullSimplify**

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\text{Sinh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Tanh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Csch}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right]\right\} \text{ // Simplify}$$

$$\left\{-i z, \frac{1-z^2+i z\sqrt{1-z^2}}{i z+\sqrt{1-z^2}}, -\frac{-1+\left(i z+\sqrt{1-z^2}\right)^2}{1+\left(i z+\sqrt{1-z^2}\right)^2}, -\frac{1+\left(i z+\sqrt{1-z^2}\right)^2}{-1+\left(i z+\sqrt{1-z^2}\right)^2}, \frac{i}{z}, \frac{2\left(i z+\sqrt{1-z^2}\right)}{1+\left(i z+\sqrt{1-z^2}\right)^2}\right\}$$

$$\left\{\text{Sinh}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right], \text{Tanh}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right], \text{Csch}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[i z+\sqrt{1-z^2}\right]\right]\right\} \text{ // FullSimplify}$$

$$\left\{-i z, \sqrt{1-z^2}, -\frac{i z}{\sqrt{1-z^2}}, \frac{i\sqrt{1-z^2}}{z}, \frac{i}{z}, \frac{1}{\sqrt{1-z^2}}\right\}$$

## Operations performed by specialized *Mathematica* functions

### Series expansions

Calculating the series expansion of hyperbolic functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sinh[z], {z, 0, 5}]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 3}]
```

$$\left\{z+\frac{z^3}{6}+O[z]^4, 1+\frac{z^2}{2}+O[z]^4, z-\frac{z^3}{3}+O[z]^4, \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+O[z]^4, \frac{1}{z}-\frac{z}{6}+\frac{7 z^3}{360}+O[z]^4, 1-\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Coth[z], {z, 0, 100}] // Timing
```

$$\left\{0.79\text{ Second}, \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2 z^5}{945}-\frac{z^7}{4725}+\frac{2 z^9}{93\,555}-\frac{1382 z^{11}}{638\,512\,875}+\frac{4 z^{13}}{18\,243\,225}-\frac{3617 z^{15}}{162\,820\,783\,125}+\frac{87\,734 z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222 z^{19}}{1\,531\,329\,465\,290\,625}+\right.$$

$$\frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125} - \frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875} + \frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125} -$$
$$\frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875} + \frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625} -$$
$$\frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125} + \frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625} -$$
$$\frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875} + \frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375} -$$
$$\frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625} +$$
$$\frac{6\,080\,390\,575\,672\,283\,210\,764\,z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} -$$
$$\frac{10\,121\,188\,937\,927\,645\,176\,372\,z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} +$$
$$\frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} -$$
$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} +$$
$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} -$$
$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} +$$
$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} -$$
$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} +$$
$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} -$$
$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} +$$
$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$
$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} +$$
$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65}\right) \Big/$$
$$86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 - \left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67}\right) \Big/$$
$$801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 + \left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69}\right) \Big/$$
$$5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 - \left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71}\right) \Big/$$
$$3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793$$

458 452 606 201 171 875 +

(3 692 153 220 456 342 488 035 683 646 645 690 290 452 790 030 604 $z^{73}$) /

11 359 005 221 796 317 918 049 302 062 760 294 302 183 889 391 189 419 445 133 951 612 582 060 536
346 435 546 875 − (5 190 545 015 986 394 254 249 936 008 544 252 611 445 319 542 919 116 $z^{75}$) /

157 606 197 452 423 911 112 934 066 120 799 083 442 801 465 302 753 194 801 233 578 624 576 089
941 806 793 212 890 625 +

(255 290 071 123 323 586 643 187 098 799 718 199 072 122 692 536 861 835 992 $z^{77}$) /

76 505 736 228 426 953 173 738 238 352 183 101 801 688 392 812 244 485 181 277 127 930 109 049 138
257 655 704 498 291 015 625 −

(9 207 568 598 958 915 293 871 149 938 038 093 699 588 515 745 502 577 839 313 734 $z^{79}$) /

27 233 582 984 369 795 892 070 228 410 001 578 355 986 013 571 390 071 723 225 259 349 721 067 988
068 852 863 296 604 156 494 140 625 +

(163 611 136 505 867 886 519 332 147 296 221 453 678 803 514 884 902 772 183 572 $z^{81}$) /

4 776 089 171 877 348 057 451 105 924 101 750 653 118 402 745 283 825 543 113 171 217 116 857 704
024 700 607 798 175 811 767 578 125 −

(8 098 304 783 741 161 440 924 524 640 446 924 039 959 669 564 792 363 509 124 335 729 908 $z^{83}$) /

2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393
530 725 143 500 144 033 328 342 437 744 140 625 +

(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 $z^{85}$) /

349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231
069 047 713 981 709 496 784 210 205 078 125 −

(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 $z^{87}$) /

13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114
528 962 242 087 062 453 152 690 410 614 013 671 875 +

(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
$z^{89}$) /

522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471
400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 −

(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
$z^{91}$) /

1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743
269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 +

(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
$z^{93}$) /

11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254
692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 −

(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336
699 974 518 $z^{95}$) /

112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237
591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 +

(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229
244 $z^{97}$) /

14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225
810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 −

(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581
660 788 990 844 $z^{99}$) /

9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359
080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + $O[z]^{101}$]

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following $n^{\text{th}}$ term for odd hyperbolic functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, n}] z^n
```

$$\left\{z^n \text{If}\left[\text{Odd}[n], \frac{1}{n!}, 0\right], z^n \text{If}\left[\text{Even}[n], \frac{1}{n!}, 0\right], z^n \text{If}\left[\text{Odd}[n], \frac{2^{1+n}\left(-1+2^{1+n}\right)\text{BernoulliB}[1+n]}{(1+n)!}, 0\right], \frac{2^{1+n} z^n \text{BernoulliB}[1+n]}{(1+n)!}, \frac{2^{1+n} z^n \text{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}, \frac{z^n \text{EulerE}[n]}{n!}\right\}$$

Here is a quick check of the last result.

This series should be evaluated to {sinh($z$), cosh($z$), tanh($z$), coth($z$), csch($z$), sech($z$)}, which can be concluded from the following relation.

```
Sum[#, {n, 0, 100}] & /@ % -
 Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 100}]
```

$$\left\{O[z]^{101}, O[z]^{101}, O[z]^{101}, -\frac{1}{z} - 1 + O[z]^{101}, -\frac{1}{z} + O[z]^{101}, O[z]^{101}\right\}$$

### Differentiation

*Mathematica* can evaluate derivatives of hyperbolic functions of an arbitrary positive integer order.

```
D[Sinh[z], z]
```

```
Cosh[z]
```

```
Sinh[z] // D[#, z] &
```

```
Cosh[z]
```

$\partial_z$ `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z]\,\text{Csch}[z], -\text{Sech}[z]\,\text{Tanh}[z]\right\}$$

$\partial_{\{z,2\}}$ `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\text{Sinh}[z], \text{Cosh}[z], -2\,\text{Sech}[z]^2\,\text{Tanh}[z], 2\,\text{Coth}[z]\,\text{Csch}[z]^2, \text{Coth}[z]^2\,\text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z]\,\text{Tanh}[z]^2\right\}$$

```
Table[D[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, n}], {n, 4}]
```

$$\{\{\text{Cosh}[z],\ \text{Sinh}[z],\ \text{Sech}[z]^2,\ -\text{Csch}[z]^2,\ -\text{Coth}[z]\,\text{Csch}[z],\ -\text{Sech}[z]\,\text{Tanh}[z]\},$$
$$\{\text{Sinh}[z],\ \text{Cosh}[z],\ -2\,\text{Sech}[z]^2\,\text{Tanh}[z],\ 2\,\text{Coth}[z]\,\text{Csch}[z]^2,$$
$$\text{Coth}[z]^2\,\text{Csch}[z]+\text{Csch}[z]^3,\ -\text{Sech}[z]^3+\text{Sech}[z]\,\text{Tanh}[z]^2\},$$
$$\{\text{Cosh}[z],\ \text{Sinh}[z],\ -2\,\text{Sech}[z]^4+4\,\text{Sech}[z]^2\,\text{Tanh}[z]^2,\ -4\,\text{Coth}[z]^2\,\text{Csch}[z]^2-2\,\text{Csch}[z]^4,$$
$$-\text{Coth}[z]^3\,\text{Csch}[z]-5\,\text{Coth}[z]\,\text{Csch}[z]^3,\ 5\,\text{Sech}[z]^3\,\text{Tanh}[z]-\text{Sech}[z]\,\text{Tanh}[z]^3\},$$
$$\{\text{Sinh}[z],\ \text{Cosh}[z],\ 16\,\text{Sech}[z]^4\,\text{Tanh}[z]-8\,\text{Sech}[z]^2\,\text{Tanh}[z]^3,$$
$$8\,\text{Coth}[z]^3\,\text{Csch}[z]^2+16\,\text{Coth}[z]\,\text{Csch}[z]^4,\ \text{Coth}[z]^4\,\text{Csch}[z]+18\,\text{Coth}[z]^2\,\text{Csch}[z]^3+$$
$$5\,\text{Csch}[z]^5,\ 5\,\text{Sech}[z]^5-18\,\text{Sech}[z]^3\,\text{Tanh}[z]^2+\text{Sech}[z]\,\text{Tanh}[z]^4\}\}$$

### Finite summation

*Mathematica* can calculate finite sums that contain hyperbolic functions. Here are two examples.

```
Sum[Sinh[a k], {k, 0, n}]
```

$$\frac{-1+e^{a+a\,n}}{2\,(-1+e^a)}-\frac{e^{-a\,n}\,(-1+e^{a+a\,n})}{2\,(-1+e^a)}$$

$$\sum_{k=0}^{n}(-1)^k\,\text{Sinh}[a\,k]$$

$$-\frac{e^a+(-e^{-a})^n}{2\,(1+e^a)}+\frac{1+e^a\,(-e^a)^n}{2\,(1+e^a)}$$

### Infinite summation

*Mathematica* can calculate infinite sums that contain hyperbolic functions. Here are some examples.

$$\sum_{k=1}^{\infty}z^k\,\text{Sinh}[k\,x]$$

$$-\frac{z}{2\,(e^x-z)}-\frac{e^x\,z}{2\,(-1+e^x\,z)}$$

$$\sum_{k=1}^{\infty}\frac{\text{Sinh}[k\,x]}{k\,!}$$

$$\frac{1}{2}\left(1-e^{e^{-x}}\right)+\frac{1}{2}\left(-1+e^{e^x}\right)$$

$$\sum_{k=1}^{\infty}\frac{\text{Cosh}[k\,x]}{k}$$

$$-\frac{1}{2}\,\text{Log}[1-e^{-x}]-\frac{1}{2}\,\text{Log}[1-e^x]$$

### Finite products

*Mathematica* can calculate some finite symbolic products that contain the hyperbolic functions. Here are two examples.

$$\prod_{k=1}^{n-1} \mathrm{Sinh}\left[\frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \mathrm{Cosh}\left[z + \frac{\pi k i}{n}\right]$$

$$-(-1)^n 2^{1-n} \mathrm{Sech}[z] \mathrm{Sin}\left[\frac{1}{2} n (\pi + 2 i z)\right]$$

### Infinite products

*Mathematica* can calculate infinite products that contain hyperbolic functions. Here are some examples.

$$\prod_{k=1}^{\infty} \mathrm{Exp}\left[z^k \mathrm{Sinh}[k x]\right]$$

$$e^{-\frac{(-1+e^{2 x}) z}{2 (e^{x}-z) (-1+e^{x} z)}}$$

$$\prod_{k=1}^{\infty} \mathrm{Exp}\left[\frac{\mathrm{Cosh}[k x]}{k!}\right]$$

$$e^{\frac{1}{2} \left(-2+e^{e^{-x}}+e^{e^{x}}\right)}$$

### Indefinite integration

*Mathematica* can calculate a huge set of doable indefinite integrals that contain hyperbolic functions. Here are some examples.

$$\int \mathrm{Sinh}[7 z]\, dz$$

$$\frac{1}{7} \mathrm{Cosh}[7 z]$$

$$\int \{\{\mathrm{Sinh}[z], \mathrm{Sinh}[z]^a\}, \{\mathrm{Cosh}[z], \mathrm{Cosh}[z]^a\}, \{\mathrm{Tanh}[z], \mathrm{Tanh}[z]^a\}, \{\mathrm{Coth}[z], \mathrm{Coth}[z]^a\}, \{\mathrm{Csch}[z], \mathrm{Csch}[z]^a\}, \{\mathrm{Sech}[z], \mathrm{Sech}[z]^a\}\}\, dz$$

$$\left\{\left\{\text{Cosh}[z],\; -\text{Cosh}[z]\ \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]^{1+a} \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1-a)}\right\},\right.$$

$$\left\{\text{Sinh}[z],\; -\frac{\text{Cosh}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]}{(1+a)\sqrt{-\text{Sinh}[z]^2}}\right\},$$

$$\left\{\text{Log}[\text{Cosh}[z]],\; \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Tanh}[z]^2\right] \text{Tanh}[z]^{1+a}}{1+a}\right\},$$

$$\left\{\text{Log}[\text{Sinh}[z]],\; \frac{\text{Coth}[z]^{1+a}\ \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Coth}[z]^2\right]}{1+a}\right\},$$

$$\left\{-\text{Log}\left[\text{Cosh}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sinh}\left[\frac{z}{2}\right]\right],\; -\text{Cosh}[z]\ \text{Csch}[z]^{-1+a}\ \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1+a)}\right\},$$

$$\left.\left\{2\,\text{ArcTan}\left[\text{Tanh}\left[\frac{z}{2}\right]\right],\; -\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}[z]^2\right] \text{Sech}[z]^{-1+a}\ \text{Sinh}[z]}{(1-a)\sqrt{-\text{Sinh}[z]^2}}\right\}\right\}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain hyperbolic functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sinh}[z]}\, dz$$

$$-\frac{(-1)^{1/3}\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{3}\right]}{2\,\text{Gamma}\left[\frac{7}{6}\right]} + (-1)^{1/3}\,\text{Cosh}\left[\frac{\pi}{2}\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{2}\right]^2\right]$$

$$\int_1^{\pi/2} \left\{\sqrt{\text{Sinh}[z]}, \sqrt{\text{Cosh}[z]}, \sqrt{\text{Tanh}[z]}, \sqrt{\text{Coth}[z]}, \sqrt{\text{Csch}[z]}, \sqrt{\text{Sech}[z]}\right\} dz$$

$$\Big\{2\,(-1)^{1/4}\,\text{EllipticE}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{1/4}\,\text{EllipticE}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticE}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticE}\Big[\frac{i\,\pi}{4},\,2\Big],$$

$$\frac{1}{2}\Bigg(i\,\text{Log}\Big[1-i\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-i\,\text{Log}\Big[1+i\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-$$

$$i\,\text{Log}\Big[1-i\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+i\,\text{Log}\Big[1+i\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+\text{Log}\Big[1-\sqrt{\text{Tanh}[1]}\Big]-$$

$$\text{Log}\Big[1+\sqrt{\text{Tanh}[1]}\Big]-\text{Log}\Big[1-\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]+\text{Log}\Big[1+\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$\frac{1}{2}\,i\,\Bigg(\text{Log}\Big[1-i\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1+i\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1-i\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]+$$

$$\text{Log}\Big[1+i\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]-i\,\text{Log}\Big[-1+\sqrt{\text{Coth}[1]}\Big]+i\,\text{Log}\Big[1+\sqrt{\text{Coth}[1]}\Big]+$$

$$i\,\text{Log}\Big[-1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]-i\,\text{Log}\Big[1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$2\,(-1)^{3/4}\,\text{EllipticF}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{3/4}\,\text{EllipticF}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticF}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticF}\Big[\frac{i\,\pi}{4},\,2\Big]\Big\}$$

$$\int_1^{\frac{\pi}{4}}\Big\{\{\mathbf{Sinh}[\mathbf{z}],\,\mathbf{Sinh}[\mathbf{z}]^{\mathbf{a}}\},\,\{\mathbf{Cosh}[\mathbf{z}],\,\mathbf{Cosh}[\mathbf{z}]^{\mathbf{a}}\},\,\{\mathbf{Tanh}[\mathbf{z}],\,\mathbf{Tanh}[\mathbf{z}]^{\mathbf{a}}\},$$

$$\{\mathbf{Coth}[\mathbf{z}],\,\mathbf{Coth}[\mathbf{z}]^{\mathbf{a}}\},\,\{\mathbf{Csch}[\mathbf{z}],\,\mathbf{Csch}[\mathbf{z}]^{\mathbf{a}}\},\,\{\mathbf{Sech}[\mathbf{z}],\,\mathbf{Sech}[\mathbf{z}]^{\mathbf{a}}\}\Big\}\,d\mathbf{z}$$

$$\Big\{\Big\{-\text{Cosh}[1]+\text{Cosh}\Big[\frac{\pi}{4}\Big],$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\text{Cosh}[1]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}-$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}\Big\},$$
$$\Big\{-\text{Sinh}[1]+\text{Sinh}\Big[\frac{\pi}{4}\Big],\ -\frac{i\,\text{Cosh}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}[1]^2\Big]}{1+a}+$$
$$\frac{i\,\text{Cosh}\Big[\frac{\pi}{4}\Big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Cosh}[1]]+\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{4}\Big]\Big],\ -\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}[1]^2\Big]\,\text{Tanh}[1]^{1+a}}{1+a}+$$
$$\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Tanh}\Big[\frac{\pi}{4}\Big]^{1+a}}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Sinh}[1]]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{4}\Big]\Big],\ -\frac{\text{Coth}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}[1]^2\Big]}{1+a}+$$
$$\frac{\text{Coth}\Big[\frac{\pi}{4}\Big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}\Big[\frac{\pi}{4}\Big]^2\Big]}{1+a}\Big\},$$
$$\Big\{\text{Log}\Big[\text{Cosh}\Big[\frac{1}{2}\Big]\Big]-\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{8}\Big]\Big]-\text{Log}\Big[\text{Sinh}\Big[\frac{1}{2}\Big]\Big]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\text{Cosh}[1]\,\text{Csch}[1]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}-$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Csch}\Big[\frac{\pi}{4}\Big]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}\Big\},$$
$$\Big\{-2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{1}{2}\Big]\Big]+2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}[1]^2\Big]\,\text{Sech}[1]^{-1+a}}{-1+a}-$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sech}\Big[\frac{\pi}{4}\Big]^{-1+a}}{-1+a}\Big\}\Big\}$$

$$\int_0^\infty\Big\{\frac{1}{a+b\,\text{Sinh}[z]},\frac{1}{a+b\,\text{Cosh}[z]},$$
$$\frac{1}{a+b\,\text{Tanh}[z]},\frac{1}{a+b\,\text{Coth}[z]},\frac{1}{a+b\,\text{Csch}[z]},\frac{1}{a+b\,\text{Sech}[z]}\Big\}\,dz$$

$$\left\{-\frac{1}{\sqrt{-a^2-b^2}}\left(i\left(\text{Log}\left[1-\frac{i\,a}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[1+\frac{i\,a}{\sqrt{-a^2-b^2}}\right]+\text{Log}\left[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]\right)\right),\right.$$

$$-\frac{1}{\sqrt{-a^2+b^2}}\left(i\left(\text{Log}\left[1-\frac{i\,a}{\sqrt{-a^2+b^2}}\right]-\text{Log}\left[1+\frac{i\,a}{\sqrt{-a^2+b^2}}\right]-\text{Log}\left[\frac{-i\,a-i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\right]+\text{Log}\left[\frac{i\,a+i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\right]\right)\right),$$

$$\frac{b\,(\text{Log}[2\,a]-\text{Log}[a+b])}{a^2-b^2},\ \frac{b\,(\text{Log}[-a-b]-\text{Log}[-2\,b])}{-a^2+b^2},$$

$$\frac{1}{a\sqrt{-a^2-b^2}}\left(i\,b\left(\text{Log}\left[\frac{-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]-\text{Log}\left[\frac{i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]+\text{Log}\left[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\right]\right)\right),$$

$$\left.\frac{1}{a\sqrt{a^2-b^2}}\left(i\,b\left(\text{Log}\left[1-\frac{i\,b}{\sqrt{a^2-b^2}}\right]-\text{Log}\left[1+\frac{i\,b}{\sqrt{a^2-b^2}}\right]-\text{Log}\left[\frac{-i\,a-i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right]+\text{Log}\left[\frac{i\,a+i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right]\right)\right)\right\}$$

### Limit operation

*Mathematica* can calculate limits that contain hyperbolic functions. Here are some examples.

$$\textbf{Limit}\left[\frac{\textbf{Sinh}[\textbf{z}]}{\textbf{z}}+\textbf{Cosh}[\textbf{z}]^3,\ \textbf{z}\to\textbf{0}\right]$$

2

$$\textbf{Limit}\left[\left(\frac{\textbf{Tanh}[\textbf{x}]}{\textbf{x}}\right)^{\frac{1}{x^2}},\ \textbf{x}\to\textbf{0}\right]$$

$$\frac{1}{e^{1/3}}$$

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> 1]
```

```
-1
```

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> -1]
```

```
1
```

### Solving equations

The next input solves equations that contain hyperbolic functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

```
Solve[Tanh[z]^2 + 3 Sinh[z + Pi/6] == 4, z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{z -> -ArcSech[
    3 (-2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2
          #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1] - 3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 +
          9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 -
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
          18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 +
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
          18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^5 +
       2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
          9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^2 Sinh[π/6] +
       6 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
          9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^4 Sinh[π/6] -
       3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
```

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,1]^5\,\text{Sinh}[\tfrac{\pi}{6}]^2)]]\},$$

$$\{z\to\text{ArcSech}[3\,(-2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^3-$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^3+$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^5+$$

$$2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^2\,\text{Sinh}[\tfrac{\pi}{6}]+$$

$$6\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^4\,\text{Sinh}[\tfrac{\pi}{6}]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,2]^5\,\text{Sinh}[\tfrac{\pi}{6}]^2)]]\},$$

$$\{z\to\text{ArcSech}[3\,(-2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]-$$

$$3\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5-$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6+9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^3-$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1+6\,\#1^2-6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3+9\,\#1^4+9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4-$$

$$18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,3\right]^3+$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,3\right]^5+$$

$$2\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,3\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]+$$

$$6\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,3\right]^4\mathrm{Sinh}\left[\frac{\pi}{6}\right]-$$

$$3\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,3\right]^5\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big\},$$

$$\Big\{z\to\mathrm{ArcSech}\Big[3\,\Big(-2\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,4\right]-$$

$$3\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,4\right]^3-$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,4\right]^3+$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,4\right]^5+$$

$$2\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^6+9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\#1^6\,\&,\,4\right]^2\mathrm{Sinh}\left[\frac{\pi}{6}\right]+$$

$$6\,\mathrm{Root}\left[1+6\,\#1^2-6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2\#1^4-18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$
\begin{aligned}
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 4\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right] - \\
&3\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 4\right]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big\}, \\
&\Big\{z \to -\text{ArcSech}\Big[3\,\Big(-2\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
&18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right] - \\
&3\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^3 - \\
&3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
&18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^3 + \\
&3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
&18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^5 + \\
&2\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right] + \\
&6\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right] - \\
&3\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
&9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 5\right]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big\}, \\
&\Big\{z \to -\text{ArcSech}\Big[3\,\Big(-2\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
&18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\, \&,\, 6\right] - \\
&3\,\text{Root}\left[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right.
\end{aligned}
$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^3-$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^3+$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4-\right.$$

$$\left.18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^5+$$

$$2\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right]+$$

$$6\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right]-$$

$$3\,\text{Root}\left[1+6\,\#1^2-6\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^3+9\,\#1^4+9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4-18\,\text{Sinh}\left[\frac{\pi}{6}\right]\#1^5-\right.$$

$$\left.9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6+9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\,\&,\,6\right]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big]\Big\}\Big\}$$

Complete solutions can be obtained by using the function `Reduce`.

**`Reduce[Sinh[x] == a, x] // InputForm`**

```
// InputForm = C[1] ∈ Integers &&
  (x == I * Pi - ArcSinh[a] + (2 * I) * Pi * C[1] || x == ArcSinh[a] + (2 * I) * Pi * C[1])
```

**`Reduce[Cosh[x] == a, x] // InputForm`**

```
// InputForm =
C[1] ∈ Integers && (x == -ArcCosh[a] + (2 * I) * Pi * C[1] || x == ArcCosh[a] + (2 * I) * Pi * C[1])
```

**`Reduce[Tanh[x] == a, x] // InputForm`**

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcTanh[a] + I * Pi * C[1]
```

**`Reduce[Coth[x] == a, x] // InputForm`**

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcCoth[a] + I * Pi * C[1]
```

**`Reduce[Csch[x] == a, x] // InputForm`**

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == I * Pi - ArcSinh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcSinh[a^(-1)] + (2 * I) * Pi * C[1])
```

```
Reduce[Sech[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCosh[a^(-1)] + (2*I)*Pi*C[1] || x == ArcCosh[a^(-1)] + (2*I)*Pi*C[1])
```

#### Solving differential equations

Here are differential equations whose linear-independent solutions are hyperbolic functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh($z$) and cosh($z$).

```
{DSolve[w''[z] - w[z] == 0, w[z], z],
  DSolve[w'[z] + w[z]^2 - 1 == 0, w[z], z]} // (ExpToTrig //@ #) &
```

$$\left\{\{\{w[z] \to C[1]\,\mathrm{Cosh}[z] + C[2]\,\mathrm{Cosh}[z] + C[1]\,\mathrm{Sinh}[z] - C[2]\,\mathrm{Sinh}[z]\}\},\ \left\{\left\{w[z] \to \frac{\mathrm{Cosh}[2\,z] + \mathrm{Cosh}[2\,C[1]] + \mathrm{Sinh}[2\,z] + \mathrm{Sinh}[2\,C[1]]}{\mathrm{Cosh}[2\,z] - \mathrm{Cosh}[2\,C[1]] + \mathrm{Sinh}[2\,z] - \mathrm{Sinh}[2\,C[1]]}\right\}\right\}\right\}$$

All hyperbolic functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

$$\mathtt{DSolve}\left[\left\{w'[z] == \sqrt{1 + w[z]^2},\ w[0] == 0\right\},\ w[z],\ z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Sinh[z]}}
```

$$\mathtt{DSolve}\left[\left\{w'[z] == \sqrt{-1 + w[z]^2},\ w[0] == 1\right\},\ w[z],\ z\right]\ \mathtt{//\ FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Cosh[z]}}
```

$$\mathtt{DSolve}\left[\left\{w'[z] + w[z]^2 - 1 == 0,\ w[0] == 0\right\},\ w[z],\ z\right]\ \mathtt{//\ FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Tanh[z]}}
```

$$\mathtt{DSolve}\left[\left\{w'[z] - w[z]^2 + 1 == 0,\ w\left[\frac{\pi\, i}{2}\right] == 0\right\},\ w[z],\ z\right]\ \mathtt{//\ FullSimplify}$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → -Coth[z]}}
```

#### Integral transforms

*Mathematica* supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results containing classical or generalized functions. Here are some transforms of hyperbolic functions.

```
LaplaceTransform[Sinh[t], t, s]
```

$$\frac{1}{-1+s^2}$$

```
LaplaceTransform[Cosh[t], t, s]
```

$$\frac{s}{-1+s^2}$$

```
FourierTransform[Csch[t], t, s]
```

$$i\sqrt{\frac{\pi}{2}}\ \mathrm{Tanh}\left[\frac{\pi s}{2}\right]$$

```
FourierTransform[Sech[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \mathrm{Sech}\left[\frac{\pi s}{2}\right]$$

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

$$\mathrm{Plot}\left[\mathrm{Sin}\left[\mathrm{Sinh}\left[\sum_{k=0}^{5} z^k\right]\right], \{z, -3/2, 4/5\}, \mathrm{PlotRange} \to \mathrm{All}, \mathrm{PlotPoints} \to 120\right];$$

```
Plot3D[Re[Tanh[x + ⅈ y]], {x, -2, 2}, {y, -2, 2},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\mathrm{ContourPlot}\left[\mathrm{Arg}\left[\mathrm{Sech}\left[\frac{1}{x+i\,y}\right]\right], \left\{x, -\frac{1}{4}, \frac{1}{4}\right\}, \left\{y, -\frac{1}{3}, \frac{1}{3}\right\},\right.$$
$$\mathrm{PlotPoints} \to 400, \mathrm{PlotRange} \to \{-\pi, \pi\}, \mathrm{FrameLabel} \to \{\text{"x"}, \text{"y"}, \mathrm{None}, \mathrm{None}\},$$
$$\left.\mathrm{ColorFunction} \to \mathrm{Hue}, \mathrm{ContourLines} \to \mathrm{False}, \mathrm{Contours} \to 200\right];$$

# Introduction to the Hyperbolic Tangent Function in *Mathematica*

## Overview

The following shows how the hyperbolic tangent function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic tangent function or return it are shown. These involve numeric and symbolic calculations and plots.

## Notations

### *Mathematica* forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the hyperbolic tangent function in `StandardForm`.

```
Tanh[z]
```

```
Tanh[z]
```

This shows the hyperbolic tangent function in `TraditionalForm`.

```
% // TraditionalForm
```

tanh($z$)

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the hyperbolic tangent function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Tanh[2 π z]], TeXForm[Tanh[2 π z]], FortranForm[Tanh[2 π z]]}
```

```
{Tanh(2*Pi*z), \tanh(2\, \pi \, z), Tanh(2*Pi*z)}
```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For the exact argument $z = \pi i / 4$, *Mathematica* returns an exact result.

$$\text{Tanh}\left[\frac{\pi i}{4}\right]$$

$i$

$$\text{Tanh}[z] \;/.\; z \to \frac{\pi i}{4}$$

$i$

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

```
Tanh[4.]
```

```
0.999329
```

```
Tanh[z] /. z → 2.
```

```
0.964028
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Tanh[z] /. z → 1, 100]
```

```
0.761594155955764888119458282604793590412768597257936551596810500121953244576638483 4
 589475216736767144
```

```
N[Tanh[2], 100]
```

```
0.9640275800758168839464137241009231502550299762409347760482632174131079463176102025
 594748500452076891
```

```
Tanh[2] // N[#, 100] &
```

```
0.9640275800758168839464137241009231502550299762409347760482632174131079463176102025
 594748500452076891
```

It is possible to calculate thousands of digits for the hyperbolic tangent function within a second. The next input calculates 10000 digits for tanh(1) and analyzes the frequency of the digit $k$ in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]] ,
 N[{Tanh[z]} /. z → 1, 10 000]]
```

```
{{{0, 971}, {1, 1023}, {2, 1016}, {3, 970},
  {4, 949}, {5, 1052}, {6, 981}, {7, 1056}, {8, 1010}, {9, 972}}}
```

Here is a 50-digit approximation to the hyperbolic tangent function at the complex argument $z = 3 - 2\,i$.

```
N[Tanh[3 - 2 ⅈ], 50]
```

```
1.0032386273536098014463585978219272598077897241071 +
 0.0037640256415042482927512211303226908396306202016581 ⅈ
```

```
{N[Tanh[z] /. z → 3 - 2 ⅈ, 50], Tanh[3 - 2 ⅈ] // N[#, 50] &}
```

```
{1.0032386273536098014463585978219272598077897241071 +
  0.0037640256415042482927512211303226908396306202016581 ⅈ,
 1.0032386273536098014463585978219272598077897241071 +
  0.0037640256415042482927512211303226908396306202016581 ⅈ}
```

*Mathematica* automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Tanh[3.], N[Tanh[3]], N[Tanh[3], 16], N[Tanh[3], 5], N[Tanh[3], 20]}
```

```
{0.995055, 0.995055, 0.995055, 0.995055, 0.99505475368673045133}
```

```
% // InputForm
```

```
{0.9950547536867305, 0.9950547536867305, 0.9950547536867305, 0.9950547536867305,
 0.9950547536867304513318801852554837279`20}
```

### Simplification of the argument

*Mathematica* knows the symmetry and periodicity of the hyperbolic tangent function. Here are some examples.

```
Tanh[-3]
```

```
-Tanh[3]
```

```
{Tanh[-z], Tanh[z + π ⅈ], Tanh[z + 2 π ⅈ], Tanh[-z + 21 π ⅈ]}
```

```
{-Tanh[z], Tanh[z], Tanh[z], -Tanh[z]}
```

*Mathematica* automatically simplifies the composition of the direct and the inverse hyperbolic tangent functions into its argument.

```
Tanh[ArcTanh[z]]
```

```
z
```

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

```
{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]],
 Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1+z^2}},\ \frac{\sqrt{\frac{-1+z}{1+z}}\,(1+z)}{z},\ z,\ \frac{1}{z},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\,z},\ \sqrt{1-\frac{1}{z^2}}\,z\right\}$$

If the argument has the structure $\pi k i/2+z$ or $\pi k i/2-z$, and $\pi k i/2+i z$ or $\pi k i/2-i z$ with integer $k$, the hyperbolic tangent function can be automatically transformed into hyperbolic or trigonometric tangent or cotangent functions.

$$\text{Tanh}\left[\frac{\pi i}{2}+4\right]$$

```
Coth[4]
```

$$\left\{\text{Tanh}\left[\frac{\pi i}{2}-z\right],\ \text{Tanh}\left[\frac{\pi i}{2}+z\right],\ \text{Tanh}\left[-\frac{\pi i}{2}-z\right],\ \text{Tanh}\left[-\frac{\pi i}{2}+z\right],\ \text{Tanh}[\pi i-z],\ \text{Tanh}[\pi i+z]\right\}$$

```
{-Coth[z], Coth[z], -Coth[z], Coth[z], -Tanh[z], Tanh[z]}
```

```
Tanh[i 5]
```

```
i Tan[5]
```

$$\left\{\text{Tanh}[i z],\ \text{Tanh}\left[\frac{\pi i}{2}-i z\right],\ \text{Tanh}\left[\frac{\pi i}{2}+i z\right],\ \text{Tanh}[\pi i-i z],\ \text{Tanh}[\pi i+i z]\right\}$$

```
{i Tan[z], i Cot[z], -i Cot[z], -i Tan[z], i Tan[z]}
```

### Simplification of simple expressions containing the hyperbolic tangent function

Sometimes simple arithmetic operations containing the hyperbolic tangent function can automatically produce other hyperbolic functions.

```
1 / Tanh[4]
```

```
Coth[4]
```

```
{1 / Tanh[z], 1 / Tanh[π i / 2 - z], Tanh[π i / 2 - z] / Tanh[z],
 Tanh[z] / Tanh[π i / 2 - z], 1 / Tanh[π i / 2 - z], Tanh[π i / 2 - z] / Tanh[z]^2}
```

$$\{\mathrm{Coth}[z], -\mathrm{Tanh}[z], -\mathrm{Coth}[z]^2, -\mathrm{Tanh}[z]^2, -\mathrm{Tanh}[z], -\mathrm{Coth}[z]^3\}$$

### The hyperbolic tangent function arising as special cases from more general functions

The hyperbolic tangent function can be treated as a particular case of other, more general special functions. For example, $\tanh(z)$ can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions or their ratios for appropriate parameters.

$$\left\{\mathbf{BesselI}\left[\frac{1}{2}, z\right] \Big/ \mathbf{BesselI}\left[-\frac{1}{2}, z\right], \frac{\mathbf{MathieuS}[1, 0, i z]}{\mathbf{MathieuC}[1, 0, i z]}, \mathbf{JacobiSC}[i z, 0], \right.$$
$$\mathbf{JacobiCS}\left[\frac{\pi}{2} - i z, 0\right], -i\,\mathbf{JacobiSN}[z, 1], i\,\mathbf{JacobiNS}\left[\frac{\pi i}{2} - z, 1\right],$$
$$\mathbf{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right] \Big/ \mathbf{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}\right\}, \frac{z^2}{4}\right],$$
$$\left.\mathbf{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, -\frac{z^2}{4}\right] \Big/ \mathbf{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{-\frac{1}{2}\right\}, \{0\}\right\}, -\frac{z^2}{4}\right]\right\}$$

$$\left\{\mathrm{Tanh}[z], i\,\mathrm{Tanh}[z], i\,\mathrm{Tanh}[z], i\,\mathrm{Tanh}[z], -i\,\mathrm{Tanh}[z], -i\,\mathrm{Tanh}[z], \frac{\mathrm{Tanh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, -\frac{1}{2} z\,\mathrm{Tanh}[z]\right\}$$

## Equivalence transformations carried out by specialized *Mathematica* functions

### General remarks

Almost everybody prefers using $i - \tanh(z)$ instead of $\tanh(\pi i - z) + \tanh(\pi i / 4)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\tanh(\pi i / 16)$ should not be automatically expanded into the more complicated expression $i\left(\left(2 - \left(2 + 2^{1/2}\right)^{1/2}\right) \Big/ \left(2 + \left(2 + 2^{1/2}\right)\right)\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Tanh[x - y]]
```

$$\frac{\mathrm{Cosh}[y]\,\mathrm{Sinh}[x]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y] - \mathrm{Sinh}[x]\,\mathrm{Sinh}[y]} - \frac{\mathrm{Cosh}[x]\,\mathrm{Sinh}[y]}{\mathrm{Cosh}[x]\,\mathrm{Cosh}[y] - \mathrm{Sinh}[x]\,\mathrm{Sinh}[y]}$$

```
Tanh[4 z] // TrigExpand
```

$$\frac{4\,\mathrm{Cosh}[z]^3\,\mathrm{Sinh}[z]}{\mathrm{Cosh}[z]^4 + 6\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]^2 + \mathrm{Sinh}[z]^4} + \frac{4\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^3}{\mathrm{Cosh}[z]^4 + 6\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]^2 + \mathrm{Sinh}[z]^4}$$

```
Tanh[2 z]^2 // TrigExpand
```

$$-\frac{1}{2\left(\mathrm{Cosh}[z]^2+\mathrm{Sinh}[z]^2\right)^2}+\frac{\mathrm{Cosh}[z]^4}{2\left(\mathrm{Cosh}[z]^2+\mathrm{Sinh}[z]^2\right)^2}+\frac{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]^2}{\left(\mathrm{Cosh}[z]^2+\mathrm{Sinh}[z]^2\right)^2}+\frac{\mathrm{Sinh}[z]^4}{2\left(\mathrm{Cosh}[z]^2+\mathrm{Sinh}[z]^2\right)^2}$$

```
TrigExpand[{Tanh[x + y + z], Tanh[3 z]}]
```

```
{(Cosh[y] Cosh[z] Sinh[x]) / (Cosh[x] Cosh[y] Cosh[z] +
      Cosh[z] Sinh[x] Sinh[y] + Cosh[y] Sinh[x] Sinh[z] + Cosh[x] Sinh[y] Sinh[z]) +
  (Cosh[x] Cosh[z] Sinh[y]) / (Cosh[x] Cosh[y] Cosh[z] + Cosh[z] Sinh[x] Sinh[y] +
      Cosh[y] Sinh[x] Sinh[z] + Cosh[x] Sinh[y] Sinh[z]) +
  (Cosh[x] Cosh[y] Sinh[z]) / (Cosh[x] Cosh[y] Cosh[z] + Cosh[z] Sinh[x] Sinh[y] +
      Cosh[y] Sinh[x] Sinh[z] + Cosh[x] Sinh[y] Sinh[z]) +
  (Sinh[x] Sinh[y] Sinh[z]) / (Cosh[x] Cosh[y] Cosh[z] + Cosh[z] Sinh[x] Sinh[y] +
      Cosh[y] Sinh[x] Sinh[z] + Cosh[x] Sinh[y] Sinh[z]),
```

$$\frac{3\,\mathrm{Cosh}[z]^2\,\mathrm{Sinh}[z]}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}+\frac{\mathrm{Sinh}[z]^3}{\mathrm{Cosh}[z]^3+3\,\mathrm{Cosh}[z]\,\mathrm{Sinh}[z]^2}\Big\}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Tanh[x] + Tanh[y]]
```

```
Sech[x] Sech[y] Sinh[x + y]
```

```
Tanh[x] - Coth[y] // TrigFactor
```

```
-Cosh[x - y] Csch[y] Sech[x]
```

### TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Tanh[x] Tanh[y]]
```

$$\frac{-\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y]}{\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y]}$$

```
Tanh[x] Coth[y] // TrigReduce
```

$$\frac{-\mathrm{Sinh}[x-y]-\mathrm{Sinh}[x+y]}{\mathrm{Sinh}[x-y]-\mathrm{Sinh}[x+y]}$$

```
Table[TrigReduce[Tanh[z]^n], {n, 2, 5}]
```

$$\left\{\frac{-1+\mathrm{Cosh}[2z]}{1+\mathrm{Cosh}[2z]}, \frac{-3\,\mathrm{Sinh}[z]+\mathrm{Sinh}[3z]}{3\,\mathrm{Cosh}[z]+\mathrm{Cosh}[3z]}, \frac{3-4\,\mathrm{Cosh}[2z]+\mathrm{Cosh}[4z]}{3+4\,\mathrm{Cosh}[2z]+\mathrm{Cosh}[4z]}, \frac{10\,\mathrm{Sinh}[z]-5\,\mathrm{Sinh}[3z]+\mathrm{Sinh}[5z]}{10\,\mathrm{Cosh}[z]+5\,\mathrm{Cosh}[3z]+\mathrm{Cosh}[5z]}\right\}$$

```
TrigReduce[TrigExpand[{Tanh[x + y + z], Tanh[3 z], Tanh[x] Tanh[y]}]]
```

$$\left\{\mathrm{Tanh}[x+y+z], \mathrm{Tanh}[3z], \frac{-\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y]}{\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y]}\right\}$$

```
TrigFactor[Tanh[x] + Tanh[y]] // TrigReduce
```

$$\frac{2\,\mathrm{Sinh}[x+y]}{\mathrm{Cosh}[x-y]+\mathrm{Cosh}[x+y]}$$

### TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Tanh[z]]
```

$$\frac{-e^{-z}+e^{z}}{e^{-z}+e^{z}}$$

```
Tanh[a z] + Tanh[b z] // TrigToExp
```

$$-\frac{e^{-a z}}{e^{-a z}+e^{a z}}+\frac{e^{a z}}{e^{-a z}+e^{a z}}-\frac{e^{-b z}}{e^{-b z}+e^{b z}}+\frac{e^{b z}}{e^{-b z}+e^{b z}}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Tanh[z]]]
```

Tanh[z]

$$\left\{\alpha\, e^{-x\beta}+\alpha\, e^{x\beta}\big/\left(\alpha\, e^{-x\beta}+\gamma\, e^{x\beta}\right)\right\} \;//\; \mathbf{ExpToTrig}$$

$$\left\{\alpha\,\mathrm{Cosh}[x\beta]-\alpha\,\mathrm{Sinh}[x\beta]+\frac{\alpha\,(\mathrm{Cosh}[x\beta]+\mathrm{Sinh}[x\beta])}{\alpha\,\mathrm{Cosh}[x\beta]+\gamma\,\mathrm{Cosh}[x\beta]-\alpha\,\mathrm{Sinh}[x\beta]+\gamma\,\mathrm{Sinh}[x\beta]}\right\}$$

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

**ComplexExpand[Tanh[x + ⅈ y]]**

$$\frac{i\, \text{Sin}[2\, y]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}+\frac{\text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**Tanh[x + ⅈ y] + Tanh[x - ⅈ y] // ComplexExpand**

$$\frac{2\, \text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**ComplexExpand[Re[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}]**

$$\frac{\text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**ComplexExpand[Im[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}]**

$$\frac{\text{Sin}[2\, y]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**ComplexExpand[Abs[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}]**

$$\sqrt{\frac{\text{Sin}[2\, y]^2}{(\text{Cos}[2\, y]+\text{Cosh}[2\, x])^2}+\frac{\text{Sinh}[2\, x]^2}{(\text{Cos}[2\, y]+\text{Cosh}[2\, x])^2}}$$

**ComplexExpand[Abs[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}] //**
**Simplify[#, {x, y} ∈ Reals] &**

$$\frac{\sqrt{\text{Sin}[2\, y]^2+\text{Sinh}[2\, x]^2}}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**ComplexExpand[Re[Tanh[x + ⅈ y]] + Im[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}]**

$$\frac{\text{Sin}[2\, y]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}+\frac{\text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

**ComplexExpand[Arg[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}]**

$$\text{ArcTan}\left[\frac{\text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}, \frac{\text{Sin}[2\, y]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}\right]$$

**ComplexExpand[Arg[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}] //**
**Simplify[#, {x, y} ∈ Reals] &**

ArcTan[Sinh[2 x], Sin[2 y]]

**ComplexExpand[Conjugate[Tanh[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify**

$$\frac{-i\, \text{Sin}[2\, y]+\text{Sinh}[2\, x]}{\text{Cos}[2\, y]+\text{Cosh}[2\, x]}$$

## Simplify

The function `Simplify` performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\frac{\text{Tanh}[z_1] + \text{Tanh}[z_2] + \text{Tanh}[z_3] + \text{Tanh}[z_1]\,\text{Tanh}[z_2]\,\text{Tanh}[z_3]}{1 + \text{Tanh}[z_1]\,\text{Tanh}[z_2] + \text{Tanh}[z_1]\,\text{Tanh}[z_3] + \text{Tanh}[z_2]\,\text{Tanh}[z_3]} \; \text{// Simplify}$$

$$\text{Tanh}[z_1 + z_2 + z_3]$$

$$\text{Simplify}\left[\text{Tanh}\left[z - \frac{\pi\, i}{3}\right] \text{Tanh}\left[\frac{\pi\, i}{3} + z\right] + \text{Tanh}\left[z - \frac{\pi\, i}{3}\right] \text{Tanh}[z] + \text{Tanh}[z]\, \text{Tanh}\left[\frac{\pi\, i}{3} + z\right]\right]$$

3

Here is a collection of hyperbolic identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#]\ \&\ \text{/@} \Bigg(\text{Tanh}[2\,z]\,\left(1 + \text{Tanh}[z]^2\right) == 2\,\text{Tanh}[z] \bigwedge$$
$$\text{Tanh}[z]^2 == \frac{\text{Cosh}[2\,z] - 1}{\text{Cosh}[2\,z] + 1} \bigwedge \text{Tanh}[z]^3 == -\frac{3\,\text{Sinh}[z] - \text{Sinh}[3\,z]}{3\,\text{Cosh}[z] + \text{Cosh}[3\,z]} \bigwedge$$
$$\text{Tanh}[a] + \text{Tanh}[b] == \frac{\text{Sinh}[a + b]}{\text{Cosh}[a]\,\text{Cosh}[b]} \bigwedge \text{Tanh}[a] - \text{Tanh}[b] == \frac{\text{Sinh}[a - b]}{\text{Cosh}[a]\,\text{Cosh}[b]} \bigwedge$$
$$\text{Tanh}\left[\frac{z}{2}\right] == -\frac{1 - \text{Cosh}[z]}{\text{Sinh}[z]} == \frac{\text{Sinh}[z]}{1 + \text{Cosh}[z]} \bigwedge$$
$$\text{Tanh}[a]^2 - \text{Tanh}[b]^2 == \text{Sech}[a]^2\,\text{Sech}[b]^2\,\text{Sinh}[a - b]\,\text{Sinh}[a + b]\Bigg)$$

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of hyperbolic functions for the symbolic integer coefficient $k$ of $k\,\pi\,i$.

```
Simplify[{Tanh[z + 2 k π ⅈ], Tanh[z + k π ⅈ] / Tanh[z]}, k ∈ Integers]
```

```
{Tanh[z], 1}
```

*Mathematica* also knows that the composition of the inverse and the direct hyperbolic functions reproduces the inner argument under the corresponding restriction.

```
ArcTanh[Tanh[z]]
```

```
ArcTanh[Tanh[z]]
```

```
Simplify[ArcTanh[Tanh[z]], -π / 2 < Im[z] < π / 2]
```

```
z
```

### FunctionExpand (and Together)

While the hyperbolic tangent function auto-evaluates for simple fractions of $\pi\,i$, for more complicated cases it stays as a hyperbolic tangent function to avoid the build up of large expressions. Using the function `FunctionExpand`, the hyperbolic tangent function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\mathbf{Tanh}\left[\frac{\pi\, i}{16}\right],\ \mathbf{Tanh}\left[\frac{\pi\, i}{60}\right]\right\}$$

$$\left\{i\,\mathrm{Tan}\left[\frac{\pi}{16}\right],\ i\,\mathrm{Tan}\left[\frac{\pi}{60}\right]\right\}$$

**FunctionExpand[%]**

$$\left\{i\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \frac{i\left(-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}+\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}\right)}{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}-\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}\right\}$$

**Together**[%]

$$\left\{i\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ -\frac{i\left(1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers deeply inside the expression (meaning that the imaginary number $i=\sqrt{-1}$ appears unavoidably).

$$\left\{\mathbf{Tanh}\left[\frac{\pi\, i}{9}\right]\right\}$$

$$\left\{i\,\mathrm{Tan}\left[\frac{\pi}{9}\right]\right\}$$

**FunctionExpand[%] // Together**

$$\left\{\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as the roots of polynomial equations.

**RootReduce**[**Simplify**[%]]

{Root[3 + 27 #1^2 + 33 #1^4 + #1^6 &, 4]}

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including inverse hyperbolic functions, to simpler ones. Here are some examples.

$$\left\{\mathbf{Tanh}\left[\sqrt{z^2}\right],\ \mathbf{Tanh}\left[\frac{\mathbf{ArcTanh}[z]}{2}\right],\ \mathbf{Tanh}[2\,\mathbf{ArcTanh}[z]],\ \mathbf{Tanh}[3\,\mathbf{ArcSinh}[z]]\right\}\ \text{// }\mathbf{FunctionExpand}$$

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\mathrm{Tanh}[z]}{z},\ \frac{z}{1+\sqrt{1-z}\,\sqrt{1+z}},\ -\frac{2\,(-1+z)\,z\,(1+z)}{\left(1-z^2\right)\left(1+z^2\right)},\right.$$
$$\left.-\frac{i\left(i z^3+3\,i z\left(1+z^2\right)\right)}{3\,z^2\sqrt{i\,(-i+z)}\,\sqrt{-i\,(i+z)}+(i\,(-i+z))^{3/2}\,(-i\,(i+z))^{3/2}}\right\}$$

Applying `Simplify` to the previous expression gives a more compact result.

**Simplify[%]**

$$\left\{\frac{\sqrt{z^2}\,\mathrm{Tanh}[z]}{z},\ \frac{z}{1+\sqrt{1-z^2}},\ \frac{2\,z}{1+z^2},\ \frac{z\left(3+4\,z^2\right)}{\sqrt{1+z^2}\left(1+4\,z^2\right)}\right\}$$

### FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that compare the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

$$\mathbf{set1}=\left\{\mathrm{Tanh}\left[\mathrm{Log}\left[z+\sqrt{1+z^2}\right]\right],\ \mathrm{Tanh}\left[\frac{\pi i}{2}-\mathrm{Log}\left[z+\sqrt{1+z^2}\right]\right],\right.$$
$$\mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}[1-z]-\frac{1}{2}\mathrm{Log}[1+z]\right],\ \mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}\left[1-\frac{1}{z}\right]-\frac{1}{2}\mathrm{Log}\left[1+\frac{1}{z}\right]\right],$$
$$\left.\mathrm{Tanh}\left[\mathrm{Log}\left[\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right]\right],\ \mathrm{Tanh}\left[\frac{\pi i}{2}+\mathrm{Log}\left[\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right]\right]\right\}$$

$$\left\{\frac{-1+\left(z+\sqrt{1+z^2}\right)^2}{1+\left(z+\sqrt{1+z^2}\right)^2},\ -\frac{1+\left(z+\sqrt{1+z^2}\right)^2}{-1+\left(z+\sqrt{1+z^2}\right)^2},\ \mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}[1-z]-\frac{1}{2}\mathrm{Log}[1+z]\right],\right.$$

$$\left.\mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}\left[1-\frac{1}{z}\right]-\frac{1}{2}\mathrm{Log}\left[1+\frac{1}{z}\right]\right],\ \frac{-1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}{1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2},\ \frac{1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}{-1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}\right\}$$

**set1 // Simplify**

$$\left\{\frac{z\left(z+\sqrt{1+z^2}\right)}{1+z^2+z\sqrt{1+z^2}},\ -\frac{1+\left(z+\sqrt{1+z^2}\right)^2}{-1+\left(z+\sqrt{1+z^2}\right)^2},\ \text{Tanh}\left[\frac{1}{2}\left(\text{Log}[1-z]-\text{Log}[1+z]\right)\right],\right.$$

$$\left.\text{Tanh}\left[\frac{1}{2}\left(-\text{Log}\left[1+\frac{1}{z}\right]+\text{Log}\left[\frac{-1+z}{z}\right]\right)\right],\ \frac{1+\sqrt{1+\frac{1}{z^2}}\ z}{1+\sqrt{1+\frac{1}{z^2}}\ z+z^2},\ \frac{1+\sqrt{1+\frac{1}{z^2}}\ z+z^2}{1+\sqrt{1+\frac{1}{z^2}}\ z}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\frac{z}{\sqrt{1+z^2}},\ -\frac{\sqrt{1+z^2}}{z},\ -z,\ -\frac{1}{z},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\ z},\ \sqrt{1+\frac{1}{z^2}}\ z\right\}$$

## Operations carried out by specialized *Mathematica* functions

### Series expansions

Calculating the series expansion of a hyperbolic tangent function to hundreds of terms can be done in seconds.

```
Series[Tanh[z], {z, 0, 3}]
```

$$z-\frac{z^3}{3}+O[z]^4$$

```
Normal[%]
```

$$z-\frac{z^3}{3}$$

```
Series[Tanh[z], {z, 0, 100}] // Timing
```

$$\left\{0.841\ \text{Second},\ z-\frac{z^3}{3}+\frac{2z^5}{15}-\frac{17z^7}{315}+\frac{62z^9}{2835}-\frac{1382z^{11}}{155\,925}+\frac{21\,844z^{13}}{6\,081\,075}-\right.$$
$$\frac{929\,569z^{15}}{638\,512\,875}+\frac{6\,404\,582z^{17}}{10\,854\,718\,875}-\frac{443\,861\,162z^{19}}{1\,856\,156\,927\,625}+\frac{18\,888\,466\,084z^{21}}{194\,896\,477\,400\,625}-$$
$$\frac{113\,927\,491\,862z^{23}}{2\,900\,518\,163\,668\,125}+\frac{58\,870\,668\,456\,604z^{25}}{3\,698\,160\,658\,676\,859\,375}-\frac{8\,374\,643\,517\,010\,684z^{27}}{1\,298\,054\,391\,195\,577\,640\,625}+$$
$$\frac{689\,005\,380\,505\,609\,448z^{29}}{263\,505\,041\,412\,702\,261\,046\,875}-\frac{129\,848\,163\,681\,107\,301\,953z^{31}}{122\,529\,844\,256\,906\,551\,386\,796\,875}+$$
$$\frac{1\,736\,640\,792\,209\,901\,647\,222z^{33}}{4\,043\,484\,860\,477\,916\,195\,764\,296\,875}-\frac{418\,781\,231\,495\,293\,038\,913\,922z^{35}}{2\,405\,873\,491\,984\,360\,136\,479\,756\,640\,625}+$$
$$\frac{56\,518\,638\,202\,982\,204\,522\,669\,764z^{37}}{801\,155\,872\,830\,791\,925\,447\,758\,961\,328\,125}-\frac{32\,207\,686\,319\,158\,956\,594\,455\,462z^{39}}{1\,126\,482\,925\,555\,250\,126\,673\,224\,649\,609\,375}+$$

$$\frac{1\,410\,211\,493\,828\,985\,228\,276\,049\,834\,684\,z^{41}}{121\,699\,582\,862\,361\,447\,435\,141\,825\,020\,548\,828\,125} - \frac{516\,098\,083\,439\,704\,913\,515\,348\,955\,653\,804\,z^{43}}{109\,894\,723\,324\,712\,387\,033\,933\,067\,993\,555\,591\,796\,875} + \frac{103\,537\,504\,005\,512\,749\,467\,288\,942\,622\,106\,408\,z^{45}}{54\,397\,888\,045\,732\,631\,581\,796\,868\,656\,810\,017\,939\,453\,125} - \frac{45\,361\,105\,584\,983\,995\,647\,044\,252\,937\,847\,808\,918\,z^{47}}{58\,804\,116\,977\,436\,974\,739\,922\,415\,018\,011\,629\,392\,548\,828\,125} + \frac{87\,176\,517\,890\,549\,500\,795\,745\,183\,943\,750\,553\,204\,z^{49}}{278\,845\,328\,893\,007\,589\,895\,761\,129\,278\,958\,371\,635\,634\,765\,625} - \frac{1\,396\,470\,103\,398\,938\,597\,044\,980\,843\,456\,514\,101\,088\,564\,z^{51}}{11\,021\,361\,624\,496\,124\,990\,629\,958\,634\,750\,829\,638\,898\,464\,111\,328\,125} + \frac{389\,951\,962\,465\,960\,362\,323\,362\,101\,491\,789\,115\,193\,414\,088\,z^{53}}{7\,593\,718\,159\,277\,830\,118\,544\,041\,499\,343\,321\,621\,201\,041\,772\,705\,078\,125} - \frac{321\,055\,735\,622\,680\,218\,266\,276\,441\,690\,024\,211\,623\,548\,948\,z^{55}}{15\,426\,363\,155\,304\,483\,893\,348\,702\,635\,464\,887\,287\,939\,188\,826\,904\,296\,875} + \frac{37\,951\,675\,284\,166\,717\,133\,668\,639\,194\,471\,627\,545\,621\,910\,540\,728\,z^{57}}{4\,499\,391\,915\,144\,503\,512\,689\,122\,748\,983\,408\,210\,385\,935\,265\,954\,349\,365\,234\,375} - \frac{641\,885\,182\,338\,872\,430\,017\,276\,041\,951\,405\,742\,741\,480\,681\,339\,128\,z^{59}}{187\,767\,306\,507\,615\,744\,151\,489\,976\,183\,185\,645\,072\,447\,200\,976\,777\,847\,900\,390\,625} + \frac{9\,759\,387\,159\,544\,076\,997\,817\,707\,959\,584\,835\,600\,439\,155\,088\,202\,173\,136\,z^{61}}{7\,044\,090\,503\,633\,204\,641\,843\,146\,456\,512\,209\,474\,892\,856\,744\,643\,820\,963\,983\,154\,296\,875} - \frac{7\,724\,760\,729\,208\,487\,305\,545\,342\,963\,324\,697\,288\,405\,380\,586\,579\,904\,269\,441\,z^{63}}{13\,757\,108\,753\,595\,648\,665\,519\,665\,029\,568\,345\,104\,465\,749\,222\,289\,382\,342\,659\,100\,341\,796\,875} + \frac{203\,497\,294\,113\,685\,566\,585\,581\,532\,155\,905\,318\,177\,648\,366\,860\,283\,186\,794\,902\,z^{65}}{894\,212\,068\,983\,717\,163\,258\,778\,226\,921\,942\,431\,790\,273\,699\,448\,809\,852\,272\,841\,522\,216\,796\,875} - \left(4\,928\,399\,482\,219\,367\,719\,407\,723\,316\,603\,414\,444\,925\,848\,984\,020\,131\,054\,275\,466\,z^{67}\right) / 53\,435\,213\,095\,216\,179\,674\,734\,017\,830\,389\,586\,937\,521\,490\,526\,522\,123\,875\,006\,827\,178\,955\,078\,125 + \left(65\,033\,291\,600\,604\,926\,267\,730\,204\,296\,537\,787\,351\,363\,679\,128\,255\,646\,162\,810\,228\,z^{69}\right) / 1\,739\,791\,210\,461\,723\,491\,420\,203\,381\,737\,988\,344\,092\,550\,795\,455\,053\,787\,171\,171\,272\,613\,525\,390\,625 - \left(5\,135\,746\,785\,881\,293\,900\,665\,825\,337\,251\,063\,912\,099\,812\,018\,333\,491\,820\,410\,850\,621\,042\,z^{71}\right) / 339\,003\,946\,094\,735\,963\,173\,345\,929\,589\,646\,769\,396\,958\,527\,805\,814\,290\,898\,437\,688\,022\,862\,701\,416\,015\,625 + \left(23\,247\,600\,823\,869\,669\,181\,617\,874\,661\,621\,842\,533\,234\,313\,612\,312\,895\,049\,759\,227\,683\,259\,644\,z^{73}\right) / 3\,786\,335\,073\,932\,105\,972\,683\,100\,687\,586\,764\,767\,394\,629\,797\,063\,139\,815\,044\,650\,537\,527\,353\,512\,115\,478\,515\,625 - \left(26\,145\,766\,198\,741\,584\,025\,528\,698\,683\,516\,199\,629\,583\,197\,662\,307\,446\,174\,936\,102\,767\,991\,445\,644\,z^{75}\right) / 10\,507\,079\,830\,161\,594\,074\,195\,604\,408\,053\,272\,229\,520\,097\,686\,850\,212\,986\,748\,905\,241\,638\,405\,996\,120\,452\,880\,859\,375 + \left(15\,502\,650\,114\,137\,077\,692\,879\,282\,322\,945\,671\,281\,504\,464\,245\,814\,336\,939\,030\,473\,090\,718\,667\,221 \right.$$

928 $z^{77}$) /
15 371 857 791 526 412 130 548 169 248 981 937 271 787 902 915 861 861 599 613 648 368 516 987 972
324 222 564 697 265 625 −
(624 447 880 395 344 915 327 575 701 011 165 339 822 237 764 093 445 803 722 686 391 628 220 033
123 558 $z^{79}$) /
1 527 764 317 925 576 637 878 029 337 293 978 991 431 565 447 863 561 148 013 214 536 238 736 772
346 159 023 284 912 109 375 +
(3 177 409 273 870 478 888 667 047 675 148 588 648 707 554 958 779 304 377 142 474 306 979 107 045
033 591 884 $z^{81}$) /
19 181 081 011 555 614 688 558 658 329 725 906 237 423 304 197 927 010 213 305 908 502 477 340 176
806 026 537 342 071 533 203 125 −
(4 382 231 878 630 427 838 203 781 834 402 677 719 332 895 725 070 795 917 855 918 399 166 899 403
692 359 715 604 $z^{83}$) /
65 273 218 682 323 756 785 165 114 296 057 258 925 951 504 185 545 615 755 880 006 633 930 388 621
670 908 306 575 069 427 490 234 375 +
(3 170 252 255 497 465 850 441 721 151 634 611 336 636 763 918 620 746 439 504 134 202 802 452 839
465 833 038 132 808 $z^{85}$) /
116 512 695 347 947 905 861 519 729 018 462 207 182 823 434 971 198 924 124 245 811 841 565 743
689 682 571 327 236 498 928 070 068 359 375 −
(282 743 645 351 878 196 427 175 381 372 737 723 603 024 007 898 834 815 645 760 821 235 876 739
738 607 970 302 830 604 $z^{87}$) /
25 639 646 664 510 183 283 996 782 721 062 771 592 408 380 601 603 245 596 988 446 005 841 026 302
535 441 137 364 220 146 465 301 513 671 875 +
(26 125 334 033 648 605 299 760 215 770 950 877 021 992 145 130 248 999 739 712 305 337 867 298 482
114 198 536 057 875 576 $z^{89}$) /
5 845 488 211 471 821 649 254 225 408 584 215 172 773 324 360 992 915 294 118 886 395 550 852 065
110 922 559 577 434 464 350 986 480 712 890 625 −
(3 165 183 288 800 001 305 552 844 563 646 295 861 445 487 620 103 190 581 092 424 979 876 576 485
235 244 926 077 451 042 788 728 $z^{91}$) /
1 747 421 018 496 329 004 719 811 872 515 122 362 672 993 717 853 417 133 447 429 304 653 993 962
083 933 635 347 280 371 600 762 143 611 907 958 984 375 +
(2 743 910 203 329 295 441 771 249 659 819 135 452 881 820 248 213 784 360 868 316 092 063 767 619
714 125 477 901 042 867 863 808 976 $z^{93}$) /
3 737 733 558 563 647 741 095 677 595 309 846 733 757 533 562 488 459 248 444 051 282 654 893 084
897 534 046 007 832 714 854 030 225 185 871 124 267 578 125 −
(4 965 369 860 827 668 851 290 623 237 994 135 971 062 634 507 133 697 075 181 739 578 164 128 821
938 437 552 932 557 133 865 324 691 926 $z^{95}$) /
16 688 980 338 986 687 163 992 200 463 058 465 666 227 387 356 510 970 544 302 688 977 054 097 624
067 489 515 424 973 071 823 244 955 454 914 569 854 736 328 125 +
(13 618 722 892 337 243 626 196 029 509 843 989 171 050 070 079 708 515 121 451 737 330 953 176 267
769 526 553 238 319 205 989 401 876 548 $z^{97}$) /
112 941 704 154 537 813 133 063 496 156 977 058 345 864 412 110 341 684 381 211 220 751 691 683
921 014 870 906 713 189 858 152 657 721 799 538 135 528 564 453 125 −
(905 838 570 048 586 218 745 173 742 117 616 558 174 626 778 700 773 083 971 608 582 082 083 300
800 057 692 087 180 696 588 351 163 326 044 $z^{99}$) /
18 535 679 696 858 777 383 843 519 947 971 924 100 345 314 960 922 504 303 800 151 196 111 426 769
580 058 982 725 432 267 570 131 653 944 398 944 377 899 169 921 875 + O[z]$^{101}$}

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following $n^{\text{th}}$ term of tanh($z$) can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[Tanh[z], {z, 0, n}] z^n
```

$$z^n \,\mathrm{If}\left[\mathrm{Odd}[n],\ \frac{2^{1+n}\left(-1+2^{1+n}\right)\mathrm{BernoulliB}[1+n]}{(1+n)\,!},\ 0\right]$$

This result can be verified by the following process. The previous expression is not zero for $n == 2\,k+1$ and the corresponding sum has the following value.

$$\mathrm{Sum}\left[\mathrm{Evaluate}\left[\frac{2^{1+n}\left(-1+2^{1+n}\right)\mathrm{BernoulliB}[1+n]}{(1+n)\,!}\,z^n \mathbin{/.} \{n\to 2\,k+1\}\right],\ \{k,\ 0,\ \infty\}\right] \mathbin{//} \mathrm{FunctionExpand}$$

```
Tanh[z]
```

## Differentiation

*Mathematica* can evaluate derivatives of the hyperbolic tangent function of an arbitrary positive integer order.

$$\partial_z \mathrm{Tanh}[z]$$

$$\mathrm{Sech}[z]^2$$

$$\partial_{\{z,2\}} \mathrm{Tanh}[z]$$

$$-2\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]$$

```
Table[D[Tanh[z], {z, n}], {n, 10}]
```

$$\{\mathrm{Sech}[z]^2,\ -2\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z],$$
$$-2\,\mathrm{Sech}[z]^4+4\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^2,\ 16\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]-8\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^3,$$
$$16\,\mathrm{Sech}[z]^6-88\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^2+16\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^4,$$
$$-272\,\mathrm{Sech}[z]^6\,\mathrm{Tanh}[z]+416\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^3-32\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^5,$$
$$-272\,\mathrm{Sech}[z]^8+2880\,\mathrm{Sech}[z]^6\,\mathrm{Tanh}[z]^2-1824\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^4+64\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^6,$$
$$7936\,\mathrm{Sech}[z]^8\,\mathrm{Tanh}[z]-24\,576\,\mathrm{Sech}[z]^6\,\mathrm{Tanh}[z]^3+7680\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^5-$$
$$128\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^7,\ 7936\,\mathrm{Sech}[z]^{10}-137\,216\,\mathrm{Sech}[z]^8\,\mathrm{Tanh}[z]^2+$$
$$185\,856\,\mathrm{Sech}[z]^6\,\mathrm{Tanh}[z]^4-31\,616\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^6+256\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^8,$$
$$-353\,792\,\mathrm{Sech}[z]^{10}\,\mathrm{Tanh}[z]+1\,841\,152\,\mathrm{Sech}[z]^8\,\mathrm{Tanh}[z]^3-$$
$$1\,304\,832\,\mathrm{Sech}[z]^6\,\mathrm{Tanh}[z]^5+128\,512\,\mathrm{Sech}[z]^4\,\mathrm{Tanh}[z]^7-512\,\mathrm{Sech}[z]^2\,\mathrm{Tanh}[z]^9\}$$

## Indefinite integration

*Mathematica* can calculate a huge set of doable indefinite integrals that contain the hyperbolic tangent function. Here are some examples.

$$\int \mathrm{Tanh}[z]\,dz$$

$$\text{Log}[\text{Cosh}[z]]$$

$$\int \text{Tanh}[z]^a \, dz$$

$$\frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Tanh}[z]^2\right] \text{Tanh}[z]^{1+a}}{1+a}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain the hyperbolic tangent function. Here are some examples.

$$\int_0^{\pi/2} \sqrt{\text{Tanh}[z]} \, dz$$

$$\frac{1}{2}\left(-i \,\text{Log}\left[1 - i\sqrt{\frac{-1+e^{\pi}}{1+e^{\pi}}}\right] + i \,\text{Log}\left[1 + i\sqrt{\frac{-1+e^{\pi}}{1+e^{\pi}}}\right] - \text{Log}\left[1 - \sqrt{\text{Tanh}\left[\frac{\pi}{2}\right]}\right] + \text{Log}\left[1 + \sqrt{\text{Tanh}\left[\frac{\pi}{2}\right]}\right]\right)$$

$$\int_0^{\frac{\pi}{2}} \text{Tanh}[z]^a \, dz$$

$$\frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Tanh}\left[\frac{\pi}{2}\right]^2\right] \text{Tanh}\left[\frac{\pi}{2}\right]^{1+a}}{1+a}$$

### Limit operation

*Mathematica* can calculate limits that contain the hyperbolic tangent function. Here are some examples.

$$\text{Limit}\left[\frac{\text{Tanh}[3\,z]}{z}, z \to 0\right]$$

$$3$$

$$\text{Limit}\left[\frac{\text{Tanh}\left[\sqrt{z^2}\right]}{z}, z \to 0, \text{Direction} \to 1\right]$$

$$-1$$

$$\text{Limit}\left[\frac{\text{Tanh}\left[\sqrt{z^2}\right]}{z}, z \to 0, \text{Direction} \to -1\right]$$

$$1$$

### Solving equations

The next inputs solve two equations that contain the hyperbolic tangent function. Because of the multivalued nature of the inverse hyperbolic tangent function, a message is printed indicating that only some of the possible solutions are returned.

$$\text{Solve}\left[\text{Tanh}[z]^2 + 3\,\text{Tanh}[z + \text{Pi}\, i / 6] == 4, z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

$$\left\{\left\{z \to -\text{ArcCosh}\left[-\sqrt{-\frac{5}{114}+\frac{i\sqrt{3}}{19}-\frac{1}{114}\sqrt{\frac{1}{2}\left(233+51\,i\sqrt{3}\right)}}\right]\right\},\right.$$

$$\left\{z \to -\text{ArcCosh}\left[\sqrt{-\frac{5}{114}+\frac{i\sqrt{3}}{19}-\frac{1}{114}\sqrt{\frac{1}{2}\left(233+51\,i\sqrt{3}\right)}}\right]\right\},$$

$$\left\{z \to \text{ArcCosh}\left[-\sqrt{-\frac{5}{114}+\frac{i\sqrt{3}}{19}+\frac{1}{114}\sqrt{\frac{1}{2}\left(233+51\,i\sqrt{3}\right)}}\right]\right\},$$

$$\left.\left\{z \to \text{ArcCosh}\left[\sqrt{-\frac{5}{114}+\frac{i\sqrt{3}}{19}+\frac{1}{114}\sqrt{\frac{1}{2}\left(233+51\,i\sqrt{3}\right)}}\right]\right\}\right\}$$

```
Solve[Tanh[x] == a, x]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{x → ArcTanh[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Tanh[x] == a, x] // InputForm
```

```
C[1] ∈ Integers && -1 + a^2 != 0 && x == ArcTanh[a] + I*Pi*C[1]
```

### Solving differential equations

Here is a linear inhomogeneous differential equation whose independent solutions include the hyperbolic tangent function.

```
DSolve[w′[z] == Sech[z]^2, w[z], z]
```

```
{{w[z] → C[1] + Tanh[z]}}
```

Here is a nonlinear differential equation whose solution is the hyperbolic tangent function.

```
DSolve[{w′[z] + w[z]^2 - 1 == 0, w[0] == 0}, w[z], z] // FullSimplify
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Tanh[z]}}
```

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Tanh[Sin[z]], {z, -6, 6}];
```

```
Plot3D[Re[Tanh[x + ⅈ y]], {x, -3, 3}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\text{ContourPlot}\left[\text{Arg}\left[\text{Tanh}\left[\frac{1}{x + i\, y}\right]\right], \left\{x, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{y, -\frac{1}{2}, \frac{1}{2}\right\},\right.$$

```
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → (Hue[0.78 #] &), ContourLines → False, Contours → 200];
```

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